Astrophys Space Sci (2010) 327: 299–308
DOI 10.1007/s10509-010-0339-5

ORIGINAL ARTICLE

Analysis on the stability of triangular points in the perturbed photogravitational restricted three-body problem with variable masses

Jagadish Singh · Oni Leke · Umar Aishetu

Received: 27 January 2010 / Accepted: 21 February 2010 / Published online: 30 March 2010

Abstract This paper examines the effect of a constant κ of a particular integral of the Gylden-Meshcherskii problem on the stability of the triangular points in the restricted three-body problem under the influence of small perturbations in the Coriolis and centrifugal forces, together with the effects of radiation pressure of the bigger primary, when the masses of the primaries vary in accordance with the unified Meshcherskii law. The triangular points of the autonomized system are found to be conditionally stable due to κ. We observed further that the stabilizing or destabilizing tendency of the Coriolis and centrifugal forces is controlled by κ, while the destabilizing effects of the radiation pressure remain unchanged but can be made strong or weak due to κ. The condition that the region of stability is increasing, decreasing or does not exist depend on this constant. The motion around the triangular points $L_{4,5}$ varying with time is studied using the Lyapunov Characteristic Numbers, and are found to be generally unstable.

Keywords Celestial mechanics · Variable masses · Radiation pressure

1 Introduction

The circular restricted three-body problem (CRTBP) describes the motion of an infinitesimal mass moving under the gravitational effect of the two finite masses, called primaries, which move in circular orbits around their center of mass on account of their mutual attraction and the infinitesimal mass not influencing the motion of the primaries.

The classical RTBP assumes that the infinitesimal mass moves under only the mutual gravitational force of the primaries, but in practice, Coriolis and centrifugal forces are effective and small perturbations affect these forces. Perturbations can well arise from the causes such as the lack of sphericity (oblateness), the atmospheric drag, the solar wind, and the actions of other bodies. An example is the motion of a close artificial satellite of the Earth perturbed by the atmospheric friction and the oblateness of the Earth. This motivates many researchers to study the RTBP by taking into account the effect of a small perturbation in the Coriolis and centrifugal forces.

Wintner (1941) showed that the stability of the triangular points is due to the existence of the Coriolis terms in the equations of motion when they are written in a rotating coordinate system. The absence of the Coriolis force renders the triangular solution unstable according to Wintner (1941), so that the oscillatory solution of the linearized equations of motion is replaced by exponential terms with real characteristic exponents.

The effect of a small perturbation in the Coriolis force on the stability of the equilibrium points, keeping the centrifugal force constant, was studied by Szebehely (1967b). He maintained that the collinear points remain unstable and obtained for the stability of the triangular points a relation between the critical value of the mass parameter μ_c and the change ε in the Coriolis force

$$\mu_c = \mu_0 + \frac{16\varepsilon}{3\sqrt{69}}.$$

J. Singh · O. Leke (✉) · U. Aishetu
Department of Mathematics, Faculty of Science, Ahmadu Bello University, Zaria, Nigeria
e-mail: lekkyonix4ree@yahoo.com

J. Singh
e-mail: jgds2004@yahoo.com

U. Aishetu
e-mail: umaraishetu33@yahoo.com

Then, he concluded that the Coriolis force is a stabilizing force.

This work was extended by Bhatnagar and Hallan (1978), by considering the effect of perturbations ε and ε' in the Coriolis and centrifugal forces, respectively, and found that collinear points remain unstable; for the triangular points they obtained the relation

$$\mu_c = \mu_0 + \frac{4(36\varepsilon - 19\varepsilon')}{27\sqrt{69}}.$$

They inferred that the range of stability increases or decreases depending on whether the points $(\varepsilon, \varepsilon')$ lie in one or the other of the two parts in which the $(\varepsilon, \varepsilon')$ plane is divided by the line $36\varepsilon - 19\varepsilon' = 0$. The combined effect of perturbations, radiation and oblateness on the stability of equilibrium points in the restricted three-body problem was studied by AbdulRaheem and Singh (2006). They found that the collinear points remain unstable, while the triangular points are conditionally stable, and they further observed that the Coriolis force has a stabilizing tendency, while the centrifugal force, radiation and oblateness of the primaries have destabilizing effects; consequently the overall effect is that the range of stability of the triangular points decreases.

The classical CRTBP is not suited to discuss the case when at least one of the interacting bodies is an intense emitter of radiation. According to Radzievskii (1950, 1953), the problem in such a statement is called the photogravitational problem. In certain stellar dynamics problems it is altogether inadequate to consider solely the gravitational interaction force. For example, when a star acts upon a particle in a cloud of gas and dust, the dominant factor is by no means gravity, but the repulsive force of the radiation pressure. Since a large fraction of all stars belong to binary systems, the particle motion in the field of a double star offers special interest. If a satellite flies high enough above the Earth and is large enough in size, but at the same time has sufficiently small mass, then the radiation pressure has a very strong effect on its motion. The distance of the satellite to the Sun practically is unaltered, and so the magnitude of the radiation pressure is practically constant.

Further, the classical model assumes that the masses of the bodies are constant, but there are numerous practical problems where the mass does not remain constant. There is a decrease in stellar mass, on account of light emission or corpuscularly. A satellite moving around a radiating star surrounded by cloud varies its mass due to particles of this cloud. Comets lose part of their mass traveling around the Sun (or other stars) due to their interaction with the solar wind, which blows off particles from their surfaces. The restricted problem dealing with variable mass of one, two or three bodies under different aspects have been studied by Orlov (1939), Gel'fgat (1973), Singh and Ishwar (1984, 1985), Lu (1990), Luk'yanov (1989a, 1990), El-Shaboury (1990), Bekov (1988, 1991), Bekov et al. (2005), Singh (2009), and, Singh and Oni (2010).

Gasanov (2008) investigated seven libration points and the general case in the problem of the motion of a star inside a layered inhomogeneous elliptical galaxy with variable mass. He examined the stability of these points using the Lyapunov Characteristic Numbers (LCN), and concluded that solutions with negative exponents are stable.

Our aim is to investigate the effect produced by a constant κ of a particular integral of the Gylden-Meshcherskii problem on the location and stability of the triangular points of a small particle in the restricted three-body problem under the influence of a small perturbation in the Coriolis and centrifugal forces, together with the effects of radiation pressure of the bigger primary, when both primaries vary their masses in accordance with the unified Meshcherskii law (1952) and with their motion governed by the Gylden-Meshcherskii problem (Gylden 1884; Meshcherskii 1952).

2 Equations of motion

The equations of motion of the infinitesimal body in the rotating barycentric coordinate system Oxyz, the x–y plane of which coincides with the plane of motion of the primaries, and the x-axis of which always passes through these points, as in the work by (Singh and Oni 2010), with the consideration that the bigger primary is an intense emitter of radiation have the form

$$\begin{aligned}
\ddot{x} - 2\omega\dot{y} &= \omega^2 x + \dot{\omega} y - \mu_1 q \frac{(x - x_1)}{r_1^3} - \mu_2 \frac{(x - x_2)}{r_2^3},\\
\ddot{y} + 2\omega\dot{x} &= \omega^2 y - \dot{\omega} x - \mu_1 q \frac{y}{r_1^3} - \mu_2 \frac{y}{r_2^3},\\
\ddot{z} &= -\mu_1 q \frac{z}{r_1^3} - \mu_2 \frac{z}{r_2^3},
\end{aligned} \tag{1}$$

with

$$r_i^2 = (x - x_i)^2 + y^2 + z^2, \quad (i = 1, 2)$$

where r_1 and r_2 are distances of the infinitesimal mass from these primaries m_1 and m_2 positioned at $(x_1, 0, 0)$ and $(x_2, 0, 0)$ respectively, r is the distance between the primaries, and ω is their angular velocity. μ_1 and μ_2 are the product of the masses of the primaries and the gravitational constant f. q is the radiation factor of the bigger primary, and is given (Radzievskii 1950) by $q = 1 - (F_p/F_g)$ such that $0 < 1 - q \ll 1$, where F_g is the gravitational attraction and F_p is the radiation repulsive force. A dot denotes differentiation with respect to time t.

Introducing small perturbations ε' and ε'' in the Coriolis and the centrifugal forces with the help of the parameters φ and ψ respectively such that

$$\varphi = 1 + \varepsilon', \quad |\varepsilon'| \ll 1 \quad \text{and} \quad \psi = 1 + \varepsilon'' \quad |\varepsilon''| \ll 1,$$

the system (1) now has the form

$$\begin{aligned} &\ddot{x} - 2\omega\dot{y}\varphi - \dot{\omega}\varphi y = \omega^2 x\psi - \frac{\mu_1 q(x - x_1)}{r_1^3} - \frac{\mu_2(x - x_2)}{r_2^3}, \\ &\ddot{y} + 2\omega\dot{x}\varphi + \dot{\omega}\varphi x = \omega^2 y\psi - \frac{\mu_1 q y}{r_1^3} - \frac{\mu_2 y}{r_2^3}, \\ &\ddot{z} = -\mu_1 \frac{qz}{r_1^3} - \mu_2 \frac{z}{r_2^3}. \end{aligned} \tag{2}$$

The equations of motion of system (2) are non-integrable differential equations with variable coefficients. We transform (2) from (x, y, z, t) to (ξ, η, ζ, τ). Following Luk'yanov (1989a), we use the Meshcherskii's (1952) transformation:

$$\begin{aligned} &x = \xi R(t), \qquad y = \eta R(t), \qquad \frac{\mathrm{d}t}{\mathrm{d}\tau} = R^2(t), \\ &r = \rho_{12} R(t), \qquad r_i = \rho_i R(t), \quad (i = 1, 2), \end{aligned} \tag{3}$$

the Meshcherskii unified law (1952),

$$\mu(t) = \frac{\mu_0}{R(t)}, \qquad \mu_1(t) = \frac{\mu_{10}}{R(t)}, \qquad \mu_2(t) = \frac{\mu_{20}}{R(t)},$$

$$\mu(t) = \mu_1(t) + \mu_2(t)$$

where $R(t) = \sqrt{\alpha t^2 + 2\beta t + \gamma}$: $\alpha, \beta, \gamma, \mu_0, \mu_{10}$ and μ_{20} are constants, and the particular solutions of the Gylden-Meshcherskii problem (Gylden 1884; Meshcherskii 1952)

$$\begin{aligned} &\omega(t) = \frac{\omega_0}{R^2(t)}, \quad x_1 = \xi_1 R(t), \quad x_2 = \xi_2 R(t), \\ &C = \rho_{12}^2 \omega_0 \end{aligned} \tag{4}$$

The system (2) is reduced to the autonomous form:

$$\begin{aligned} &\xi'' - 2\omega_0\varphi\eta' = \frac{\partial \Omega}{\partial \xi}, \qquad \eta'' + 2\omega_0\varphi\xi' = \frac{\partial \Omega}{\partial \eta}, \\ &\zeta'' = \frac{\partial \Omega}{\partial \zeta} \end{aligned} \tag{5}$$

where

$$\Omega = \frac{(\xi^2 + \eta^2)(\omega_0^2 \psi + \Delta)}{2} + \frac{\Delta \zeta^2}{2} + \frac{q\mu_{10}}{\rho_1} + \frac{\mu_{20}}{\rho_2},$$

$$\rho_i^2 = (\xi - \xi_i)^2 + \eta^2 + \zeta^2, \quad (i = 1, 2),$$

$$\Delta = \beta^2 - \alpha\gamma = \omega_0^2(\kappa - 1)$$

$\xi_1 = \frac{-\mu_{20}}{\mu_0}\rho_{12}, \xi_2 = \frac{\mu_{10}}{\mu_0}\rho_{12}$: ρ_{12} a constant, κ is an arbitrary dimensionless constant of a particular integral $r\mu = \kappa C^2$ (Gel'fgat 1973) of the Gylden-Meshcherskii problem (Gylden 1884; Meshcherskii 1952), C is a constant of the area integral, and prime denotes a differentiation with respect to the new independent time τ. The relation $\rho_{12}\mu_0 = \kappa C^2$ connects ρ_{12} and the parameter κ.

Following Luk'yanov (1989a), we choose units for the sum of the masses, distance and time, such that at initial time t_0, $\mu_0 = f$, $\rho_{12} = 1$, $\omega_0 = C = 1$, respectively.

Consequently

$$\kappa = \beta^2 - \alpha\gamma + 1, \tag{6}$$

The ranges of variation of the parameter κ are as follows.

(i) If $\beta^2 - \alpha\gamma = 0$, we would have $\kappa = 1$
(ii) If $\beta^2 - \alpha\gamma > 0$, this implies $1 < \kappa < \infty$
(iii) If $\beta^2 - \alpha\gamma < 0$, this implies $0 < \kappa < 1$

These are obtained from the conditions that the distances and masses are positive and $R(t)$ is real.

Introducing the mass parameter υ, expressed as $\frac{\mu_{10}}{\mu_0} = 1 - \upsilon$, $\frac{\mu_{20}}{\mu_0} = \upsilon$ (Luk'yanov 1989a), where $0 < \upsilon \le \frac{1}{2}$ and where υ is the ratio of the mass of the smaller primary to the total mass of the primaries.

The autonomized system (5) becomes

$$\xi'' - 2\varphi\eta' = \frac{\partial \Omega}{\partial \xi}, \qquad \eta'' + 2\varphi\xi' = \frac{\partial \Omega}{\partial \eta}, \qquad \zeta'' = \frac{\partial \Omega}{\partial \zeta} \tag{7}$$

where

$$\Omega = \frac{(\xi^2 + \eta^2)(\psi + \kappa - 1)}{2} + \frac{(\kappa - 1)\zeta^2}{2} + \frac{q\kappa(1 - \upsilon)}{\rho_1} + \frac{\kappa\upsilon}{\rho_2},$$

$$\rho_1 = \sqrt{(\xi + \upsilon)^2 + \eta^2}, \qquad \rho_2 = \sqrt{(\xi + \upsilon - 1)^2 + \eta^2}.$$

For $\kappa = 1$, system (7) is fully analogous to that of Devi and Singh (1994). If further we ignore the centrifugal force, i.e. $\psi = 1$, then the system (7) is fully analogous to that of Bhatnagar and Chawla (1979).

In the problem under consideration, we consider motion only in the ξ–η plane, though other particular solutions of the system (7), e.g. coplanar solutions $L_{6,7}$ exist for $\kappa > 1$ (Bekov 1988; Luk'yanov 1989a; Singh and Oni 2010). Infinitely remote solutions $L_{\pm\infty}$ (Luk'yanov 1988) exist for $\xi = \eta = 0$ and $\zeta = \pm\infty$ only for $\kappa = 1$, i.e. only for the first Mescherskii law (1952).

3 Location of equilibrium points of the autonomized system

The equilibrium points are the solutions of the equations

$$\frac{\partial \Omega}{\partial \xi}=0, \qquad \frac{\partial \Omega}{\partial \eta}=0$$

The particular solutions and/or stability of the restricted three-body problem for the isotropic case of mass variation have been considered in different respects by Gel'fgat (1973), Bekov (1988), Luk'yanov (1989a, 1990), and Singh and Oni (2010).

For the perturbed case, the triangular points of the autonomized system are the solutions of (7), i.e.

$$(\psi+\kappa-1)\xi-\frac{q\kappa(1-\upsilon)(\xi+\upsilon)}{\rho_1^3}-\frac{\kappa\upsilon(\xi+\upsilon-1)}{\rho_2^3}=0$$

and

$$\left(\psi+\kappa-1-\frac{q\kappa(1-\upsilon)}{\rho_1^3}-\frac{\kappa\upsilon}{\rho_2^3}\right)\eta=0, \quad \eta\neq 0. \tag{8}$$

Solving (8), we have

$$\rho_1=\left(\frac{q\kappa}{\psi+\kappa-1}\right)^{\frac{1}{3}}, \qquad \rho_2=\left(\frac{\kappa}{\psi+\kappa-1}\right)^{\frac{1}{3}}.$$

Hence, the coordinates of the triangular points are

$$\xi=\left[\frac{\kappa^{2/3}(q^{2/3}-1)}{2(\psi+\kappa-1)^{2/3}}-\upsilon+\frac{1}{2}\right],$$
$$\eta=\pm\left\{\frac{\kappa^{2/3}(q^{2/3}+1)}{2(\psi+\kappa-1)^{2/3}}-\left(\frac{\kappa^{2/3}(q^{2/3}-1)}{2(\psi+\kappa-1)^{2/3}}\right)^2-\frac{1}{4}\right\}^{\frac{1}{2}} \tag{9}$$

where the positive sign corresponds to L_4 and the negative to L_5. These points form simple triangles with the line joining the primaries, different from the classical problem where these points make equilateral triangles.

For the system of equations with variable coefficients, the equilibrium points are determined (Luk'yanov 1990) from the transformation (3) and the particular solutions (9) in the form

$$x^{(i)}=\xi^{(i)}R(t), \qquad y^{(i)}=\eta^{(i)}R(t), \quad i=4,5 \tag{10}$$

where $\xi^{(i)}(\tau), \eta^{(i)}(\tau)$ are the libration points of the system with constant coefficients. Consequently, the triangular points $L_{4,5}$ of the time dependent dynamical system (2) differ from those of the autonomized system (7) only by the function $R(t)$.

4 Stability of the triangular points of the autonomous system

The stability of linear systems of ordinary differential equations with constant coefficients is determined by the eigen values. Let the infinitesimal mass be displaced a little from the equilibrium point by giving it a small displacement with a small velocity. If its motion is a rapid departure from the vicinity of the point, we can call such a position of the equilibrium an "unstable one"; if, however, the body merely oscillates about the point, it is said to be a "stable position".

We denote the equilibrium points and their positions as $L(\xi_0,\eta_0)$. Let a small displacement in (ξ_0,η_0) be (u,v). Then we can write

$$\xi=\xi_0+u, \qquad \eta=\eta_0+v, \tag{11}$$

Substituting these values in (7) we obtain the variational equations

$$\begin{aligned} u''-2v'&=\left(\Omega^0_{\xi\xi}\right)u+\left(\Omega^0_{\xi\eta}\right)v,\\ v''+2u'&=\left(\Omega^0_{\xi\eta}\right)u+\left(\Omega^0_{\eta\eta}\right)v \end{aligned} \tag{12}$$

where the superscript 0 indicates that the partial derivatives are evaluated at the equilibrium points (ξ_0,η_0).

In a computation of these derivatives, we will substitute $q=1-\varepsilon$, $\varphi=1+\varepsilon'$, $\psi=1+\varepsilon''$: $|\varepsilon|\ll 1$, $|\varepsilon'|\ll 1$, $|\varepsilon''|\ll 1$ and neglect the second and higher order terms in $\varepsilon,\varepsilon',\varepsilon''$ and their products.

In the case of triangular points of the autonomized system, we have

$$\Omega^o_{\xi\xi}=\frac{1}{4}\left(3\kappa-2\kappa\varepsilon+6\kappa\upsilon\varepsilon+5\varepsilon''\right), \tag{13}$$

$$\Omega^o_{\eta\eta}=\frac{1}{4}\left(9\kappa+2\kappa\varepsilon-6\kappa\upsilon\varepsilon+7\varepsilon''\right), \tag{14}$$

$$\Omega^o_{\xi\eta}=\left(\frac{3\kappa-6\kappa\upsilon-2\kappa\upsilon\varepsilon+5\varepsilon''-10\upsilon\varepsilon''}{4}\right)\times\left(\frac{9\kappa-8\varepsilon''-4\kappa\varepsilon}{3\kappa}\right)^{1/2}. \tag{15}$$

Using the first two equations of system (12) and the above derivatives, the characteristic equation in the triangular case is

$$\lambda^4-\left(3\kappa-4+3\varepsilon''-8\varepsilon'\right)\lambda^2+\frac{3}{4}\kappa\left(9\kappa+22\varepsilon''+2\kappa\varepsilon\right)\upsilon(1-\upsilon)=0. \tag{16}$$

The roots of (16) are given by

$$\lambda^2_{1,2}=\frac{-P\pm\sqrt{D}}{2} \tag{17}$$

where

$$P = 4 - 3\kappa - 3\varepsilon'' + 8\varepsilon', \qquad D = P^2 - 4Q,$$

$$Q = \frac{3\kappa}{4}\big(9\kappa + 22\varepsilon'' + 2\kappa\varepsilon\big)\upsilon(1-\upsilon) > 0.$$

Consequently, the roots of the characteristic equation depend on the value of the mass parameter υ, the radiation parameter ε, perturbations $\varepsilon, \varepsilon'$ and the constant κ. So the nature of these roots is controlled by the constant κ and the sign of the discriminant D, given by

$$D = 3\kappa\big(9\kappa + 22\varepsilon'' + 2\kappa\varepsilon\big)\upsilon^2 - 3\kappa\big(9\kappa + 22\varepsilon'' + 2\kappa\varepsilon\big)\upsilon + 9\kappa^2 - 24\kappa + 16 + (4 - 3\kappa)\big(16\varepsilon' - 6\varepsilon''\big) \qquad (18)$$

Since D is a monotonous function of υ in the interval $(0, 1/2]$ and has values opposite in signs at the endpoints, for $0 < \kappa < 10$, there are many values of υ, say υ_{C_κ} in the interval $0 < \upsilon \le \frac{1}{2}$ for which the discriminant is zero. We consider the three regions of the value of υ coupled with the changes in P which is solely due to κ.

1. When $0 < \upsilon < \upsilon_{C_\kappa}$, $P > 0$, $D > 0$, in this case all λ_i $(i = 1, 2, 3, 4)$ are pure imaginary and given as $\lambda_{1,2,3,4} = \pm i\Lambda_n$ $(n = 1, 2)$ where

$$\Lambda_{1,2} = \sqrt{\frac{1}{2}(-P \pm \sqrt{D})} \qquad (19)$$

Consequently, the triangular point is stable in this case.

The solution is written (Szebehely 1967a, 1967b), as

$$\begin{aligned} u &= A_1 \cos\Lambda_1\tau + c_1 \sin\Lambda_1\tau + A_2\cos\Lambda_2\tau + c_2\sin\Lambda_2\tau, \\ v &= \bar{A}_1\cos\Lambda_1\tau + \bar{C}_1\sin\Lambda_1\tau + \bar{A}_2\cos\Lambda_2\tau + \bar{C}_2\sin\Lambda_2\tau, \end{aligned} \qquad (20)$$

where $A_i, \bar{A}_i, c_i$ and $\bar{C}_i (i = 1, 2)$ are constants.

2. When $P < 0, 0 < \upsilon < \upsilon_{C_\kappa}, D > 0$ and in this case the roots are real and distinct and can be written as

$$\lambda_{1,2} = \pm U_1, \qquad \lambda_{34} = \pm U_2$$

where

$$U_{1,2} = \left(\frac{1}{2}(P \pm \sqrt{D})\right)^{\frac{1}{2}}. \qquad (21)$$

The general solution for real roots with the condition $P < 0, D > 0$ can be represented as

$$\begin{aligned} u &= A_1e^{U_1\tau} + A_2e^{-U_1\tau} + A_3e^{U_2\tau} + A_4e^{-U_2\tau}, \\ v &= c_1A_1e^{U_1\tau} + c_1A_2e^{-U_1\tau} + c_2A_3e^{U_2\tau} + c_2A_4e^{-U_2\tau}, \end{aligned} \qquad (22)$$

where c_1, c_2, A_1 and A_2 are constants. The positive roots induce instability at the triangular points.

3. When $\upsilon_{C_\kappa} < \upsilon \le \frac{1}{2}, D < 0, P < 0, P > 0$, the real parts of two of the values of λ are positive and equal. Therefore, the triangular point is unstable.
4. When $\upsilon = \upsilon_{C_\kappa}, D = 0$ the double roots give secular terms in the solution of the equations of motion. Therefore, the triangular point is unstable.

4.1 Critical mass parameter

The value of the mass ratio υ, when $D = 0$, denoted υ_{C_κ} is given by

$$\upsilon_{C_\kappa} = \upsilon_{0_\kappa} + \upsilon_{r_\kappa} + \upsilon_{p1_\kappa} + \upsilon_{p2_\kappa} \qquad (23)$$

where

$$\upsilon_{0_\kappa} = \frac{1}{2} - \frac{1}{6\kappa\sqrt{3}}\sqrt{96\kappa - 9\kappa^2 - 64},$$

$$\upsilon_{r_\kappa} = \frac{2(24\kappa - 9\kappa^2 - 16)}{27\kappa\sqrt{3}\sqrt{96\kappa - 9\kappa^2 - 64}}\varepsilon,$$

$$\upsilon_{p1_\kappa} = \frac{16(4 - 3\kappa)}{3\kappa\sqrt{3}\sqrt{96\kappa - 9\kappa^2 - 64}}\varepsilon',$$

$$\upsilon_{p2_\kappa} = \frac{4(78\kappa - 9\kappa^2 - 88)}{27\kappa^2\sqrt{3}\sqrt{96\kappa - 9\kappa^2 - 64}}\varepsilon''.$$

Equation (23) represents the effects of the constant κ of a particular integral of the Gylden-Meshcherskii problem, perturbations and radiation on the critical mass values υ_{C_κ}. However, for $\kappa = 1, 2$ the value of $\upsilon_{0_{1,2}} = 0.038520$ and it coincides with the classical value given by Szebehely (1967a), but differs for $2 < \kappa < 10$ and does not exist for $0 < \kappa \le 0.714531$, $\kappa = \frac{4}{3}$ and $\kappa \ge 10$. In the absence of radiation and perturbations (i.e. $\varepsilon = \varepsilon' = \varepsilon'' = 0$) and $\kappa = 1, 2$, the values $\upsilon_{C_{1,2}}$ fully coincide with the classical case of Szebehely (1967a).

If there is no perturbation in the centrifugal force, (i.e.$\varepsilon'' = 0$),

$$\upsilon_{C_\kappa} = \upsilon_{0_\kappa} + \upsilon_{r_\kappa} + \frac{16(4 - 3\kappa)}{3\kappa\sqrt{3}\sqrt{96\kappa - 9\kappa^2 - 64}}\varepsilon'. \qquad (24)$$

From (24), we find that $\upsilon_{C_\kappa} < \upsilon_{0_\kappa}$. If $\upsilon_{r_\kappa} = 0$, then $\upsilon_{C_\kappa} > \upsilon_0$. Thus, keeping the centrifugal force constant, in the absence of radiation of the bigger primary and $0.714532 \le \kappa < \frac{4}{3}$, the Coriolis force is a stabilizing force, which agrees with the result of Szebehely (1967b), but becomes a destabilizing force, when $\frac{4}{3} < \kappa < 10$, and does not exist for, $0 < \kappa \le 0.714531$, $\kappa = \frac{4}{3}$ and $\kappa \ge 10$.

If there is no perturbation in the Coriolis force, (23), becomes

$$\upsilon_{C_\kappa} = \upsilon_{0_\kappa} + \upsilon_{r_\kappa} + \frac{4(78\kappa - 9\kappa^2 - 88)}{27\kappa^2\sqrt{3}\sqrt{96\kappa - 9\kappa^2 - 64}}\varepsilon''. \qquad (25)$$

Here, we find that $\upsilon_{C_\kappa} < \upsilon_{0_\kappa}$ irrespective of whether $\upsilon_{r_\kappa} = 0$. So, keeping the Coriolis force constant and $0.714532 \le \kappa < \frac{4}{3}$, the centrifugal force is a destabilizing force, but it becomes stabilizing once κ is in the range $\frac{4}{3} < \kappa \le 7$, and again destabilizing when $7 < \kappa < 10$.

The critical mass values, υ_{C_κ}, for the values of κ ($\kappa = 0.714532, 0.72, 1, \ldots, 9.952000$) are

$$\begin{aligned}
\upsilon_{C_{0.714532}} &= 0.498889 - 24.9788500\varepsilon + 322.9320\varepsilon' \\
&\quad - 186.96007\varepsilon'', \\
\upsilon_{C_{0.720000}} &= 0.409910 - 0.29832496\varepsilon + 11.67358\varepsilon' \\
&\quad - 2.2338342\varepsilon'', \\
\upsilon_{C_{0.750000}} &= 0.280104 - 0.1018899\varepsilon + 4.1920451\varepsilon' \\
&\quad - 3.0664033\varepsilon'', \\
\upsilon_{C_{0.999900}} &= 0.038553 - 0.00892522\varepsilon + 0.6465822\varepsilon' \\
&\quad - 0.3390962\varepsilon'', \\
\upsilon_{C_1} &= 0.038520 - 0.00891747\varepsilon + 0.6420578\varepsilon' \\
&\quad - 0.3388638\varepsilon'', \\
\upsilon_{C_{1.1}} &= 0.015230 - 0.00343770\varepsilon + 0.35359290\varepsilon' \\
&\quad - 0.1669744\varepsilon'', \\
\upsilon_{C_{1.3}} &= 0.000219 - 0.00004872\varepsilon + 0.03508002\varepsilon' \\
&\quad - 0.0135672\varepsilon'',
\end{aligned} \tag{26}$$

$$\begin{aligned}
\upsilon_{C_{4/3}} &= 0.000000 - 0.00000000\varepsilon + 0.00000000\varepsilon' \\
&\quad - 0.0000000\varepsilon'',
\end{aligned} \tag{27}$$

$$\begin{aligned}
\upsilon_{C_{1.34}} &= 0.000008 - 0.0000018\varepsilon - 0.00041253\varepsilon' \\
&\quad + 0.00246017\varepsilon'', \\
\upsilon_{C_{1..5}} &= 0.004132 - 0.0009221\varepsilon - 0.13278465\varepsilon' \\
&\quad + 0.04303206\varepsilon'', \\
\upsilon_{C_{1..8}} &= 0.022930 - 0.0052182\varepsilon - 0.26836631\varepsilon' \\
&\quad + 0.06874816\varepsilon'', \\
\upsilon_{C_2} &= 0.038520 - 0.00891747\varepsilon - 0.3210289\varepsilon' \\
&\quad + 0.0713397\varepsilon'', \\
\upsilon_{C_3} &= 0.116438 - 0.02980274\varepsilon - 0.4291595\varepsilon' \\
&\quad + 0.0516580\varepsilon'', \\
\upsilon_{C_4} &= 0.180857 - 0.17106674\varepsilon - 1.5396007\varepsilon' \\
&\quad + 0.1069167\varepsilon'', \\
\upsilon_{C_5} &= 0.234028 - 0.08726458\varepsilon - 0.4010457\varepsilon' \\
&\quad + 0.0235181\varepsilon'', \\
\upsilon_{C_6} &= 0.280104 - 0.10188998\varepsilon - 0.5240056\varepsilon' \\
&\quad + 0.0097038\varepsilon'', \\
\upsilon_{C_7} &= 0.322356 - 0.13663034\varepsilon - 0.5786697\varepsilon' \\
&\quad + 0.0022963\varepsilon'',
\end{aligned} \tag{28}$$

$$\begin{aligned}
\upsilon_{C_8} &= 0.363917 - 0.18900383\varepsilon - 0.6804138\varepsilon' \\
&\quad - 0.0047250\varepsilon'', \\
\upsilon_{C_9} &= 0.409910 - 0.29832496\varepsilon - 0.9338868\varepsilon' \\
&\quad - 0.0144118\varepsilon'', \\
\upsilon_{C_{9.500}} &= 0.439437 - 0.45193517\varepsilon - 1.3281360\varepsilon' \\
&\quad - 0.0252423\varepsilon'', \\
\upsilon_{C_{9.900}} &= 0.479821 - 1.37435213\varepsilon - 3.85032504\varepsilon' \\
&\quad - 0.0831860\varepsilon'', \\
\upsilon_{C_{9.950}} &= 0.495897 - 6.81717495\varepsilon - 18.9878760\varepsilon' \\
&\quad - 0.4161217\varepsilon'', \\
\upsilon_{C_{9.952}} &= 0.498973 - 27.0689614\varepsilon - 75.377677\varepsilon' \\
&\quad - 1.65284206\varepsilon''.
\end{aligned} \tag{29}$$

The critical mass value υ_{C_1}, for the case $\kappa = 1$ fully coincides with the relation obtained by Bhatnagar and Hallan (1978) in the absence of radiation pressure of the bigger primary. Ignoring perturbations in the Coriolis and centrifugal forces, in all the system of (26), (27), (28) and (29) above shows that the radiation pressure of the bigger primary always has a destabilizing tendency and this agrees with the result of Bhatnagar and Chawla (1979), Singh and Ishwar (1999), AbdulRaheem and Singh (2006), and, Singh and Oni (2010).

Let us now consider perturbing effects on the problem influenced by radiation pressure and $\kappa = 1$. The graph of the equation $36\varepsilon' - 19\varepsilon'' = 0$ is the straight line POQ (Fig. 1), which divides the plane $(\varepsilon', \varepsilon'')$ into two parts, π_1 and π_2. Standing at O and looking toward P, π_1 is on our right, and π_2 on the left.

For the points belonging to $\pi_1, 36\varepsilon' - 19\varepsilon'' > 0$ and $\upsilon_{C_1} < \upsilon_{0_1}$. This implies that the range of stability decreases. For the points belonging to $\pi_2, 36\varepsilon' - 19\varepsilon'' < 0$ and $\upsilon_{C_1} < \upsilon_{0_1}$. This also implies that the range of stability is decreasing. For a point lying on the line, $36\varepsilon' - 19\varepsilon'' = 0$ and

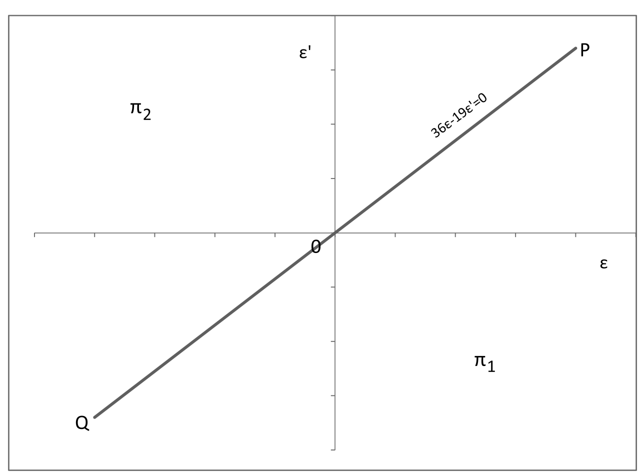

Fig. 1 Two parts of the $(\varepsilon', \varepsilon'')$-plane

$\upsilon_{C_1} < \upsilon_{0_1}$. This again implies that the range of stability decreases. For points lying on the ε'-axis, $\varepsilon'' = 0$; that is, there is no perturbation in the centrifugal force. Then

$$\upsilon_{C_1} = \upsilon_{0_1} + \upsilon_{r_1} + \frac{16\varepsilon'}{3\sqrt{69}}. \tag{30}$$

Here we find that $\upsilon_{C_1} > \upsilon_{0_1}$. Thus, keeping the centrifugal force constant in the absence or presence of radiation, the Coriolis force remains a stabilizing force and confirms Szebehely's (1967b) result.

For points lying on the ε''-axis, $\varepsilon' = 0$; that is, there is no perturbation in the Coriolis force. Then

$$\upsilon_{C_1} = \upsilon_{0_1} + \upsilon_{r_1} - \frac{76\varepsilon'}{27\sqrt{69}}. \tag{31}$$

Here we find that $\upsilon_{C_1} < \upsilon_{0_1}$. So, keeping the Coriolis force constant and in the absence or presence of radiation pressure, the centrifugal force is always a destabilizing force. In this case the radiation pressure of the bigger primary increases the destabilizing tendency of the centrifugal force. This confirms AbdulRaheem and Singh's (2006) result in the absence of radiation pressure of the smaller primary and oblateness of both primaries.

We conclude that the triangular points of the autonomized system are stable for $0 < \upsilon < \upsilon_{C_\kappa}, 0 < \kappa < \frac{4}{3}$ and $\varepsilon, \varepsilon', \varepsilon''$ very small, and, unstable for $\upsilon_C \le \upsilon \le \frac{1}{2}, 0 < \kappa < 10$.

5 Stability of triangular points of the non-autonomous system

Stability of non-autonomous solutions is related to the Lyapunov Characteristic Numbers, which govern the long-time asymptotic exponential behaviors of the solutions. The analysis of the stability of the triangular libration points $L_{4,5}$ of the non-autonomous system would solely depend on the methods applied, since these libration points are themselves time dependent, which means that a change in time would result in a change in the positions of the libration points. For example, using the concept of definition of the theorem of Lyapunov (1956) and taking the limit as t is tending to infinity, we have in the triangular case

$$\lim_{t\to\infty} x^{(4,5)}(t) = \frac{1}{2}\lim_{t\to\infty}\left[\frac{\kappa^{2/3}(q^{2/3}-1)}{(\psi+\kappa-1)^{2/3}} - 2\upsilon + 1\right]R(t). \tag{32}$$

Hence,

$$\lim_{t\to\infty} x(t) = \infty.$$

This at once proves the instability of the solutions $x(t)$, and similarly for $y(t)$, according to the Lyapunov theorem, and verifies the result of Luk'yanov (1990).

The relationships between the old and new independent variable, t and τ based on Poincaré's (1911) representation of the angular velocity $\omega(t)$ are given (Singh and Oni 2010) as

$$0 < \lim_{t\to\infty} \tau < \infty \tag{33}$$

and

$$\lim_{t\to\infty} \frac{\tau}{t} = \Gamma < S^2 \tag{34}$$

where $S = 2\pi\kappa$, and finite.

Equation (33) implies that as t is approaching ∞, τ is always approaching a finite value, and this confirms the result of Luk'yanov (1990), while (34) shows that $\lim_{t\to\infty}\frac{\tau}{t}$ always approaches a non-negative finite value Γ.

The system of (7) with constant coefficients and the reducible system are regular. The system (2) is reducible due to the transformation (3). The reducible systems are regular because the characteristic numbers are invariant with respect to transformation, and consequently we can apply the theorem of Lyapunov, using the Lyapunov Characteristic Numbers (LCN) on the stability of the perturbed motion to the triangular steady-state solutions of the system (2). The calculations of the Lyapunov characteristic numbers here are limited to finding the maximum LCN. This produces an easily computed value that can be used as a metric to give a qualitative indication of how stability varies over the solutions. The calculation of the LCN as in Chetaev (1952) and MaLklin (2004) is used here.

Calculating the LCN of the triangular solutions varying with time with the consideration that as $t \to \infty$, τ is approaching a finite value, we have

$$L_{4,5}\big[x(t)\big] = -\lim_{t\to\infty}\frac{1}{t}\ln\left|\frac{1}{2}\left[\frac{\kappa^{2/3}(q^{2/3}-1)}{(\psi+\kappa-1)^{2/3}} - 2\upsilon + 1\right]R(t)\right| = 0$$

and similarly,

$$L_{4,5}[y(t)] = 0. \tag{35}$$

Thus, the Lyapunov characteristic number is zero for triangular solutions; therefore, the stability or instability of the perturbed motion cannot be determined directly from the triangular libration points.

Using system (11), the particular solutions of the system of equations with variable coefficients (2) can be represented, given transformation (3) and solutions (20), by

$$\begin{aligned}
&x_1 = A_1 \cos \Lambda_1 \tau R(t), \qquad x_2 = C_1 \sin \Lambda_1 \tau R(t), \\
&x_3 = A_2 \cos \Lambda_2 \tau R(t), \qquad x_4 = C_2 \sin \Lambda_2 \tau R(t), \\
&x_5 = \xi_0 R(t), \qquad y_1 = \bar{A}_1 \cos \Lambda_1 \tau R(t), \\
&y_2 = \bar{C}_1 \sin \Lambda_1 \tau R(t), \qquad y_3 = \bar{A}_2 \cos \Lambda_2 \tau R(t), \\
&y_4 = \bar{C}_2 \sin \Lambda_2 \tau R(t), \qquad y_5 = \eta_0 R(t)
\end{aligned} \tag{36}$$

where ξ_0, η_0 are coordinates of the infinitesimal mass.

The solutions (36) correspond to the region where $0 < \upsilon < \upsilon_{C_\kappa}$, $P > 0$ i.e. $\kappa < \frac{4}{3} - \varepsilon' + \frac{8}{3}\varepsilon$.

Similarly, the particular solutions of (2) with conditions $0 < \upsilon < \upsilon_{C_\kappa}$, $P < 0$ using (11) and solutions (22) can be represented as

$$\begin{aligned}
&x_4 = e^{\pm U_1 \tau} R(t), \quad x_5 = e^{\pm U_2 \tau} R(t), \quad x_6 = \xi_0 R(t) \\
&y_4 = c_1 e^{\pm U_1 \tau} R(t), \quad y_5 = c_2 e^{\pm U_2 \tau} R(t), \quad y_6 = \eta_0 R(t).
\end{aligned} \tag{37}$$

These solutions (37) correspond to the region where $0 < \frac{4}{3} - \varepsilon' + \frac{8}{3}\varepsilon < \kappa$. We have chosen these regions since it contains the region where the triangular points for the autonomized equations are stable, as well as where they are unstable. This region is solely determined by the parameter κ, and perturbations in the Coriolis and centrifugal forces. Since in both cases $0 < \upsilon < \upsilon_{C_\kappa}$, $(D > 0)$.

For the solutions (36), their LCN's are

$$L(x_1) = -\lim_{t \to \infty} \frac{1}{t} \ln |\cos \Lambda_1 \tau R(t)| = 0.$$

Hence

$$L(x_1) = L(x_2) = L(x_3) = L(y_1) = L(y_2) = L(y_3) = 0. \tag{38}$$

In view of the particular solutions (37), their LCN's are

$$L(x_4) = -\lim_{t \to \infty} \frac{1}{t} \ln |e^{\pm U_1 \tau} R(t)| = \mp U_1 \Gamma.$$

Thus

$$L(x_4) = \mp U_1 \Gamma, \quad L(x_5) = \mp U_2 \Gamma, \quad L(x_6) = 0 \tag{39}$$

Similarly

$$L(y_4) = \mp U_1 \Gamma, \quad L(y_5) = \mp U_2 \Gamma, \quad L(y_6) = 0. \tag{40}$$

Hence the LCN are positive for solutions with negative exponents, negative for solutions with positive exponents and, zero for solutions with imaginary exponents and constant solutions. We make the following observations.

1. If roots of characteristic equation corresponding to triangular solutions of the autonomized equations are positive, then the LCN of the solutions varying with time is negative, and consequently the solutions are unstable according to the Lyapunov theorem.
2. If roots of characteristic equation corresponding to triangular solutions of the autonomized equations are pure imaginary numbers, then the LCN of the corresponding solution varying with time is zero, and in this case the stability or instability of the solutions cannot be determined.
3. If roots of characteristic equation corresponding to triangular solutions of the autonomized equations are negative, then the LCN of corresponding solutions varying with time is positive and consequently the solutions are stable.

6 Discussion

Equation (2) of the perturbed motion are different from those obtained by Gel'fgat (1973), Bekov (1988), and Luk'yanov (1990) due to the presence of radiation pressure of the bigger primary and perturbations in the Coriolis and centrifugal forces. However, if these are ignored, (2) will fully coincide with those as used by them. The positions of the triangular points of the autonomized system, $L_{4,5}$, in system (9) are affected by the factors which appear due to radiation pressure of the bigger primary, the perturbation in the centrifugal force and the parameter κ. If the perturbation is ignored, i.e., $\psi = 1$, the points are fully analogous to the photogravitational case of Bhatnagar and Chawla (1979). If, further, we ignore the radiation effect of the bigger primary, i.e. $q_1 = 1$, the points become fully analogous to the classical case. The characteristic equation (16) of triangular points of the autonomized system differs from that obtained by Bhatnagar and Hallan (1978) by the presence of the radiation factor of the bigger primary and the parameter κ. That of Bhatnagar and Chawla (1979) differs also by κ and the perturbations in the Coriolis and the centrifugal forces, while that of Devi and Singh (1994) differs only by the presence of the constant κ of a particular integral of the Gylden-Meschcherskii problem (Gylden 1884; Meshcherskii 1952). If the bigger primary is nonradiating, and there are no perturbations in the Coriolis and centrifugal forces, with $\kappa = 1$, (16) corresponds to the characteristic equation of the classical restricted problem.

Equation (23) represents the effects of the constant κ of a particular integral of the Gylden-Meschcherskii problem, perturbations and radiation on the critical mass values υ_{C_κ} of the autonomized restricted problem. For $\kappa = 1$, the critical value υ_{C_1} differs from that obtained by Devi and Singh

(1994) in the sense that here, the radiation coefficient of the bigger primary always has a destabilizing tendency. When perturbations are ignored, (i.e. $\varepsilon' = \varepsilon'' = 0$) and $\kappa = 1$, the value υ_{C_1} verifies the results of Bhatnagar and Chawla (1979).

The critical mass value υ_{C_1}, for the case $\kappa = 1$ fully coincides with the relation obtained by Bhatnagar and Hallan (1978) in the absence of radiation pressure of the bigger primary. Ignoring perturbations in the Coriolis and centrifugal forces, in the whole system of (26), (27), (28) and (29), shows that the radiation pressure of the bigger primary always has a destabilizing tendency and this agrees with the result of Bhatnagar and Chawla (1979), Singh and Ishwar (1999), AbdulRaheem and Singh (2006), and, Singh and Oni (2010). We observe that, for, $\kappa = 1, 2$ the value of $\upsilon_{0_{1,2}} = 0.038520$, and this coincide with the classical value given by Szebehely (1967a).

When κ is in the range $0.714532 \le \kappa < 1.333333$ (system (26)), the Coriolis force always has a strong stabilizing tendency that counters the destabilizing effect of the radiation pressure of the bigger primary and that of the centrifugal force. In this range of κ, every $\upsilon_{C_\kappa} > \upsilon_{0_\kappa}, |\varepsilon| \ll 1, |\varepsilon'| \ll 1, |\varepsilon''| \ll 1$, so that the region of stability of the triangular points of the autonomized system increases. For $\kappa = \frac{4}{3}$, the critical mass value $\upsilon_{C_{4/3}}$ is zero, and so are all the parameters involved. For the range $1.333333 < \kappa \le 7$ (system (28)), the centrifugal force has a stabilizing tendency, while the radiation pressure and the Coriolis force both have destabilizing tendency. In this range of κ, every $\upsilon_{C_\kappa} < \upsilon_{0_\kappa}, |\varepsilon| \ll 1, |\varepsilon'| \ll 1, |\varepsilon''| \ll 1$, so that region of stability decreases. Further, in the range $8 \le \kappa \le 9.952$ (system (29)), the Coriolis and centrifugal forces κ, and the radiation pressure, all have destabilizing effects. In this range of κ, the region of stability decreases fast.

By keeping the centrifugal force constant (i.e., $\varepsilon'' = 0$), (23) gives the relationship among the critical mass value, radiation coefficients and to the change ε' in the Coriolis force. Here for $0.714532 \le \kappa < 1.333333$, the Coriolis force has a stronger stabilizing force, which confirms the result of Szebehely (1967b), but becomes destabilizing when $\frac{4}{3} < \kappa < 10$, and does not exist for $0 < \kappa \le 0.714531, \kappa = \frac{4}{3}$ and $\kappa \ge 10$. Also, if the Coriolis force is kept constant (i.e., $\varepsilon' = 0$), (23) provides the relationship of the critical mass value, radiation coefficients and to the change ε'' in the centrifugal force. In the range $0.714532 \le \kappa < \frac{4}{3}$, the centrifugal force is a destabilizing force, does not exist for $\kappa = \frac{4}{3}$, has a stabilizing tendency when $\frac{4}{3} < \kappa < 8$, and again is destabilizing for $8 \le \kappa \le 9.952$. We note that the region of stability does not exist when $0 < \kappa \le 0.714531, \kappa = \frac{4}{3}$ and $\kappa \ge 10$, so that when the restricted problem of variable masses evolves into the problem with constant masses, the region of stability of the triangular points does not exist for these values of κ. The overall effect is that the increase, decrease or non-existence in the region of stability of the triangular points depends on the constant κ of a particular integral of the Glyden-Mescherskii problem (Gylden 1884; Meshcherskii 1952).

Equation (33) implies that as t is approaching ∞, τ is always approaching a finite value, and this validates the result of Luk'yanov (1990). Further, that the limits of the solutions $x(t)$ and $y(t)$ approaches infinity, as t is approaching infinity in (32), verifies the result of Luk'yanov (1990), that these solutions are unstable.

7 Concluding remarks

The equations of the system (2), which govern the motion of an infinitesimal mass in the gravitational field of two variable mass bodies, under the influence of small perturbations in the Coriolis and centrifugal forces, together with the effects of radiation pressure of the bigger primary, are non-integrable differential equations with variable coefficients. Therefore, with the help of the particular solutions (4) of the Gylden-Meshcherskii problem (Gylden 1884; Meshcherskii 1952), the unified Meshcherskii law (1952) and a Meshcherskii transformation (1952), the system (2) is reduced to a system of equations with constant coefficients. Hence, the search for the equilibrium solutions $L_{4,5}$ of (2) boils down to that of the system (7). Analogous triangular points for the system of equations with variable coefficients $(L_{4,5})R(t)$ are obtained with the help of the transformation (3) and the solutions $L_{4,5}$ of system (7).

The triangular points of the autonomized system are found to be stable for $0 < \upsilon < \upsilon_{C_\kappa}$ when $\kappa < \frac{4}{3} - \varepsilon' + \frac{8}{3}\varepsilon$, and unstable for $0 < \upsilon < \upsilon_{C_\kappa}, \frac{4}{3} - \varepsilon' + \frac{8}{3}\varepsilon < \kappa < 10$, and $\upsilon_c \le \upsilon \le \frac{1}{2}$ for any κ in the range $0 < \kappa < 10$, where υ_{C_κ} is given by (23). We observed further that the stabilizing or destabilizing tendency of the Coriolis and centrifugal forces can be altered by κ, while the destabilizing tendency of the radiation pressure cannot be altered by this constant, though its presence can cause the stabilizing or destabilizing tendency of the Coriolis and centrifugal forces, and the destabilizing effects of the radiation pressure, to be strong or weak. The condition that the region of stability is increasing, decreasing or does not exist, depends on this constant.

For the stability of the triangular points varying with time, for some initial conditions, we observe according to the Lyapunov theorem, that solutions with negative exponents consequently having a positive LCN are stable, those with positive exponents having a negative LCN are unstable, while the stability or instability of constant and pure oscillatory solutions having zero LCN's cannot be determined. In the case when the solutions have zero LCN's, the region of stability or instability does not exist due to κ.

We further observe that as κ increases, the LCN of solutions with negative exponents increases. On the other hand, for solutions with positive exponents the range of instability of these solutions increases.

Since the stability of the triangular equilibrium solutions of the non-autonomous system cannot be determined when $0 < \upsilon < \upsilon_{C_\kappa}$, but since it has unstable solutions in the same range of the mass parameter due to a change in the range of the constant κ, we conclude that motion around the triangular points for the perturbed photogravitational restricted three-body problem with variable masses is in general unstable.

References

AbdulRaheem, A., Singh, J.: Astron. J. **131**, 1880 (2006)
Bekov, A.A.: Sov. Astron. **32**, 106 (1988)
Bekov, A.A.: Sov. Astron. **35**, 103 (1991)
Bekov, A.A., Beysekov, A.N., Aldibaeva, L.T.: Astron. Astrophys. Trans. **24**, 311 (2005)
Bhatnagar, K.B., Hallan, P.P.: Celest. Mech. **18**, 105 (1978)
Bhatnagar, K.B., Chawla, J.M.: Indian J. Pure Appl. Math. **10**, 1443 (1979)
Chetaev, N.G.: Stability of motion GITTL. Moscow (1952)
Devi, G.S., Singh, R.: Bull. Astron. Soc. India **22**, 433 (1994)
El-Shaboury, S.M.: Astrophys. Space Sci. **174**, 151 (1990)
Gasanov, S.A.: Astron. Lett. **34**, 179 (2008)
Gel'fgat, B.E.: Modern Problems of Celestial Mechanics and Astrodynamics, p. 7. Nauka, Moscow (1973)
Gylden, H.: Astron. Nachr. **109**, 1 (1884)
Lu, T.-W.: Publ. Purple Mt. Obs. **9**, 290 (1990)
Luk'yanov, L.G.: Sov. Astron. **32**, 682 (1988)
Luk'yanov, L.G.: Sov. Astron. **33**, 92 (1989)
Luk'yanov, L.G.: Sov. Astron. **34**, 87 (1990)
Lyapunov, A.M.: A General Problem of Stability of Motion. Acad. Sci. USSR, Moscow (1956)
Meshcherskii, I.V.: Works on the Mechanics of Bodies of Variable mass. GITTL, Moscow (1952)
MaLklin, J.G.: Theory of the Stability of Motion, (Editorial) Moscow (2004)
Orlov, A.A.: Astron. J. Acad. Sci. Mosc. **16**, 52 (1939)
Poincaré, H.: Leçons sur les Hypothèsis Cosmogoniques (1911)
Radzievskii, V.V.: Astron. J. **27**, 249 (1950)
Radzievskii, V.V.: Astron. Zh. **30**, 225 (1953)
Singh, J., Ishwar, B.: Celest. Mech. **32**, 297 (1984)
Singh, J., Ishwar, B.: Celest. Mech. **35**, 201 (1985)
Singh, J., Ishwar, B.: Bull. Astron. Soc. India **27**, 415 (1999)
Singh, J.: Astrophys. Space Sci. **321**, 127 (2009)
Singh, J., Oni, L.: Astrophys. Space Sci. **326**, 305 (2010)
Szebehely, V.G.: Theory of Orbits. Academic Press, New York (1967a)
Szebehely, V.: Astron. J. **72**, 7 (1967b)
Wintner, A.: The Analytical Foundations of Celestial Mechanics, pp. 372–373. Princeton University Press, New Jersey (1941)