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ORIGINAL ARTICLE

Some Robertson-Walker models with time dependent G and Λ

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Abstract Einstein field equations with variable gravitational and cosmological constants are considered in the presence of perfect fluid for Robertson-Walker universe by assuming the cosmological term proportional to the Hubble parameter. This variation law for vacuum density has recently been proposed by Schützhold on the basis of quantum field estimations in the curved and expanding background. The cosmological term tends asymptotically to a genuine cosmological constant and the model tends to a deSitter universe. We obtain that the present universe is accelerating with a large fraction of cosmological density in the form of cosmological term.

Keywords Cosmology · R-W universe · Variable cosmological term

1 Introduction

The problem of cosmological constant is one of the most salient and unsettled problem in cosmology. To resolve the problem of huge difference between the effective cosmological constant observed today and the vacuum energy density predicted by quantum field theory, several mechanisms have been proposed by Weinberg (1989). A possible way is to consider a varying cosmological term. Due to the coupling of dynamic degree of freedom with the matter fields of the universe, Λ relaxes to its present small value through the expansion of the universe and creation of particles. From this point of view, the constant is small because the universe is old. Models with dynamically decaying cosmological term representing the energy density of vacuum have been studied by several authors. Some authors have argued for the dependence $\Lambda \sim t^{-2}$, see, e.g., Endo and Fukui (1977), Canuto et al. (1977), Lau (1985), Berman (1991a, 1991b). Keeping in mind the dimensional considerations in the spirit of quantum cosmology, Chen and Wu (1990) considered Λ varying as R^{-2}, Carvalho and Lima (1992) generalized it by taking $\Lambda = \alpha R^{-2} + \beta H^2$, where R is the scale factor of Robertson-Walker metric, H is the Hubble parameter and α and β are adjustable dimensionless parameters on the basis of quantum field estimations in the curved expanding background. Schützhold (2002a, 2002b) recently proposed a vacuum density proportional to the Hubble parameter this leads to a vacuum energy density decaying as $\Lambda \approx m^3 H$, where $m \approx 150$ MeV is the energy scale of the chiral phase transmission of QCD. In a recent paper, Borges and Carnerio (2005) have considered an isotropic and homogeneous flat space filled with matter and cosmological term proportional to H, obeying the equation of state of the vacuum.

The idea of gravitational constant G in the framework of general relativity was first proposed by Dirac (1937). Lau (1983) working in the frame work of general relativity proposed modification linking the variation of G with that of Lambda. This modification allows us to use Einsteins field equations formally unchanged since variations in Lambda is accompanied by a variation in Arbab (1997, 1998) has considered cosmological models with viscous fluid considering G and Λ. A lot of work have been done by Saha (2005, 2006a, 2006b), Vishwakarma and Abdussattar (1996a, 1996b, 1999), and Vishwakarma (2000, 2001) in studying the FRW models and Bianchi type-I cosmological model in general relativity with varying G and Λ. Singh and Tiwari (2008) and Tiwari (2008) have studied perfect

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fluid Bianchi type-I model with variable G and Λ by taking expansion scalar is proportional to shear scalar. Recently I have (Tiwari 2008) considered as cosmological term is proportional to Hubble parameter in Bianchi type-I model with variable G and Lambda.

In this paper we study homogeneous Robertson-Walker space-time with variable G and Λ containing matter in form of perfect fluid. We obtain solution of the Einstein equations assuming that cosmological term proportional to the Hubble parameter. The paper is organized as follows. Basic equations of the model in Sect. 2 and solution in given Sect. 3. We discuss the model and conclude our results in Sect. 4.

2 Model and field equations

We consider a spatially homogeneous and isotropic R-W line element

$$ds^2 = dt^2 - R^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right] \quad (1)$$

where $R(t)$ is the scale factor and $k = -1$, 0 or $+1$ is the curvature parameter for open, flat and closed universe, respectively.

We assume that the cosmic matter is represented by the energy-momentum tensor of a perfect fluid

$$T_{ij} = (\rho + p)v_i v_j - p g_{ij} \quad (2)$$

where ρ is the energy density of the cosmic matter and p is its pressure. v_i is the four velocity vector such that $v_i v^i = 1$.

We take the equation of state

$$p = (\omega - 1)\rho, \quad 1 \le \omega \le 2. \quad (3)$$

The Einstein's field equations with time dependent G and Λ given by Weinberg (1972).

$$R_{ij} - \frac{1}{2}R g_{ij} = 8\pi G(t) T_{ij} + \Lambda(t) g_{ij}. \quad (4)$$

For the metric (1) and energy-momentum tensor (2) in co-moving system of coordinates the field equation (4) yields

$$8\pi G p = -\frac{2\ddot{R}}{R} - \frac{\dot{R}^2}{R^2} - \frac{k}{R^2} + \Lambda, \quad (5)$$

$$8\pi G\rho = \frac{3\dot{R}^2}{R^2} + \frac{3k}{R^2} - \Lambda. \quad (6)$$

In a view of vanishing of divergence of Einstein tensor, we have

$$8\pi G\left\{\dot{\rho} + 3(\rho + p)\frac{\dot{R}}{R}\right\} + 8\pi\rho\dot{G} + \dot{\Lambda} = 0. \quad (7)$$

The usual energy conservation equation $T^{j}_{i;j} = 0$ yields

$$\dot{\rho} + 3(\rho + p)\frac{\dot{R}}{R} = 0. \quad (8)$$

Then (7) reduces to

$$8\pi\rho\dot{G} + \dot{\Lambda} = 0. \quad (9)$$

Implying that Λ is a constant whenever G is constant. Using (3) in (8) and then integrating we get

$$\rho = \frac{c_1}{R^{3\omega}} \quad (10)$$

where c_1 is a constant of integration.

For zero curvature ($k = 0$) (5) and (6) can be rewritten in terms of Hubble parameters H, and deceleration parameter q as

$$8\pi G p = H^2(2q - 1) + \Lambda, \quad (11)$$

$$8\pi G\rho = 3H^2 - \Lambda, \quad \text{where} \quad (12)$$

$$q = -1 - \frac{\dot{H}}{H^2} = -\frac{R\ddot{R}}{\dot{R}^2}$$

and expansion scalar $\theta = 3H = \frac{3\dot{R}}{R}$. Overduin and Cooperstock (1998) define

$$\rho_c = \frac{3H^2}{8\pi G}, \quad (13)$$

$$\rho_v = \frac{\Lambda}{8\pi G} \quad \text{and} \quad (14)$$

$$\Omega = \frac{\rho}{\rho_c} = \frac{8\pi G\rho}{3H^2} \quad (15)$$

where ρ_c, ρ_v and Ω are respectively critical density, vacuum density and density parameter.

3 Solution of the field equations

The system of equations (3), (5), (6) and (8) supply only four equations in five unknowns (R, ρ, p, G and Λ). One extra equation is needed to solve the system completely. For this purpose, we take a cosmological term is proportional to the Hubble parameter. This variation law for vacuum density has initially proposed by Schützhold (2002a, 2002b).

Recently Borges and Carnerio (2005) have considered a cosmological term proportional to H. Thus, we take decaying vacuum energy density

$$\Lambda = aH \quad (16)$$

where a is a positive constant of order m^3.

From (3), (11), (12) and (16) we get a differential equation

$$2\dot{H} + 3\omega H^2 - a\omega H = 0 \tag{17}$$

determining the time evolution of Hubble parameter, integrating (17), we get

$$H = \frac{a}{3(1 - e^{\frac{-a\omega t}{2}})} \tag{18}$$

where the integration constant is related to the choice of origin of time. Also from (17) we have

$$t = \frac{2}{a\omega} \ln\left|\frac{3H}{3H - a}\right|. \tag{19}$$

From (12) and (16) it is possible to verify that

$$\rho = (3H - a)\frac{H}{8\pi G}. \tag{20}$$

As the weak energy condition requires $\rho \geq 0$ and since, in an expanding universe, we have $H = \frac{\dot{R}}{R}$, it follows that $3H - a \geq 0$. Therefore, the solution (19) can simply be written as

$$t = \frac{2}{a\omega} \log\left(\frac{H}{H - a/3}\right). \tag{21}$$

Integrating (21) with respect to time, we obtain the scale factor:

$$R = c_2\left\{e^{\frac{a\omega t}{2}} - 1\right\}^{2/3\omega} \tag{22}$$

where c_2 is an integration constant.

From (10), (16) and (22) we obtain respectively, the cosmological term and the matter density as function of time

$$\Lambda = \frac{a^2}{3(1 - e^{-\frac{a\omega t}{2}})}, \tag{23}$$

$$\rho = \frac{c_1}{c_2^{3\omega}(e^{\frac{a\omega t}{2}} - 1)^2}. \tag{24}$$

Also gravitational parameter G, expansion scalar θ and deceleration parameter q are given by

$$G = \frac{c_2^{3\omega} a^2 e^{\frac{a\omega t}{2}}}{24\pi c_1}, \tag{25}$$

$$\theta = \frac{a}{1 - e^{-\frac{a\omega t}{2}}}, \tag{26}$$

$$q = \frac{3\omega}{2} e^{-\frac{a\omega t}{2}} - 1. \tag{27}$$

The gravitational constant G, vacuum density Λ matter density ρ and deceleration parameter q can be expressed as function of the scale factor as follows:

$$G = \frac{a^2 c_2^{3\omega}}{24\pi c_1}\left\{\left(\frac{R}{c_2}\right)^{\frac{2\omega}{2}} + 1\right\}, \tag{28}$$

$$\Lambda = \frac{a^2}{3}\left\{1 + \left(\frac{c_2}{R}\right)^{\frac{3\omega}{2}}\right\}, \tag{29}$$

$$\rho = \frac{c_1}{R^{3\omega}}, \tag{30}$$

$$q = \frac{3\omega}{2}\left\{1 + \left(\frac{R}{c_2}\right)^{\frac{2\omega}{2}}\right\}^{-1} - 1. \tag{31}$$

Critical density ρ_c, vacuum density ρ_v and the density parameter Ω are given as

$$\rho_c = \frac{c_1}{c_2^{3\omega}} \frac{e^{\frac{a\omega t}{2}}}{(e^{\frac{a\omega t}{2}} - 1)^2}, \tag{32}$$

$$\rho_v = \frac{c_1}{c_2^{3\omega}(e^{\frac{a\omega t}{2}} - 1)}, \tag{33}$$

$$\Omega = e^{\frac{-a\omega t}{2}}. \tag{34}$$

Now, we analyze for different values of ω.

3.1 The radiation era

For the radiation epoch, we have $\omega = \frac{4}{3}$. In this case, (22) is written as

$$R = c_2\left\{e^{\frac{2at}{3}} - 1\right\}^{1/2}. \tag{35}$$

In the limit of small time (at ≤ 1) this expression reduces to

$$R \approx \sqrt{2c_2^2 at/3}. \tag{36}$$

In this case

$$\rho = \frac{c_1}{R^4}, \tag{37}$$

$$\Lambda = \frac{a^2}{3}\left\{1 + \left(\frac{c_2}{R}\right)^2\right\}, \tag{38}$$

$$G = \frac{a^2 c_2^4}{24\pi c_1}\left\{\left(\frac{R}{c_2}\right)^2 + 1\right\}, \tag{39}$$

$$q = 2\left[1 + \left(\frac{R}{c_2}\right)^2\right]^{-1} - 1. \tag{40}$$

In the limit $R \to 0$ they reduces to

$$\rho = \frac{9c_1}{4c_2^4 a^2 t^2}, \tag{41}$$

$$\Lambda = \frac{a^2 c_2^2}{3R^2} = \frac{a}{2t}, \tag{42}$$

$$G = \frac{a^3 c_2^4 t}{36\pi c_1}. \tag{43}$$

From (36) and (41) one can see that scale factor and matter density have the same time dependence as in the standard model, and that the radiation density scales as R^{-4}, as should be Borges and Carnerio (2005). Also from (16) and (42) we obtain $Ht = 1/2$ and the deceleration parameter $q \approx 1$ which is in accordance with the standard model (Abdussattar and Vishwakarma 1997). Furthermore, from (41) and (42) it is clear that, for $R \to 0$ the radiation density diverges faster than the cosmological term. Thus, at early times the expansion is completely dominated by radiation. We observe that G is increasing with cosmic time and critical density, vacuum density diverges at initial epoch which is a similar result already given by Abdussattar and Vishwakarma (1997).

3.2 Dust era and the limit of late times

For the phase dominated by the dust matter, we have $\omega = 1$. In this epoch

$$\rho = \frac{c_1}{R^3}, \tag{44}$$

$$\Lambda = \frac{a^2}{3}\left[1 + \left(\frac{c_2}{R}\right)^3\right], \tag{45}$$

$$G = \frac{a^2 c_2^3}{24\pi c_1}\left[\left(\frac{R}{c_2}\right)^{3/2} + 1\right], \tag{46}$$

$$q = \frac{3}{2}\left[1 + \left(\frac{R}{c_2}\right)^{3/2}\right]^{-1} - 1, \tag{47}$$

$$\theta = \frac{a}{(1 - e^{-at/2})}. \tag{48}$$

For the energy density to be positive definite we must have $c_1 > 0$. The model has singularity at $t = 0$. The cosmic scenario starts from a Big Bang at $t = 0$ and continues till $t = \infty$. As $t = 0$, ρ, Λ, θ are all infinite and tend to zero asymptotically. G is constant initially and gradually increases and tends to infinity at $t \to \infty$. The possibility of the G increasing with time has been investigated by Abdel-Rahman (1990), Chow (1981), Levitt (1980) and Milne (1935). The deceleration parameter q is $1/2$ initially and -1 at late times which characterizes the deSitter solution (Weinberg 1972).

Also for small times (small compared to the present time), the scale factor can be approximated by

$$R = c_2(at)^{2/3}$$

which has the same time dependence as in the standard flat model with dust (Weinberg 1972). Now in the limit of large times i.e. at ≥ 1 and $R \to \infty$ we have $R = c_2 e^{at/3}$, $\Lambda = a^2/3$ and $\rho \approx 0$ i.e. as long as the cosmological term tends asymptotically to a genuine cosmological constant and our solution tends to a deSitter universe with $H = \sqrt{\frac{\Lambda}{3}} = \frac{a}{3}$ (Borges and Carnerio 2005).

4 Conclusion

We have investigated the Robertson-Walker cosmological model with a cosmological term proportional to the Hubble parameter suggested by Schützhold (2002a, 2002b) on the basis of quantum field estimations of the vacuum energy density in an accelerated expansion. In all the cases we have found that the cosmological term Λ being very large at initial times relaxes to a genuine cosmological constant at the late times, which is in accordance with the observations. The model asymptotically tends to a deSitter universe (Weinberg 1972). Gravitational constant G is increasing with cosmic time which is a similar result given by Abdussattar and Vishwakarma (1997) and others.

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