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ORIGINAL ARTICLE

Scalar-tensor theory of gravity and generalized second law of thermodynamics on the event horizon

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Abstract Here we consider our universe as homogeneous spherically symmetric FRW model and analyze the thermodynamics of this model of the universe in scalar-tensor theory. Assuming the first law of thermodynamics validity of the generalized second law of thermodynamics (GSLT) at the event horizon is examined in both the cases when the universe is filled with perfect fluid and the holographic dark energy.

Keywords Scalar tensor theory · Thermodynamics

1 Introduction

In black hole physics, semi-classical description shows that just like a black body, black hole emits thermal radiation (known as Hawking radiation) and it completes the missing link between a black hole and a thermodynamical system. The temperature (known as the Hawking temperature) and the entropy (known as Bekenstein entropy) are proportional to the surface gravity at the horizon and area of the horizon (Hawking 1975; Bekenstein 1973) respectively (i.e. related to the geometry of the horizon). Also this temperature, entropy and mass of the black hole satisfy the first law of thermodynamics (Bardeen et al. 1973). As a result, physicists start speculating about the relationship between the black hole thermodynamics and Einstein's field equations (describing the geometry of space time). It is Jacobson (1995) who first derived Einstein field equations from the first law of thermodynamics: $\delta Q = T dS$ for all local Rindler causal horizons with δQ and T as the energy flux and Unruh temperature seen by an accelerated observer just inside the horizon. Then Padmanavan (2002) was able to formulate the first law of thermodynamics on the horizon, starting from Einstein equations for a general static spherically symmetric space time.

Subsequently, this identity between Einstein equations and thermodynamical laws has been applied in the cosmological context considering universe as a thermodynamical system bounded by the apparent horizon (R_A). Using the Hawking temperature $T_A = \frac{1}{2\pi R_A}$ and Bekenstein entropy $S_A = \frac{\pi R_A^2}{G}$ at the apparent horizon, the first law of thermodynamics (on the apparent horizon) is shown to be equivalent to Friedmann equations (Cai and Kim 2005) and the generalized second law of thermodynamics (GSLT) is obeyed at the horizon. Also in higher dimensional space time the relation was established (Cai and Kim 2005; Akbar and Cai 2006) for gravity with Gauss-Bonnet term and for the Lovelock gravity theory (Lancoz 1938).

But difficulty arises if we consider universe to be bounded by event horizon. First of all, in the usual standard big bang model cosmological event horizon does not exists. However the cosmological event horizon separates from that of the apparent horizon only for the accelerating phase of the universe (dominated by dark energy). Further, Wang et al. (2005) have shown that both first and second law of thermodynamics break down at the event horizon, considering the usual definition of temperature and entropy as in the apparent horizon. According to them the applicability of the first law of thermodynamics is restricted to nearby states of local thermodynamic equilibrium while event horizon reflects the global features of space time. Also due to existence of

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the cosmological event horizon, the universe should be non-static in nature and as a result the usual definition of the thermodynamical quantities on the event horizon may not be as simple as in the static space-time. They have considered the universe bounded by the apparent horizon as a Bekenstein system as Bekenstein's entropy-mass bound: $S \le 2E\pi R_A$ and entropy-area bound: $S \le \frac{A}{4}$ are valid in this region. These Bekenstein bounds are universal in nature and all gravitationally stable special regions with weak self gravity satisfy Bekenstein bounds. Finally, they have argued that as event horizon is larger than the apparent horizon so the universe bounded by the event horizon is not a Bekenstein system.

Recently Mazumder et al. (Mazumder and Chakraborty 2009, 2010) have examined the validity of the GSLT on the event horizon assuming the validity of the first law of thermodynamics on it. Without assuming any specific choice of the entropy and temperature on the event horizon, they were able to show the validity of the GSLT both for Einstein gravity and for Gauss-Bonnet gravity. The restrictions on the matter in the universe are the following:

(i) For flat and open FRW universe the generalized second law of thermodynamics is valid if the weak energy condition $\rho + p > 0$ is satisfied.
(ii) For a closed model, the validity of the generalized second law of thermodynamics demands either the weak energy condition is satisfied and $R_A < R_H = \frac{1}{H} < R_E$ or the weak energy condition is violated and $R_A < R_E < R_H$.

In this paper, we examine the validity of GSLT in Scalar-Tensor theory when the universe is bounded by the event horizon is filled with perfect fluid or holographic dark energy. The paper is organized as follows: Sect. 2 deals with the basic equations in Scalar-Tensor theory and its equivalence to Einstein gravity. The first law of thermodynamics, Clausius relation and the time variation of entropy functions are presented in Sect. 3. The generalized second law of thermodynamics has been examined for above two fluids and corresponding restrictions are determined in Sect. 4. The paper ends with a concluding remarks at the end in Sect. 5.

2 Scalar-Tensor theory of gravity

In Scalar-Tensor theory of gravity, using Jordan frame the lagrangian is given by Akbar and Cai (2006)

$$L = \frac{1}{16\pi G} f(\phi) R - \frac{1}{2} g^{\alpha\beta} \partial_\alpha\phi \, \partial_\beta\phi - V(\phi) + L_m \tag{1}$$

where $f(\phi)$ is a positive continuous otherwise arbitrary function of the scalar field ϕ (having potential $V(\phi)$), L_m is the lagrangian for matter fields in the universe and R is the Ricci curvature scalar of the space-time. For FRW model the metric is

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2\Omega_2^2\right] \tag{2}$$

and we have

$$R = 6(\ddot{a} + \dot{a}^2).$$

Now varying the action corresponding to the Lagrangian (1) with respect to the dynamical variables $g_{\mu\nu}$ and ϕ the equation of motions are

$$G_{\alpha\beta} = \frac{8\pi G}{f(\phi)}\Big[\partial_\alpha\phi \, \partial_\beta\phi - \frac{1}{2} g_{\alpha\beta}\big(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\big) - g_{\alpha\beta}V(\phi) - g_{\alpha\beta}\nabla^2 f + \nabla_\alpha\nabla_\beta f + T^{(m)}_{\mu\nu}\Big] \tag{3}$$

and

$$\nabla^2\phi - V'(\phi) + \frac{1}{16\pi G} f'(\phi) R = 0, \tag{4}$$

where $T^{(m)}_{\mu\nu}$ is the energy-momentum tensor of the matter fields.

If we assume $T^{(m)}_{\mu\nu}$ as the form of the energy-momentum tensor for a perfect fluid i.e.

$$T^{(m)}_{\alpha\beta} = (\rho + p)U_\alpha U_\beta + p \, g_{\alpha\beta}$$

with U^μ, the four velocity of the fluid, ρ and p the energy density and the pressure of the fluid then the non-vanishing components of the modified Einstein field equations (3) and the scalar field equations (4) are

$$H^2 + \frac{k}{a^2} = \frac{8\pi G}{3f}\left[\rho + \frac{1}{2}\dot{\phi}^2 + V(\phi) - \frac{3H}{8\pi G} f'\dot{\phi}\right] \tag{5}$$

$$\dot{H} - \frac{k}{a^2} = -\frac{4\pi G}{f}\Big[(\rho + p) + \dot{\phi}^2 + \frac{1}{8\pi G}\big(f''\dot{\phi}^2 + f'\ddot{\phi} - Hf'\dot{\phi}\big)\Big] \tag{6}$$

and

$$\ddot{\phi} + 3H\dot{\phi} + \frac{dV}{d\phi} = \frac{3}{8\pi G}\big(\dot{H} + H^2\big) f' \tag{7}$$

or equivalently modified Friedmann equations can be written as

$$H^2 + \frac{k}{a^2} = 8\pi G\left[\frac{\rho}{f} + \rho_{ST}\right] = \frac{8\pi G}{3f}\rho_{eff} \tag{5a}$$

$$\dot{H} - \frac{k}{a^2} = -4\pi G\left[\frac{(\rho + p)}{f} + (\rho_{ST} + p_{ST})\right] = -4\pi G(\rho_{eff} + p_{eff}) \tag{6a}$$

where

$$\rho_{ST} = \frac{[\frac{1}{2}\dot{\phi}^2 + V(\phi) - \frac{3H}{8\pi G} f'\dot{\phi}]}{f} \tag{8}$$

$$p_{ST} = \frac{[\frac{1}{2}\dot{\phi}^2 - V(\phi) + \frac{1}{8\pi G}(f''\dot{\phi}^2 + f'\ddot{\phi} + 2Hf'\dot{\phi})]}{f} \tag{9}$$

and

$$\rho_{eff} = \frac{\rho}{f} + \rho_{ST}, \qquad p_{eff} = \frac{p}{f} + p_{ST} \tag{10}$$

The energy conservation relations are

$$\dot{\rho} + 3H(\rho + p) = \frac{\rho \dot{f}}{f} \tag{11}$$

and

$$\dot{\rho}_{ST} + 3H(\rho_{ST} + p_{ST}) = 0 \tag{12}$$

Thus gravity in scalar-tensor theory is equivalent to Einstein gravity given by the field equations (5a) and (6a) with two fluid system having energy density and pressure $(\rho_1 = \frac{\rho}{f}, p_1 = \frac{p}{f})$ and (ρ_{ST}, p_{ST}).

3 Thermodynamic study

Let us start with FRW model for which the metric (2) can be written as

$$ds^2 = h_{ab}dx^a dx^b + R^2 d\Omega_2^2 \tag{13}$$

where $R = ar$ is the area radius and $h_{ab} = \text{diag}(-1, \frac{a^2}{1-kr^2})$ with $k = 0, +1, -1$ for flat, closed and open model.

Now to apply the first law of thermodynamics we define the work density W and the energy supply vector ψ_a (Cai and Cao 2007a, 2007b) as

$$W = -\frac{1}{2}T^{ab}h_{ab} \tag{14}$$

and

$$\psi_a = T_a^b \partial_b R + W\partial_a R \tag{15}$$

Here T_{ab} is the projection of the four dimensional energy-momentum tensor in the normal direction of the 2-sphere. For universe bounded by the event horizon, W gives the work done due to a change of the event horizon while the total energy flow through the event horizon is represented by the ψ_a. For FRW model with perfect fluid as the matter content the expressions for W and ψ_a are given by

$$W = \frac{1}{2}(\rho - p), \qquad \psi_a = \left[-\frac{1}{2}(\rho + p)HR_E, \frac{1}{2}(\rho + p)a\right] \tag{16}$$

where R_E is the radius of the event horizon.

In thermodynamics, the Clausius relation $\delta Q = TdS$ gives the associated heat flow for a change in entropy. But heat flow to the system is equivalent to a change of the energy of the system. Hence the entropy of the event horizon is related to the energy supply term by the relation

$$T_E dS_E = \delta Q = -dE = -A\psi = A(\rho + p)HR_E dt \tag{17}$$

where T_E, S_E are temperature and entropy of the event horizon respectively and $A = 4\pi R_E^2$ is the area of the event horizon.

For the present scalar-tensor theory (given by field equations (5a) and (6a)) the above relation (17) becomes

$$\frac{dS_E}{dt} = \frac{4\pi R_E^3 H}{T_E}(\rho_{eff} + p_{eff}) \tag{18}$$

To find the change of entropy S_I of the matter bounded by the event horizon, we start with the Gibbs' equation (Izquierdo and Pavon 2006)

$$T_E dS_I = dE_I + pdV \tag{19}$$

where E_I is the energy of the matter distribution, V is the volume bounded by the event horizon and for the thermodynamical equilibrium, the temperature of the matter is taken as that of the event horizon i.e. T_E. Now using

$$V = \frac{4\pi R_E^3}{3}, \qquad E_I = \frac{4}{3}\pi R_E^3 \rho \tag{20}$$

We have from the Gibbs' equation

$$\frac{dS_I}{dt} = \frac{4\pi R_E^2}{T_E}\left[(\rho + p)\left(\frac{dR_E}{dt} - HR_E\right) + \frac{\rho R_E \dot{f}}{3f}\right] \tag{21}$$

4 Generalized second law of thermodynamics

We shall now examine the validity of the generalized second law of thermodynamics (GSLT) for the following two cases:

4.1 Universe filled with perfect fluid

In this case, the rate of change of the radius of the event horizon is given by Mazumder and Chakraborty (2009)

$$\frac{dR_E}{dt} = HR_E - 1$$

So from (21)

$$\frac{dS_I}{dt} = -\frac{4\pi R_E^2}{T_E}\left[(\rho + p) - \frac{\rho R_E \dot{f}}{3f}\right] \tag{21a}$$

Thus combining (20) and (21a) we have

$$\frac{d}{dt}(S_E+S_I)=\frac{4\pi R_E^2}{T_E}\left[\left(\frac{HR_E}{f}-1\right)(\rho+p)+HR_E(\rho_{ST}+p_{ST})+\frac{\rho R_E\dot{f}}{3f}\right] \tag{22}$$

Hence the validity of GSLT i.e. $\frac{1}{dt}(S_E+S_I)\geq 0$ we have the following possibilities:

I. $\rho+p>0$, $R_E>fR_H$, $\rho_{ST}+p_{ST}>0$, $\dot{f}>0$ i.e. $\rho+p>0$, $\frac{f}{R_E}<H<\frac{1}{f'\dot{\phi}}[\dot{\phi}^2(8\pi G+f'')+f'\ddot{\phi}]=H_\phi$ (say) and $\dot{f}>0$.

II. $\rho+p>0$, $R_E<fR_H$, $\rho_{ST}+p_{ST}>0$, $\dot{f}>0$ and

$$\left|\frac{\rho_{ST}+p_{ST}}{\rho+p}\right|>\frac{fR_H}{R_E}-1 \tag{23}$$

i.e. $\rho+p>0$, $H<\min[\frac{f}{R_E},\ H_\phi]$, $\dot{f}>0$ and inequality (23).

III. $\rho+p>0$, $R_E>fR_H$, $\rho_{ST}+p_{ST}<0$ and

$$\left|\frac{\rho_{ST}+p_{ST}}{\rho+p}\right|<1-\frac{fR_H}{R_E} \tag{24}$$

i.e. $\rho+p>0$, $H>\max[\frac{f}{R_E},\ H_\phi]$, $\dot{f}>0$ and inequality (24).

IV. $\rho+p<0$, $R_E<fR_H$ and $\rho_{ST}+p_{ST}>0$, $\dot{f}>0$ i.e. $\rho+p<0$, $H<\min[\frac{f}{R_E},\ H_\phi]$, $\dot{f}>0$ and inequality (23).

V. $\rho+p<0$, $R_E>fR_H$, $\rho_{ST}+p_{ST}>0$, $\dot{f}>0$ and inequality (23) i.e. $\rho+p<0$, $\frac{f}{R_E}<H<H_\phi$, $\dot{f}>0$ and inequality (23).

VI. $\rho+p<0$, $R_E<fR_H$, $\rho_{ST}+p_{ST}<0$, $\dot{f}>0$ and inequality (24) i.e. $\rho+p<0$, $H_\phi<H<\frac{f}{R_E}$, $\dot{f}>0$ and inequality (24).

4.2 Universe filled with holographic dark energy

Recent observational evidences demand that our universe is experiencing an accelerated expansion driven by a missing energy density with negative pressure (known as dark energy). An approach to the problem of dark energy is holographic model (Li 2004; Setare and Shafei 2006; Hu and Ling 2006; Wang et al. 2005; Ito 2005; Nojiri and Odintsov 2006; Saridakis 2008a; Huang and Li 2004; Zhang 2005; Pavon and Zimdahl 2005; Kim et al. 2006; Hsu 2004; Horvat 2004). The holographic principle states that the no. of degrees of freedom for a system within a finite region should be finite and is bounded roughly by the area of its boundary. Using the effective quantum field theory the energy density for holographic dark energy can be written as (Cohen et al. 1999)

$$\rho_D=3c^2{M_p}^2L^{-2}$$

where L is an IR cut-off in units ${M_p}^2=1$ and c is any free dimensionless parameter whose value is determined by observational data (Huang and Li 2004; Chang et al. 2006; Zhang and Wu 2005, 2007; Wu et al. 2007; Shen et al. 2004; Saridakis and Setare 2008). Li (2004) has shown that $L=R_E$ gives the correct equation of state and also the desired acceleration. Thus we choose

$$\rho_D=\frac{3c^2}{R_E^2} \tag{25}$$

Then from the definition of the cosmological event horizon

$$R_E=a\int_a^\infty\frac{da}{Ha^2}=\frac{c}{(\sqrt{\Omega_D})H} \tag{26}$$

where $\Omega_D=\frac{\rho_D}{3H^2}$ is the density parameter corresponding to dark energy.

Now using (25) and the energy conservation equation (11), the time variation of R_E has the expression

$$dR_E=\frac{3}{2}R_EH(1+\omega_D)dt \tag{27}$$

where $p_D=\omega_D\rho_D$ is the equation of state of the DE and ω_D is not necessarily a constant. Using (27) in (21) we get

$$\frac{d}{dt}S_I=\frac{2\pi {R_E}^3H}{T_E}\left[(\rho_D+p_D)(3\omega_D+1)+\frac{2}{3}\frac{\rho_D\dot{f}}{Hf}\right] \tag{28}$$

Thus combining with (18)

$$\frac{d}{dt}(S_I+S_E)=\frac{2\pi {R_E}^3H}{T_E}\left[\rho_D(\omega_D+1)\left(\frac{2}{f}+3\omega_D+1\right)+\frac{2}{f}(\rho_{ST}+p_{ST})+\frac{2}{3}\frac{\rho_D\dot{f}}{Hf}\right] \tag{29}$$

Then for validity of GSLT any one of the following possibilities must be satisfied.

I. $\frac{2}{f}+3\omega_D+1>0$, $\omega_D+1>0$, $\rho_{ST}+p_{ST}>0$ and $\dot{f}>0$ i.e.

$$\frac{-1}{2}<f<1\quad \text{if } \omega_D<0\quad \text{or}\quad f<\frac{-1}{2}\quad \text{if } \omega_D<0,\ \dot{f}>0$$

and

$$H<H_\phi$$

II. $\frac{2}{f}+3\omega_D+1<0$, $\omega_D+1<0$, $\rho_{ST}+p_{ST}>0$ and $\dot{f}>0$ i.e.

$$\omega_D<\frac{2(1-f)}{3}-1,\quad f>1,\ \dot{f}>0 \text{ and } H<H_\phi$$

III. $\frac{2}{f}+3\omega_D+1>0$, $\omega_D+1>0$, $\rho_{ST}+p_{ST}<0$, $\dot{f}>0$ and

$$\left|\frac{\rho_{ST}+p_{ST}}{\rho_D+p_D}\right|<\left|1+\frac{f(3\omega_D+1)}{2}\right| \quad (30)$$

i.e.

$$\frac{-1}{2}<f<1 \quad \text{if } \omega_D<0 \quad \text{or} \quad f<\frac{-1}{2} \quad \text{if } \omega_D>0,$$

$H>H_\phi$, $\dot{f}>0$ and the inequality (30)

IV. $\frac{2}{f}+3\omega_D+1<0$, $\omega_D+1<0$ and $\rho_{ST}+p_{ST}<0$, $\dot{f}>0$ and inequality (30) i.e.

$$\omega_D<\frac{2(1-f)}{3}-1, \quad f>1 \text{ and } H>H_\phi, \ \dot{f}>0$$

inequality (30)

V. $\frac{2}{f}+3\omega_D+1>0$, $\omega_D+1<0$, $\dot{f}>0$, $\rho_{ST}+p_{ST}>0$ and

$$\left|\frac{\rho_{ST}+p_{ST}}{\rho_D+p_D}\right|>\left|1+\frac{f(3\omega_D+1)}{2}\right| \quad (31)$$

i.e.

$$\frac{3}{2}(1-f)-1<\omega_D<-1, \quad f>1, \ H<H_\phi, \ \dot{f}>0$$

and inequality (31).

IV. $\frac{2}{f}+3\omega_D+1<0$, $\omega_D+1>0$ and $\rho_{ST}+p_{ST}>0$, $\dot{f}>0$ and inequality (31) i.e.

$$-1<\omega_D<\frac{3(1-f)}{2}-1, \quad f<1, \ \dot{f}>0, \ H<H_\phi$$

and inequality (31)

5 Conclusions

From the study of the validity of GSLT in Scalar-Tensor gravity theory in the previous sections we have seen that as in other gravity theory the results do not depend on the specific choice of entropy function at the horizon—the only thing that we need is the validity of the first law of thermodynamics at the event horizon. For both the fluids, the geometric and physical quantities are restricted one by the other for the validity of the GSLT. For example, in case of perfect fluid the Hubble parameter is restricted by geometry in one hand and by the scalar field on the other. For dark energy model, both the equation of state parameter and the Hubble parameter are restricted by the scalar field for the validity of GSLT. At phantom divide line we have $\rho+p=0$ (or $\rho_D+p_D=0$ in the case of DE) and GSLT will be valid if $\rho_{ST}+p_{ST}>0$ for both the matter. Finally, it is interesting to note that if $f\to 1$ and $\phi\to 0$ then the conditions for the validity of GSLT at the event horizon becomes identical as in Einstein gravity (Mazumder and Chakraborty 2009, 2010). For future work, it will be nice to infer about the entropy function at the event horizon assuming the validity of GSLT there.

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