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ORIGINAL ARTICLE

Stability of equilibrium points in the generalized perturbed restricted three-body problem

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Abstract This paper studies the existence and stability of equilibrium points under the influence of small perturbations in the Coriolis and the centrifugal forces, together with the non-sphericity of the primaries. The problem is generalized in the sense that the bigger and smaller primaries are respectively triaxial and oblate spheroidal bodies. It is found that the locations of equilibrium points are affected by the non-sphericity of the bodies and the change in the centrifugal force. It is also seen that the triangular points are stable for $0 < \mu < \mu_c$ and unstable for $\mu_c \leq \mu < \frac{1}{2}$, where μ_c is the critical mass parameter depending on the above perturbations, triaxiality and oblateness. It is further observed that collinear points remain unstable.

Keywords Celestial mechanics · Three-body problem · Equilibrium points · Stability

1 Introduction

It is well known that there are five equilibrium points in the restricted three-body problem. Three are collinear with the primaries and the other two are in equilateral triangular configuration with the primaries. At these equilibrium points, the infinitesimal mass can be at rest in a rotating coordinate frame where the gravitational and centrifugal forces just balance each other. The collinear points $L_{1,2,3}$ are unstable, while the triangular points $L_{4,5}$ are stable for the mass ratio $\mu < 0.03852$ (Szebehely 1967a). Their stability occurs in spite of the fact that the potential energy has a maximum rather than a minimum at $L_{4,5}$. The stability is actually achieved through the influence of the Coriolis force, because the coordinate system is rotating (Wintner 1941; Contopoulos 2002).

In the classical problem, the effects of the gravitational attraction of the infinitesimal body and other perturbations have been ignored. Perturbations can well arise from the causes such as from the lack of the sphericity, or the triaxiality, oblateness, and radiation forces of the bodies, variation of the masses, the atmospheric drag, the solar wind, Poynting Robertson effect and the action of other bodies. The Kirkwood gaps in the ring of the asteroid's orbits lying between the orbits of the Mars and Jupiter are examples of the perturbation produced by Jupiter on an asteroid. This enables many researchers to study the restricted problem by taking into account the effects of small perturbations in the Coriolis and the centrifugal forces, radiation, oblateness and triaxiality of the bodies (Szebehely 1967a; Subbarao and Sharma 1975; Bhatnagar and Hallan 1978; Sharma et al. 2001; Abdul Raheem and Singh 2006; Singh 2009).

By Perturbations in the Coriolis and centrifugal forces we mean small changes caused by minor disturbances such as rotation of the coordinate axes tending to perturb the motion (coordinates, stability,etc) of the infinitesimal body. These effects are discussed by Szebehely (1967a), Bhatnagar and Hallan (1978), Abdul Raheem and Singh (2006), Singh (2009) and others. The same concept is being used in this paper. Szebehely (1967a) investigated the effect of the perturbation of the Coriolis force on the stability of the equilibrium points by keeping the centrifugal force constant. For the stability of the triangular points he asserted that the Coriolis force is a stabilizing force, whereas Subbarao and Sharma (1975) observed that the oblateness of the primary resulted in an increase in both the Coriolis and the centrifu-

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gal forces, thereby concluding that the Coriolis force is not always a stabilizing force. Bhatnagar and Hallan (1978) extended the above work of Szebehely by considering the effect of perturbations ε and ε' in the Coriolis and centrifugal forces respectively, on the stability of libration points. They showed that for the triangular points, the range of stability increases or decreases depending upon whether the point $(\varepsilon, \varepsilon')$ lies in one or the other of the two parts in which the $(\varepsilon, \varepsilon')$-plane is divided by the line $36\varepsilon - 19\varepsilon' = 0$, and that the stability of the collinear points is not affirmative.

The bodies in the classical restricted three-body problem are strictly spherical in shape, but in actual situations we find that several heavenly bodies, such as Saturn and Jupiter, are sufficiently oblate. Pluto and it's moon Charon are exactly frozen in triaxial rigid configuration. The minor planets and meteoroids have irregular shapes. In these cases, on account of the small dimensions of the bodies in comparison with their distances from their respective primaries, they are considered to be point masses but in many cases the dimensions of the bodies are larger than the distances from their respective primaries. Thus, the above assumption is not justified, and the results obtained are far from a realistic approach. The lack of sphericity, triaxiality, or oblateness of the planet causes large perturbations from a two-body orbit. The motions of artificial Earth satellites are examples of this. This inspires scientists to include the shapes of the bodies in their study (Subbarao and Sharma 1975; El-Shaboury 1991; Elipe 1992; Sharma et al. 2001; Abdul Raheem and Singh 2006; Chandra and Kumar 2004).

The stability of equilibrium points under the influence of small perturbations in the Coriolis and centrifugal forces, together with the effects of oblateness and radiation of both primaries was investigated by Abdul Raheem and Singh (2006). It was found that the Coriolis force has a stabilizing tendency, while the centrifugal force, radiation, and oblateness of the primaries have destabilizing effects; the presence of any one or more of the latter makes weak the stabilizing ability of the former. The overall effect is that the range of stability of the triangular points decreases.

This paper has attempted to analyze the motion of an infinitesimal mass in the gravitational field of the two primaries in the presence of small perturbations given in the Coriolis and the centrifugal forces. The primaries are modeled as a triaxial rigid body and an oblate spheroid with one of the axes as the axis of symmetry and their equatorial plane coinciding with the plane of motion. Also, the infinitesimal body is assumed to have no influence on the motion of these primaries.

2 Equations of motion

We consider a synodic coordinate system (o, x, y, z) with the origin at the center of mass of the primaries (Szebehely 1967b). The x-axis is taken along the line joining the primaries and let r_1 and r_2 be the distances of the infinitesimal mass m from the bigger and smaller primaries with masses m_1 and m_2, respectively. For the unit of distance we take the distance between the primaries to be equal to one. We choose the unit of mass such that the sum of the masses of the primaries is taken as unity. The unit of time is so chosen as to make the gravitational constant unity.

The equations of motion of the infinitesimal mass in the dimensionless synodic coordinate system become

$$\begin{aligned} \ddot{x} - 2n\dot{y} &= n^2 x + \frac{\partial F}{\partial x}, \\ \ddot{y} + 2n\dot{x} &= n^2 y + \frac{\partial F}{\partial y}, \end{aligned} \tag{1}$$

where

$$\begin{aligned} F = {} & \frac{1-\mu}{r_1} + \frac{\mu}{r_2} + \frac{1-\mu}{2m_1 r_1^3}(I_1 + I_2 + I_3 - 3I) \\ & + \frac{\mu}{2m_2 r_2^3}(I_1' + I_2' + I_3' - 3I') \\ & \text{(McCusky 1963)}, \end{aligned} \tag{2}$$

$$\begin{aligned} r_1^2 &= (x-\mu)^2 + y^2, \\ r_2^2 &= (x+1-\mu)^2 + y^2, \end{aligned} \tag{3}$$

and μ is the ratio of the mass of the smaller primary to the total mass of the primaries and $0 < \mu \le \frac{1}{2}$. n is the mean motion of the primaries.

I_1, I_2, I_3 are the principal moments of inertia of the triaxial rigid body (bigger primary) at its center of mass, with a, b, c as lengths of its semi-axes. I is the moment of inertia about a line joining the center of the bigger primary and the infinitesimal body and is given by

$$I = I_1 l_1'^2 + I_2 m_1'^2,$$

where l_1', m_1', are the direction cosines of the line with respect to its principal axes.

I_1', I_2', I_3' are the principal moments of inertia of the oblate spheroid (smaller primary) at its center of mass, with a', b', c' as lengths of its semi-axes. I' is the moment of inertia about a line joining the center of the smaller primary and the infinitesimal body and is given by

$$I' = I_1' l_1''^2 + I_2' m_1''^2,$$

where l_1'', m_1'', are the direction cosines of the line with respect to its principal axes.

We have also assumed here that the principal axes of m_1 and m_2 are parallel to the synodic axes $oxyz$ defined by Szebehely (1967b).

Proceeding as Sharma et al. (2001), we reduce (2) to

$$F = \frac{1-\mu}{r_1} + \frac{\mu}{r_2} + \frac{1-\mu}{2r_1^3}(2\sigma_1 - \sigma_2) - \frac{3(1-\mu)}{2r_1^5}(\sigma_1 - \sigma_2) + \frac{\mu A_2}{2r_2^3}, \tag{4}$$

where

$$\sigma_1 = \frac{a^2 - c^2}{5R^2}, \qquad \sigma_2 = \frac{b^2 - c^2}{5R^2}, \qquad \sigma_1, \sigma_2 \ll 1,$$
$$\sigma_1' = \frac{a'^2 - c'^2}{5R^2}, \qquad \sigma_2 = \frac{b'^2 - c'^2}{5R^2}, \qquad \sigma_1', \sigma_2' \ll 1,$$

and R is the dimensional distance between the primaries. For an oblate body $\sigma_1' = \sigma_2'$ so that $A_2 = \sigma_1'$ is the oblateness factor of the smaller primary.

The mean motion takes the form

$$n^2 = 1 + \frac{3}{2}(2\sigma_1 - \sigma_2) + \frac{3}{2}A_2. \tag{5}$$

Now we consider perturbations in the Coriolis and centrifugal forces with the help of the parameters α and β, respectively. The unperturbed value of each is unity.

Then, (1) become

$$\begin{aligned} \ddot{x} - 2\alpha n\dot{y} &= n^2\beta x + \frac{\partial F}{\partial x}, \\ \ddot{y} + 2\alpha n\dot{x} &= n^2\beta y + \frac{\partial F}{\partial y}. \end{aligned} \tag{6}$$

Here α and β maybe taken as

$$\alpha = 1 + \varepsilon, \quad |\varepsilon| \ll 1,$$
$$\beta = 1 + \varepsilon', \quad |\varepsilon'| \ll 1,$$

where $\varepsilon, \varepsilon'$ represent small perturbations in the Coriolis and centrifugal forces respectively.

Equation (6) can be put in the form

$$\begin{aligned} \ddot{x} - 2\alpha n\dot{y} &= \frac{\partial \Omega}{\partial x}, \\ \ddot{y} + 2\alpha n\dot{x} &= \frac{\partial \Omega}{\partial y}, \end{aligned} \tag{7}$$

where

$$\Omega = \frac{n^2\beta}{2}(x^2 + y^2) + \frac{1-\mu}{r_1} + \frac{\mu}{r_2} + \frac{1-\mu}{2r_1^3}(2\sigma_1 - \sigma_2) - \frac{3(1-\mu)}{2r_1^5}(\sigma_1 - \sigma_2)y^2 + \frac{\mu A_2}{2r_2^3}, \tag{8}$$

which describe the motion of the infinitesimal mass under the combined actions of small perturbations in the Coriolis and centrifugal forces and non-sphericity(triaxiality as well as oblateness) of the primaries other than gravitational forces. Equation (5) shows that the mean motion does not change despite the introduction of the above perturbations, while (8) indicates that the force function Ω is independent of the change in the Coriolis force. Thus, the equations of motion are affected by the non-sphericity of the primaries, and small changes in the Coriolis as well as centrifugal forces.

3 Location of equilibrium points

The equilibrium points are those points at which the velocity and acceleration of the infinitesimal mass are zero. Therefore, these points are the solutions of the equations

$$\Omega_x = 0, \qquad \Omega_y = 0. \tag{9}$$

Here two cases arise:

3.1 Triangular points

The triangular points are the solutions of (9) when $y \neq 0$. That is

$$n^2\beta x - \frac{(1-\mu)(x-\mu)}{r_1^3} - \frac{\mu(x+1-\mu)}{r_2^3} - \frac{3(1-\mu)(x-\mu)}{2r_1^5}(2\sigma_1 - \sigma_2) + \frac{15(1-\mu)(x-\mu)y^2}{2r_1^7}(\sigma_1 - \sigma_2) - \frac{3\mu(x+1-\mu)}{2r_2^5}A_2 = 0$$

and

$$n^2\beta - \frac{(1-\mu)}{r_1^3} - \frac{\mu}{r_2^3} - \frac{3(1-\mu)}{2r_1^5}(4\sigma_1 - 3\sigma_2) + \frac{15(1-\mu)y^2}{2r_1^7}(\sigma_1 - \sigma_2) - \frac{3\mu}{2r_2^5}A_2 = 0. \tag{10}$$

If we take $\sigma_1 = \sigma_2 = 0 = A_2$, the solutions of system (10) can be written as

$$r_1 = r_2 = \beta^{-\frac{1}{3}}$$

and from (5), $n = 1$.

Now, we assume that the solutions of system (10) for $\sigma_1, \sigma_2, A_2 \neq 0$, are

$$r_1 = \frac{1}{\beta^{\frac{1}{3}}} + \varepsilon_1, \qquad r_2 = \frac{1}{\beta^{\frac{1}{3}}} + \varepsilon_2, \tag{11}$$

where $\varepsilon_1, \varepsilon_2 \ll 1$.

Substituting the values of r_1 and r_2 from (11) in (3), we have

$$x = \mu - \frac{1}{2} + (\varepsilon_2 - \varepsilon_1)\beta^{-\frac{1}{3}},$$
$$y = \pm\frac{\sqrt{4-\beta^{\frac{2}{3}}}}{2\beta^{\frac{1}{3}}}\left[1 + \frac{2\beta^{\frac{1}{3}}(\varepsilon_1+\varepsilon_2)}{4-\beta^{\frac{2}{3}}}\right]. \tag{12}$$

Putting the value of r_1 and r_2 from (11), x, y from (12) and n^2 from (5) in system (10) and neglecting higher order terms, we obtain

$$\varepsilon_1 = \left[\frac{-\beta^{\frac{2}{3}}-1}{\beta^{\frac{1}{3}}} + \frac{5\beta}{8}\right]\sigma_1 + \left[\frac{3\beta^{\frac{2}{3}}+1}{2\beta^{\frac{1}{3}}} - \frac{5\beta}{8}\right]\sigma_2 + \left[-\frac{1}{2\beta^{\frac{1}{3}}}\right]A_2,$$
$$\varepsilon_2 = \left[\frac{\beta^{\frac{1}{3}}}{2\mu} - \frac{\beta^{\frac{2}{3}}+2}{2\beta^{\frac{1}{3}}}\right]\sigma_1 + \left[-\frac{\beta^{\frac{1}{3}}}{2\mu} + \frac{\beta^{\frac{2}{3}}+1}{2\beta^{\frac{1}{3}}}\right]\sigma_2 + \left[-\frac{1}{2\beta^{\frac{1}{3}}} + \frac{\beta^{\frac{1}{3}}}{2}\right]A_2. \tag{13}$$

Then, we obtain the coordinates of the triangular points $L_4(x, y)$ and $L_5(x, -y)$ as

$$x = \mu - \frac{1}{2} + \left[\frac{1}{2} + \frac{1}{2\mu} - \frac{5\beta^{\frac{2}{3}}}{8}\right]\sigma_1 + \left[-1 - \frac{1}{2\mu} + \frac{5\beta^{\frac{2}{3}}}{8}\right]\sigma_2 + \frac{1}{2}A_2,$$
$$y = \frac{\sqrt{4-\beta^{\frac{2}{3}}}}{2\beta^{\frac{1}{3}}}\left[1 + \frac{2}{4-\beta^{\frac{2}{3}}}\left\{\left(-\frac{3}{2}\beta^{\frac{2}{3}} - 2 + \frac{5\beta^{\frac{4}{3}}}{8} + \frac{\beta^{\frac{2}{3}}}{2\mu}\right)\sigma_1 + \left(2\beta^{\frac{2}{3}} + 1 - \frac{5\beta^{\frac{4}{3}}}{8} - \frac{\beta^{\frac{2}{3}}}{2\mu}\right)\sigma_2 + \left(-1 + \frac{\beta^{\frac{2}{3}}}{2}\right)A_2\right\}\right]. \tag{14}$$

We see that the positions of the triangular points are affected by the triaxiality of the bigger primary and oblateness of the smaller one. They are also influenced by the small perturbation in the centrifugal force. Since $r_1 \neq r_2$, they form scalene triangles with the primaries.

3.2 Collinear points

The collinear points are the solutions of the (9) when $y = 0$. That is,

$$f(x) = n^2\beta x - \frac{(1-\mu)(x-\mu)}{r_1^3} - \frac{\mu(x+1-\mu)}{r_2^3} - \frac{3}{2r_1^5}(1-\mu)(2\sigma_1-\sigma_2)(x-\mu) - \frac{3}{2r_2^5}\mu(x+1-\mu)A_2 = 0 \tag{15}$$

where

$$r_1 = |x-\mu|, \qquad r_2 = |x+1-\mu|.$$

These points lie on the line joining the primaries (i.e., x-axis), and their abscissae are the roots of (15). By the use of a method based on the topological degree theory described in Kalantonis et al. (2001), we determine with certainty the total number of collinear equilibrium points. Now, from (15), we can obtain the number N^r of collinear equilibrium points. Instances, where we have a function $f : [a, b] \subset \mathbb{R} \to \mathbb{R}$ to be twice continuously differentiable, the total number N^r of the roots of (15) can be obtained by the scheme (Picard 1892, 1922; Kavvadias and Vrahatis 1996):

$$N^r = -\frac{\gamma}{\pi}\int_a^b \frac{f(x)f''(x) - f'^2(x)}{f^2(x) + \gamma^2 f'^2(x)}dx + \frac{1}{\pi}\arctan\left(\frac{\gamma[f(a)f'(b) - f(b)f'(a)]}{f(a)f(b) + \gamma^2 f'(a)f'(b)}\right) \tag{16}$$

where γ is a small positive real constant. The relation (16) was shown by Picard (1892, 1922) to be independent of the value of γ.

It can be observed that at the points where $x = \mu$ and $x = \mu - 1$, the denominator of (15) vanishes. So, we apply (16) to the subintervals $[a, \mu - 1 - \delta]$, $[\mu - 1 + \delta, \mu - \delta]$ and $[\mu + \delta, b]$ of the interval $[a, b]$, where δ is a small real constant proportional to the machine precision. Let us assume that the masses of the bigger and smaller primaries are 4.824×10^{24} kg and 0.126×10^{24} kg respectively, then $\mu = 0.02545$. Also, taking arbitrary values for σ_i $(i = 1, 2)$, A_2 and ε' as 1.39×10^{-4}, 1.32×10^{-4}, 4.38×10^{-4} and 1.45×10^{-4} respectively, we use the Simpson's rule to compute the integral of (16) and find the corresponding derivative by the central finite differences method. Consequently, we obtain the total number of roots of (16) as three. Next, we compute the coordinates of the collinear equilibrium points $L_{1,2,3}$ by applying the bisection method at each one of the previous mentioned subintervals. Thus, we get

$$x_{L_1} = -0.86169197, \qquad x_{L_2} = -1.09377982,$$

Table 1 Coordinates of collinear points $L_{1,2,3}$

Case	x_{L_1}	x_{L_2}	x_{L_3}
1	−0.86433899	−1.09112292	1.01060326
2	−0.86434214	−1.09111793	1.01055514
3	−0.86434755	−1.09111859	1.01067765
4	−0.86166198	−1.09381849	1.01060361
5	−0.86435546	−1.09110607	1.01055687
6	−0.86167891	−1.09379140	1.01033755
7	−0.86169197	−1.09377982	1.01033937

$x_{L_3} = 1.01033937.$

In order to show the effects of various parameters, we obtain the coordinates of the collinear equilibrium points (Table 1) for different cases with the same values for σ_i $(i = 1, 2)$, A_2, ε', and μ as used in this study. We classify the cases as follows:

Case 1. Absence of Pertubations in the Coriolis and centrifugal forces as well as sphericity of the primaries (classical case);
Case 2. Perturbations in the Coriolis and centrifugal forces only;
Case 3. Triaxiality of the bigger Primary;
Case 4. Oblateness of the smaller Primary;
Case 5. Triaxiality of the bigger Primary and Perturbations in the Coriolis and the centrifugal forces;
Case 6. Oblateness of the smaller Primary and Perturbations in the Coriolis and the centrifugal forces;
Case 7. Pertubations in the Coriolis and centrifugal forces together with triaxiality of the bigger Primary and oblateness of the smaller Primary (present problem).

4 Stability of equilibrium points

In order to study the motion near any of the equilibrium points $L(x_0, y_0)$, we write

$$x = x_0 + \xi, \qquad y = y_0 + \eta$$

where ξ and η are small displacements in (x_0, y_0).

Putting these values in the (7), we obtain the variational equations of motion as

$$\begin{aligned} \ddot{\xi} - 2\alpha n\dot{\eta} &= \Omega^0_{xx}\xi + \Omega^0_{xy}\eta, \\ \ddot{\eta} + 2\alpha n\dot{\xi} &= \Omega^0_{xy}\xi + \Omega^0_{yy}\eta. \end{aligned} \tag{17}$$

Here, only linear terms in ξ and η have been taken. The second partial derivatives of Ω are denoted by subscripts. The superscript 0 indicates that the derivatives are to be evaluated at the equilibrium points (x_0, y_0).

The characteristic equation corresponding to (17) is

$$\lambda^4 + (4\alpha^2 n^2 - \Omega^0_{xx} - \Omega^0_{yy})\lambda^2 + \Omega^0_{xx}\Omega^0_{yy} - (\Omega^0_{xy})^2 = 0 \tag{18}$$

4.1 Stability of triangular points

In the case of triangular points, we have

$$\begin{aligned} \Omega^0_{xx} &= \frac{3}{4}\left[1 + \frac{5\varepsilon'}{3} + \left(\frac{19}{4} - \frac{2}{\mu} + \frac{15\mu}{4}\right)\sigma_1 \right. \\ &\quad \left. + \left(-\frac{1}{4} + \frac{2}{\mu} - \frac{31\mu}{4}\right)\sigma_2 + \left(4\mu + \frac{1}{2}\right)A_2\right], \\ \Omega^0_{xy} &= \frac{3\sqrt{3}}{2}\left[-\frac{1}{2} + \mu - \frac{11}{9}\left(\frac{1}{2} - \mu\right)\varepsilon' \right. \\ &\quad + \left(-\frac{47}{24} + \frac{1}{3\mu} + \frac{89\mu}{24}\right)\sigma_1 + \left(\frac{3}{8} - \frac{1}{3\mu} - \frac{37\mu}{24}\right)\sigma_2 \\ &\quad \left. + \left(-\frac{7}{12} + \frac{13\mu}{6}\right)A_2\right], \\ \Omega^0_{yy} &= \frac{9}{4} + \frac{7\varepsilon'}{4} + \left(\frac{87}{16} + \frac{3}{2\mu} - \frac{45\mu}{16}\right)\sigma_1 \\ &\quad + \left(-\frac{21}{16} - \frac{3}{2\mu} + \frac{45\mu}{16}\right)\sigma_2 + \frac{33}{8}A_2. \end{aligned}$$

On substituting these values in (18) and replacing λ^2 by Λ, we obtain

$$\Lambda^2 + P\Lambda + Q = 0 \tag{19}$$

where

$$\begin{aligned} P &= 1 + 8\varepsilon - 3\varepsilon' + 3\sigma_1 - \left(\frac{9}{2} - 3\mu\right)\sigma_2 \\ &\quad + \left(\frac{3}{2} - 3\mu\right)A_2 > 0, \\ Q &= \frac{27}{4}\mu(1-\mu) + \frac{33}{2}\mu(1-\mu)\varepsilon' \\ &\quad + \frac{9}{16}(-10 + 99\mu - 89\mu^2)\sigma_1 \\ &\quad + \frac{9}{16}(10 - 47\mu + 37\mu^2)\sigma_2 + \frac{117}{4}\mu(1-\mu)A_2, \end{aligned} \tag{20}$$

and

$$\Lambda_{1,2} = \frac{1}{2}\left[-P \pm \sqrt{P^2 - 4Q}\right].$$

Thus, the roots $\lambda_1 = +\Lambda_1^{1/2}, \lambda_2 = -\Lambda_1^{1/2}, \lambda_3 = +\Lambda_2^{1/2}$ and $\lambda_4 = -\Lambda_2^{1/2}$ depend on the value of the mass parameter $\mu, A_2, \varepsilon, \varepsilon'$ and σ_i $(i = 1, 2)$.

The discriminant Δ of (19) will be zero if $P^2 - 4Q = 0$. That is,

$$\Delta = \left(27 + 66\varepsilon' + \frac{801}{4}\sigma_1 - \frac{333}{4}\sigma_2 + 117A_2\right)\mu^2 + \left(-27 - 66\varepsilon' - \frac{891}{4}\sigma_1 + \frac{447}{4}\sigma_2 - 123A_2\right)\mu + \left(1 + 16\varepsilon - 6\varepsilon' + \frac{57}{2}\sigma_1 - \frac{63}{2}\sigma_2 + 3A_2\right) = 0. \quad (21)$$

Now, we have

$$(\Delta)_{\mu=0} = 1 + 16\varepsilon - 6\varepsilon' + \frac{57}{2}\sigma_1 - \frac{63}{2}\sigma_2 + 3A_2 > 0,$$

and

$$(\Delta)_{\mu=\frac{1}{2}} = -\frac{23}{4} + 16\varepsilon - \frac{45}{2}\varepsilon' - \frac{525}{16}\sigma_1 + \frac{57}{16}\sigma_2 - \frac{117}{4}A_2 < 0.$$

The opposite sign of the discriminant Δ at $\mu = 0$ and $\mu = \frac{1}{2}$ indicates that there is only one value of μ in the open interval $(0, \frac{1}{2})$ for which Δ vanishes. This value of μ is called the critical value of the mass parameter and is denoted by μ_c.

4.2 Critical mass

The solution of (21) for μ gives the critical mass value μ_c. That is,

$$\mu_c = \mu_0 + \mu_p + \mu_t + \mu_b, \quad (22)$$

where

$$\mu_0 = \frac{1}{2}\left(1 - \sqrt{\frac{23}{27}}\right),$$

$$\mu_p = \frac{4(36\varepsilon - 19\varepsilon')}{27\sqrt{69}},$$

$$\mu_t = \frac{1}{2}\left(\frac{5}{6} + \frac{59}{9\sqrt{69}}\right)\sigma_1 - \frac{1}{2}\left(\frac{19}{18} + \frac{85}{9\sqrt{69}}\right)\sigma_2,$$

$$\mu_b = \frac{1}{9}\left(1 - \frac{13}{\sqrt{69}}\right)A_2.$$

Evidently, μ_c represents the combined effects of perturbations, triaxiality, and oblateness on the critical mass value of the restricted three-body problem. However, in the absence of perturbations and non-sphericity of the primaries, the critical mass value μ_c becomes μ_0, which corresponds to the classical restricted problem. But in the absence of perturbations only (i.e., $\mu_p = 0$) in (22), μ_c reduces to the critical mass value of the restricted problem with non-spherical primaries. This confirms the result of Sharma et al. (2001) while the smaller primary is oblate (i.e., $\sigma_1' = \sigma_2' = A_2$). In this case $\mu_c < \mu_0$, which implies that the range of stability decreases. When the primaries are spherical (i.e., $\mu_t = 0 = \mu_b$), the critical mass value verifies the result of Bhatnagar and Hallan (1978). In this case $\mu_c > \mu_0$, which implies that the range of stability increases. Here, we conclude that in the absence of perturbations in the Coriolis and centrifugal forces, the non-sphericity(triaxiality as well as oblateness) of the primaries have a destabilizing tendency.

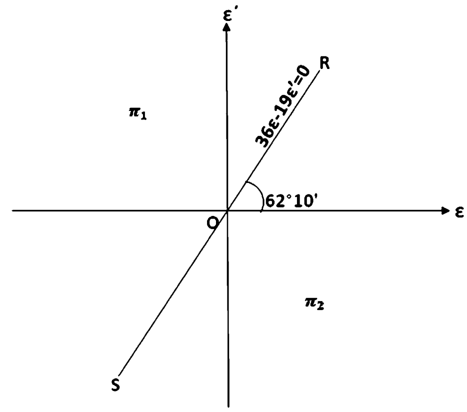

Fig. 1 Perturbations in the Coriolis and centrifugal forces

We now analyzes perturbing effects on the whole problem influenced by non-sphericity. For this, we consider the straight line ROS represented by the equation $36\varepsilon - 19\varepsilon' = 0$ in $(\varepsilon, \varepsilon')$ plane in the Fig. 1. This line divides the plane into two parts π_1 and π_2 as indicated. We call that part of the $(\varepsilon, \varepsilon')$ plane π_1 which, standing at O and looking towards OR, is on our right. π_2 is the other part. For the points in the part $\pi_1, 36\varepsilon - 19\varepsilon' > 0$ and for the points in the part $\pi_2, 36\varepsilon - 19\varepsilon' < 0$. Therefore, for a point $(\varepsilon, \varepsilon')$ in both parts, $\mu_c < \mu_0$, which implies that the range of the stability decreases. For a point $(\varepsilon, \varepsilon')$ lying on the line $36\varepsilon - 19\varepsilon' = 0$, $\mu_c < \mu_0$. This again implies that the range of stability decreases. In particular, when $\varepsilon = 0, \varepsilon' = 0$ [when the point $(\varepsilon, \varepsilon')$ coincides with the origin] and $\mu_t = 0, \mu_b = 0$, then $\mu_c = \mu_0$ and the case corresponds to the classical restricted three-body problem.

For points lying on ε-axis, $\varepsilon' = 0$, that is, there is no perturbation in the centrifugal force. Then,

$$\mu_c = \mu_0 + \mu_t + \mu_b + \frac{16\varepsilon}{3\sqrt{69}}. \quad (23)$$

Here we see that $\mu_c < \mu_0$. If $\mu_t = 0$, $\mu_b = 0$ then $\mu_c < \mu_0$. Thus, keeping the centrifugal force constant in the absence of triaxiality and oblateness, the Coriolis force remains a stabilizing force. But in the presence of either or both of them, it is no longer a stabilizing force. This is contrary to Szebehely's (1967b) result but confirms Subbarao and Sharma's (1975) result.

For points lying on ε'-axis, $\varepsilon = 0$. That is, there is no perturbation in the Coriolis force. Then,

$$\mu_c = \mu_0 + \mu_t + \mu_b - \frac{76\varepsilon'}{27\sqrt{69}}. \tag{24}$$

Here we observe that $\mu_c < \mu_0$. So, keeping the Coriolis force constant and in the absence or presence of either or both the triaxiality and oblateness, the centrifugal force is always a destabilizing force. In this case it implies that non-sphericity increase the destabilizing tendency of the centrifugal force.

Further, we can observe from (22) that the whole region of the mass parameter ($0 \le \mu \le \frac{1}{2}$) becomes unstable if the point $(\varepsilon, \varepsilon')$ lies on the line

$$4(36\varepsilon - 19\varepsilon') + 27\sqrt{69}(\mu_0 + \mu_t + \mu_b) = 0 \tag{25}$$

because in that case $\mu_c = 0$. Also, the whole region of the mass parameter except $\mu = \frac{1}{2}(0 \le \mu \le \frac{1}{2})$ is unstable if the point $(\varepsilon, \varepsilon')$ lies on the line

$$4(36\varepsilon - 19\varepsilon') + 27\sqrt{69}\left(\mu_0 + \mu_z + \mu_b - \frac{1}{2}\right) = 0 \tag{26}$$

because $\mu = \frac{1}{2}$.

Now, we consider the three regions of the values of μ separately;

(i) $0 \le \mu < \mu_c$.

In this region, we see that

$$-\frac{1}{2}P < \Lambda_1 \le 0 \quad \text{and} \quad -\frac{1}{2}P > \Lambda_2 \ge -P.$$

But $P > 0$, therefore, Λ_1 and Λ_2 are negative. So, in this case, the four roots of the characteristic equation are written as

$$\lambda_{1,2} = \pm i(-\Lambda_1)^{\frac{1}{2}} = \pm i s_1,$$
$$\lambda_{3,4} = \pm i(-\Lambda_2)^{\frac{1}{2}} = \pm i s_2,$$

showing that the triangular points are stable. Here

$$s_1 = \Big[\frac{27}{4}\mu(1-\mu) + \frac{33}{2}\mu(1-\mu)\varepsilon' + \left(-\frac{45}{8} + \frac{891}{16}\mu - \frac{801}{16}\mu^2\right)\sigma_1 + \left(\frac{45}{8} - \frac{471}{16}\mu + \frac{333}{16}\mu^2\right)\sigma_2 + \frac{117\mu}{4}(1-\mu)A_2\Big]^{\frac{1}{2}} \tag{27}$$

$$s_2 = \Big[1 + 8\varepsilon - 3\varepsilon' - \frac{27}{4}\mu(1-\mu) - \frac{33}{2}\mu(1-\mu)\varepsilon' + \left(\frac{69}{8} - \frac{891}{16}\mu + \frac{801}{16}\mu^2\right)\sigma_1 - \left(\frac{81}{8} - \frac{423}{16}\mu + \frac{333}{16}\mu^2\right)\sigma_2 + \left(\frac{3}{2} - \frac{129}{4}\mu + \frac{117}{4}\mu^2\right)A_2\Big]^{\frac{1}{2}} \tag{28}$$

Obviously, from (27) and (28), $s_1 < s_2$. We observe that both frequencies are dependent on the value of mass ratio μ and also on the triaxiality of the bigger primary and the oblateness of the smaller one. The frequency s_1 is dependent on the perturbation in the centrifugal force, while s_2 is dependent on both perturbations in the Coriolis and centrifugal forces.

(ii) $\mu_c < \mu < \frac{1}{2}$.

In this region, the discriminant of the characteristic equation is negative and $\Lambda_{1,2} = \frac{1}{2}(-P \pm \sqrt{d})$, where P is given by (20) and $d = P^2 - 4Q$. Therefore, $\Lambda_{1,2} = \frac{1}{2}(-P \pm i\delta)$, where

$$0 < \delta = +\sqrt{d} = \Big[16\varepsilon' - 1 - 16\varepsilon + 27\mu(1-\mu)\left(1 + \frac{22\varepsilon'}{9}\right) + \frac{9}{4}\left(-\frac{38}{3} + 99\mu - 89\mu^2\right)\sigma_1 + \frac{9}{4}\left(14 - \frac{149\mu}{3} + 37\mu^2\right)\sigma_2 + \frac{9}{4}\left(-\frac{4}{3} + \frac{164\mu}{3} - 52\mu^2\right)A_2\Big]^{\frac{1}{2}}. \tag{29}$$

The roots of the characteristic equation are

$$\lambda_{1,2} = \pm\Lambda_1^{1/2}, \lambda_{3,4} = \pm\Lambda_2^{1/2}$$

or

$$\lambda_1 = \frac{1}{\sqrt{2}}\sqrt{-P + i\delta} = \alpha_1 + i\beta_1,$$

$$\lambda_2 = -\frac{1}{\sqrt{2}}\sqrt{-P + i\delta} = \alpha_2 + i\beta_2,$$

$$\lambda_3 = \frac{1}{\sqrt{2}}\sqrt{-P - i\delta} = \alpha_3 + i\beta_3,$$

$$\lambda_4 = -\frac{1}{\sqrt{2}}\sqrt{-P - i\delta} = \alpha_4 + i\beta_4.$$

These roots have equal lengths, given by:

$$|\lambda| = |\lambda_{1,2,3,4}| = \frac{1}{\sqrt{2}}\left(\sqrt{P^2+\delta^2}\right)^{\frac{1}{2}},$$

where P and δ are given by (20) and (28).

The principal argument of the first root is given by

$$\theta = \theta_1 = \arctan\left[\frac{P \pm \sqrt{P^2+\delta^2}}{\delta}\right].$$

The arguments of the four roots are related by

$$\theta = \theta_1 = \theta_2 - \pi = 2\pi - \theta_3 = \pi - \theta_4.$$

The real and imaginary parts of the roots α_1 and β_1, are related by

$$\alpha = \alpha_1 = -\alpha_2 = \alpha_3 = -\alpha_4$$

and

$$\beta = \beta_1 = -\beta_2 = -\beta_3 = \beta_4,$$

where

$$\alpha = \frac{2}{2\sqrt{2|\lambda|^2+P}} > 0, \qquad \beta = \frac{\sqrt{2|\lambda|^2+P}}{2} > 0.$$

Thus, we see that the real parts of two of the characteristic roots are positive (and equal) and so the triangular points are unstable.

(iii) $\mu = \mu_c$.

In this region, the discriminant $\Delta = 0$, therefore

$$\Lambda_{1,2} = -\frac{1}{2}P \quad \text{and} \quad \lambda_1 = \lambda_3 = i\sqrt{\frac{1}{2}P},$$

$$\lambda_2 = \lambda_4 = -i\sqrt{\frac{1}{2}P}.$$

The double roots give secular terms in the solution of the equations of motion and so the triangular points are unstable.

4.3 Stability of collinear points

To examine the stability of the collinear points, we consider the points lying in the intervals $(\mu-2, \mu-1)$, $(\mu-1, 0)$ and $(\mu, \mu+1)$, respectively.

First we consider the point lying in $(\mu-2, \mu-1)$.

Here, $r_2 < 1$ and $r_1 > 1$. We have

$$\Omega^0_{xy} = 0,$$

$$\Omega^0_{xx} = n^2\beta + \frac{2(1-\mu)}{r_1^3} + \frac{2\mu}{r_2^3} + \frac{6(1-\mu)}{2r_1^5}(2\sigma_1 - \sigma_2) + \frac{6\mu A_2}{2r_2^5} > 0,$$

$$\Omega^0_{yy} = \frac{\mu\varepsilon'}{r_1} + \mu\left[\left(r_2 - \frac{1}{r_2^2}\right)\left(\frac{1}{r_2} - \frac{1}{r_1}\right)\right] + \frac{3\mu}{2r_1}(2\sigma_1 - \sigma_2) + \frac{3(1-\mu)}{2r_1^5}(-2\sigma_1 + 2\sigma_2) + \frac{3\mu A_2}{2}\left[\frac{1}{r_1}\left(1 + \frac{1}{r_2^4}\right) - \frac{1}{r_2^5}\right] < 0.$$

Similarly, for the points lying in $(\mu-1, 0)$ and $(\mu, \mu+1)$,

$$\Omega^0_{xy} = 0, \Omega^0_{xx} > 0 \quad \text{and} \quad \Omega^0_{yy} < 0.$$

Since $\Omega^0_{xx}\Omega^0_{yy} - (\Omega^0_{xy})^2 < 0$, the discriminant is positive and the four roots of the characteristic equation can be written as $\lambda_1 = s$, $\lambda_2 = -s$, $\lambda_3 = it$ and $\lambda_4 = -it$, where t and s are real. Hence, the motion around the collinear points is unbounded and therefore the collinear points are unstable.

5 Conclusion

By taking small perturbations in the Coriolis and centrifugal forces in the restricted three-body problem when the bigger primary is a triaxial rigid body and the smaller one an oblate spheroid, we have seen that the triangular points are stable for $0 < \mu < \mu_c$ and unstable for $\mu_c \le \mu < \frac{1}{2}$ where μ_c is the critical mass parameter defined in (22). We have also observed that the collinear points remain unstable despite the introduction of the various aforementioned perturbations.

In the case when $\sigma_1 = \sigma_2 = A_2 = 0 = \varepsilon = \varepsilon'$, the results obtained reduce to the classical problem (Szebehely 1967a). When $\sigma_1 = \sigma_2 = 0$ and $\varepsilon = \varepsilon' = 0$, the results agree with those of Bhatnagar and Hallan (1979). When $\sigma_1 = \sigma_2$ and $\varepsilon = \varepsilon' = 0$ the results are in agreement with those of Vidyakin (1974). When $\sigma_1 = \sigma_2 = A_2 = 0$, the results correspond to those of Bhatnagar and Hallan (1978). When $\varepsilon = \varepsilon' = 0$, the results obtained tally with those of Sharma et al. (2001) in which $\sigma_1' = \sigma_2' = A_2$. Also, we notice that both papers, present and Abdul Raheem and Singh (2006), describe the restricted problem under the influence of small perturbations in the Coriolis and the centrifugal forces together with the oblateness of the smaller primary. The bigger primary in the former is a triaxial rigid body while in the latter an oblate spheroid. Both primaries are also radiating in the latter. Thus, the results of the former when $\sigma_1 = \sigma_2$ coincide with those of the latter having non-luminous ($q_1 = q_2 = 1$) primaries.

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