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ORIGINAL ARTICLE

The effect of radiation pressure on the equilibrium points in the generalized photogravitational restricted three body problem

Badam Singh Kushvah

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Abstract The existence of equilibrium points and the effect of radiation pressure have been discussed numerically. The problem is generalized by considering bigger primary as a source of radiation and small primary as an oblate spheroid. We have also discussed the Poynting-Robertson (P-R) effect which is caused due to radiation pressure. It is found that the collinear points L_1, L_2, L_3 deviate from the axis joining the two primaries, while the triangular points L_4, L_5 are not symmetrical due to radiation pressure. We have seen that L_1, L_2, L_3 are linearly unstable while L_4, L_5 are conditionally stable in the sense of Lyapunov when P-R effect is not considered. We have found that the effect of radiation pressure reduces the linear stability zones while P-R effect induces an instability in the sense of Lyapunov.

Keywords Radiation pressure · Equilibrium points · Generalized photogravitational · RTBP · Linear stability · P-R effect

1 Introduction

Three-Body problem is a continuous source of study, since the discovery of its non-integrability due to Poincare (1892). Many of the best minds in Mathematics and Physics worked on this problem in the last century. Regular and chaotic motions have been widely investigated with any kind of tools, from analytical results to numerical explorations. The restricted three body model (RTBP) is insoluble, although specific solutions exist, like the ones in which the spacecraft is positioned at one of the Lagrangian points. The five singular points of Jacobian function are called equilibrium points, Lagrangian points and most frequently are referred to as the equidistant (triangular) and collinear (straight line) solutions. There are many periodic orbits in the restricted three body problem. One of the most famous, discovered by Lagrange, is formed by an equilateral triangle. In "Earth-Moon-Space Station" model, if the Moon is not too massive, this orbit is thought to be stable. If the space station is pushed a bit to one side (in position or velocity), it's supposed to make small oscillations around this orbit. The two kinds of triangular points are called L_4 and L_5 points. We study the motion of three finite bodies in the three body problem. The problem is restricted in the sense that one of the masses is taken to be so small that the gravitational effect on the other masses by third mass is negligible. The smaller body is known as infinitesimal mass and remaining two finite massive bodies as primaries. The classical restricted three body problem is generalized to include the force of radiation pressure, oblateness effect and Poynting-Robertson (P-R) effect. The solar radiation pressure force F_p is exactly opposite to the gravitational attraction force F_g and change with the distance by the same law it is possible to consider that the result of action of this force will lead to reducing the effective mass of the Sun or particle. It is acceptable to speak about a reduced mass of the particle as the effect of reducing its mass depends on the properties of the particle itself.

Chernikov (1970) and Schuerman (1980) discussed the position as well as the stability of the Lagrangian equilib-

B.S. Kushvah (✉)
Gwalior Engineering College, Airport Road, Maharajpura, Gwalior (M.P.) 474015, India
e-mail: bskush@gmail.com

Present address:
B.S. Kushvah
C/O Er. Prashant Anand Kushwaha, 139/I, Anupam Nagar Ext-2, Opp. Jiwaji University, City Center, Gwalior 474011, India

rium points when radiation pressure, P-R drag force are included. Murray (1994) systematically discussed the dynamical effect of general drag in the planar circular restricted three body problem. Ishwar and Kushvah (2006) examined the linear stability of triangular equilibrium points in the generalized photogravitational restricted three body problem with Poynting-Robertson drag, L_4 and L_5 points became unstable due to P-R drag which is very remarkable and important, where as they are linearly stable in classical problem when $0 < \mu < \mu_{Routh} = 0.03852$. Kushvah et al. (2007a, 2007b, 2007c) examined normalization of Hamiltonian they have also studied the nonlinear stability of triangular equilibrium points in the generalized photogravitational restricted three body problem with Poynting-Robertson drag, they have found that the triangular points are stable in the nonlinear sense except three critical mass ratios at which KAM theorem fails.

2 Equations of motion

Poynting (1903) has stated that the particle such as small meteors or cosmic dust are comparably affected by gravitational and light radiation force, as they approach luminous celestial bodies. He also suggested that infinitesimal body in solar orbit suffers a gradual loss of angular momentum and ultimately spiral into the Sun. In a system of coordinates where the Sun is at rest, radiation scattered by infinitesimal mass in the direction of motion suffers a blue shift and in the opposite direction it is red shifted. This gives rise to net drag force which opposes the direction of motion. The proper relativistic treatment of this problem was formulated by Robertson (1937) who showed that to first order in $\frac{\vec{V}}{c}$ the radiation pressure force is given by

$$\vec{F} = F_p \left\{ \frac{\vec{R}}{R} - \frac{\vec{V}.\vec{R}\vec{R}}{cR^2} - \frac{\vec{V}}{c} \right\} \tag{1}$$

Where $F_p = \frac{3Lm}{16\pi R^2 \rho s c}$ denotes the measure of the radiation pressure force, $\vec{R}$ the position vector of P with respect to radiation source Sun S, $\vec{V}$ the corresponding velocity vector and c the velocity of light. In the expression of F_p, L is luminosity of the radiating body, while m, ρ and s are the mass, density and cross section of the particle respectively.

The first term in (1) expresses the radiation pressure. The second term represents the Doppler shift of the incident radiation and the third term is due to the absorption and subsequent re-emission of the incident radiation. These last two terms taken together are the Poynting-Robertson effect. The Poynting-Robertson effect will operate to sweep small particles of the solar system into the Sun at cosmically rapid rate. We consider the barycentric rotating coordinate system $Oxyz$ relative to inertial system with angular velocity ω and common z-axis. We have taken line joining the primaries as x-axis. Let m_1, m_2 be the masses of bigger primary (Sun) and smaller primary (Earth) respectively. Let Ox, Oy in the equatorial plane of smaller primary and Oz coinciding with the polar axis of m_2. Let r_e, r_p be the equatorial and polar radii of m_2 respectively, r be the distance between primaries. Let infinitesimal mass m be placed at the point $P(x, y, 0)$. We take units such that sum of the masses and distance between primaries is unity, the unit of time i.e. time period of m_1 about m_2 consists of 2π units such that the Gaussian constant of gravitational $\Bbbk^2 = 1$. Then perturbed mean motion n of the primaries is given by $n^2 = 1 + \frac{3A_2}{2}$, where $A_2 = \frac{r_e^2 - r_p^2}{5r^2}$ is oblateness coefficient of m_2. Let $\mu = \frac{m_2}{m_1+m_2}$ then $1-\mu = \frac{m_1}{m_1+m_2}$ with $m_1 > m_2$, where μ is mass parameter. Then coordinates of m_1 and m_2 are $(-\mu, 0)$ and $(1-\mu, 0)$ respectively. Further, in our consideration, the velocity of light needs to be dimensionless, too, so consider the dimensionless velocity of light as $c_d = c$ which depends on the physical masses of the two primaries and the distance between them. In this paper, we set $c_d = 299792458$, $\mu = 0.00003$ for all numerical results.

In the above mentioned reference system the total acceleration on the particle P is as follows

$$\begin{aligned}
\vec{a}' &= \vec{a} + 2\vec{\omega} \times \vec{v} + \vec{\omega} \times (\vec{\omega} \times \vec{r}) \\
&= -\frac{(1-\mu)\vec{r}_1}{r_1^3} - \frac{\mu \vec{r}_2}{r_2^3} - \frac{3}{2}\frac{\mu A_2 \vec{r}_2}{r_2^5} \\
&\quad + \frac{(1-\mu)(1-q_1)}{r_1^2}\left\{ \frac{\vec{r}_1}{r_1} - \frac{(\dot{\vec{r}}_1 + \vec{\omega} \times \vec{r}_1).\vec{r}_1 \vec{r}_1}{c_d r_1^2} \right. \\
&\quad \left. - \frac{\dot{\vec{r}}_1 + \vec{\omega} \times \vec{r}_1}{c_d} \right\}
\end{aligned} \tag{2}$$

where

$$\begin{aligned}
&\vec{r}_1 = x\hat{i} + y\hat{j}, \qquad \vec{v} = \dot{x}\hat{i} + \dot{y}\hat{j}, \\
&\vec{a} = \ddot{x}\hat{i} + \ddot{y}\hat{j}, \qquad \vec{\omega} = n\hat{k}, \\
&\vec{r}_1 = (x+\mu)\hat{i} + y\hat{j}, \qquad \vec{r}_2 = (x+\mu-1)\hat{i} + y\hat{j}, \\
&r_1^2 = (x+\mu)^2 + y^2, \qquad r_2^2 = (x+\mu-1)^2 + y^2, \\
&\vec{\omega} \times \vec{v} = -n(\dot{y}\hat{i} - \dot{x}\hat{j}), \qquad \vec{\omega} \times (\vec{\omega} \times \vec{r}) = n^2(x\hat{i} + y\hat{j}), \\
&\dot{\vec{r}}_1 + n\hat{k} \times \vec{r}_1 = (\dot{x} - ny)\hat{i} + \left[\dot{y} + n(x+\mu)\right]\hat{j}, \\
&\left(\dot{\vec{r}}_1 + n\hat{k} \times \vec{r}_1\right).\vec{r}_1 = \left[\dot{x}(x+\mu) + y\dot{y}\right]
\end{aligned}$$

Substituting all these values in above relation (2) we get as follows:

$$\begin{aligned}
&\left(\ddot{x} - 2n\dot{y} - n^2 x\right)\hat{i} + \left(\ddot{y} + 2n\dot{x} - n^2 y\right)\hat{j} \\
&\quad = -\frac{(1-\mu)\vec{r}_1}{r_1^3} - \frac{\mu \vec{r}_2}{r_2^3} - \frac{3\mu \vec{r}_2 A_2}{2r_2^5} + \frac{(1-\mu)q_1\vec{r}_1}{r_1^3}
\end{aligned}$$

$$-\frac{(1-\mu)(1-q_1)}{r_1^2}\left\{\frac{(\vec{r_1}+\vec{\omega}\times\vec{r_1}).\vec{r_1}\vec{r_1}}{c_d r_1^2}\right.$$
$$\left.-\frac{\vec{r_1}+\vec{\omega}\times\vec{r_1}}{c_d}\right\}$$
$$=-\frac{(1-\mu)q_1\vec{r_1}}{r_1^3}-\frac{\mu\vec{r_2}}{r_2^3}-\frac{3\mu\vec{r_2}A_2}{2r_2^5}$$
$$-\frac{W_1}{c_d r_1^2}\left\{\frac{(\vec{r_1}+\vec{\omega}\times\vec{r_1}).\vec{r_1}\vec{r_1}}{r_1^2}-\frac{\vec{r_1}+\vec{\omega}\times\vec{r_1}}{1}\right\}$$

$W_1=\frac{(1-\mu)(1-q_1)}{c_d}$, substituting the values of $\vec{r_1}$, $\vec{r_2}$ and comparing the components of $\hat{i}$ and $\hat{j}$ we get the equations of motion of the infinitesimal mass particle in xy-plane.

$$\ddot{x}-2n\dot{y}=U_x, \tag{3}$$

$$\ddot{y}+2n\dot{x}=U_y \tag{4}$$

where

$$U_x=n^2x-\frac{(1-\mu)q_1(x+\mu)}{r_1^3}-\frac{\mu(x+\mu-1)}{r_2^3}$$
$$-\frac{3}{2}\frac{\mu A_2(x+\mu-1)}{r_2^5}$$
$$-\frac{W_1}{r_1^2}\left\{\frac{(x+\mu)}{r_1^2}[(x+\mu)\dot{x}+y\dot{y}]+\dot{x}-ny\right\},$$
$$U_y=n^2y-\frac{(1-\mu)q_1y}{r_1^3}-\frac{\mu y}{r_2^3}-\frac{3}{2}\frac{\mu A_2 y}{r_2^5}$$
$$-\frac{W_1}{r_1^2}\left\{\frac{y}{r_1^2}[(x+\mu)\dot{x}+y\dot{y}]+\dot{y}+n(x+\mu)\right\}$$

where

$$U=\frac{n^2(x^2+y^2)}{2}+\frac{(1-\mu)q_1}{r_1}+\frac{\mu}{r_2}+\frac{\mu A_2}{2r_2^3}$$
$$+W_1\left\{\frac{(x+\mu)\dot{x}+y\dot{y}}{2r_1^2}-n\arctan\left(\frac{y}{x+\mu}\right)\right\} \tag{5}$$

$q_1=1-\frac{F_p}{F_g}$ is a mass reduction factor expressed in terms of the particle radius **a**, density ρ radiation pressure efficiency factor χ (in C.G.S. system): $q_1=1-\frac{5.6\times10^{-5}}{\mathbf{a}\rho}\chi$. The assumption $q_1=constant$ is equivalent to neglecting fluctuations in the beam of solar radiation and the effect of the planets shadow, obviously $q_1\le 1$. The energy integral of the problem is given by $C=2U-\dot{x}^2-\dot{y}^2$, where the quantity C is the Jacobi's constant. The zero velocity curves (see (2),(3)) are given by:

$$C_i=2U(x_i,y_i) \tag{6}$$

Suffix $i=1,2,3,4,5$ correspond to respective i^{th} Lagrangian equilibrium point L_i. Using above relation we have determined $C_1\approx 3.02978$, $C_2\approx 4.04133$, $C_3\approx 3.53607$. The values of $C_4(\approx C_5)$ are shown by different curves (1–6) in Fig. 1 and Table 1 for various values of q_1, A_2.

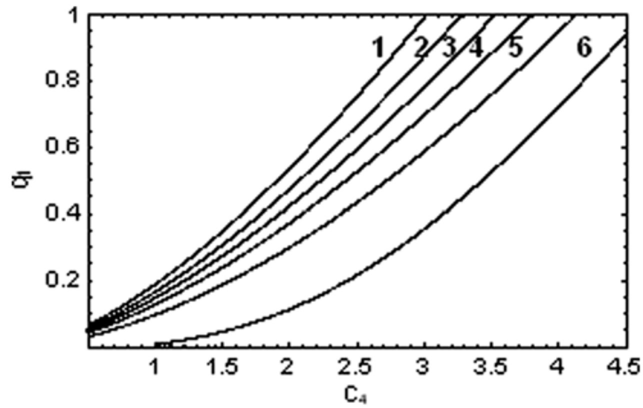

Fig. 1 The Jacobi's constant C_4–q_1 for (1) $A_2=0$, (2) $A_2=0.2$, (3) $A_2=0.4$, (4) $A_2=0.6$, (5) $A_2=0.8$, (6) $A_2=1$, $0\le q_1\le 1$ & $\mu=0.00003$

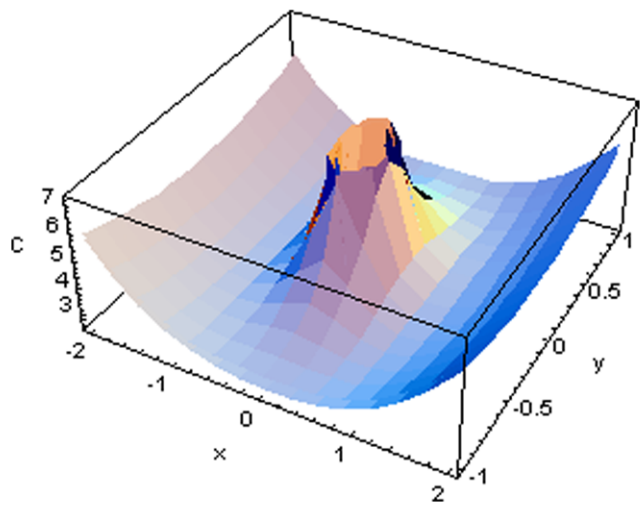

Fig. 2 C vs. x–y when $A_2=0.0024$, $0\le q_1\le 1$ & $\mu=0.00003$

3 Existence of equilibrium points

What are they? What are "Lagrange points", also known as "libration points" or "L-points" or "Equilibrium Points"? These are all jargon for places where a light third body can sit "motionless" relative to two heavier bodies that are orbiting each other, thanks to the force of gravity. The unstable Lagrange points—labeled L_1, L_2 and L_3—lie along the line connecting the two large masses. The conditionally linearly stable Lagrange points in classical case are labeled L_4 and L_5—form the apex of two equilateral triangles as in Fig. 4, that have the large masses at their vertices.

Table 1 Jacobi's constant C at L_4

A_2	$C_4: q_1 = 1$	$C_4: q_1 = 0.75$	$C_4: q_1 = 0.5$	$C_4: q_1 = 0.25$	$C_4: q_1 = 0$
0.0	2.99997	2.47643	1.88988	1.19058	Indeterminate
0.2	3.27697	2.70632	2.06659	1.30303	Indeterminate
0.4	3.51014	2.89885	2.21377	1.3961	Indeterminate
0.6	3.71975	3.0698	2.34214	1.47515	Indeterminate
0.8	3.98328	3.30328	2.53573	1.61041	Indeterminate
1	4.62593	4.06149	3.42503	2.62291	Indeterminate

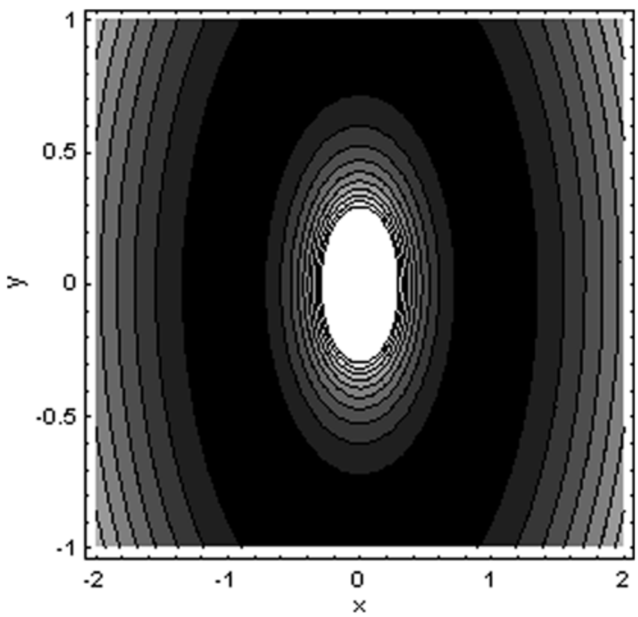

Fig. 3 Contour plot shows the Jacobi's constant C vs. x–y when $A_2 = 0.0024, 0 \le q_1 \le 1$ & $\mu = 0.00003$

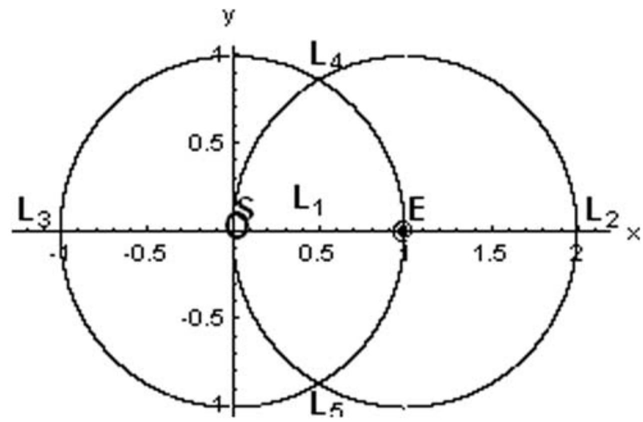

Fig. 4 Figure views the Lagrangian equilibrium points in classical case (when $A_2 = 0, q_1 = 1, \mu = 0.00003$)

3.1 Collinear equilibrium points

To investigate the Equilibrium Points, the orbital plane Oxy is divided into three parts with respect to the primaries $x \le -\mu$, $1-\mu \le x$ and $-\mu < x < 1-\mu$, $U_x = U_y = 0$ then from (3) and (4)

$$\left\{n^2 - \frac{(1-\mu)q_1}{r_1^3} - \frac{\mu}{r_2^3}\left(1 + \frac{3}{2}\frac{A_2}{r_2^2}\right)\right\}(x+\mu) + \frac{nW_1}{r_1^2}y$$

$$= \mu\left\{n^2 - \frac{1}{r_2^3}\left(1 + \frac{3A_2}{2r_2^2}\right)\right\},$$

$$\left\{n^2 - \frac{(1-\mu)q_1}{r_1^3} - \frac{\mu}{r_2^3}\left(1 + \frac{3A_2}{2r_2^2}\right)\right\}y - \frac{nW_1}{r_1^2}(x+\mu)$$

$$= 0.$$

$$\Rightarrow r_2^5\left(\frac{nW_1}{y\mu} - n^2\right) + r_2^2 + \frac{3}{2}A_2 = 0,$$

$$r_1^3\left[(1-\mu)yn^2 + nW_1\right] - n(x+\mu)W_1 r_1 = (1-\mu)q_1 y$$

i.e.

$$r_1 = \left(\frac{q_1}{n^2}\right)^{1/3}\left[1 - \frac{nW_1}{6(1-\mu)y}\right], \tag{7}$$

$$r_2 = \left[1 - \frac{nW_1}{\mu y}\left(1 - \frac{5}{2}A_2\right)\right]^{-1/3} \tag{8}$$

From above, we obtained:

$$x = -\mu \pm \left[\left(\frac{q_1}{n^2}\right)^{2/3} \times \left[1 + \frac{nW_1}{2(1-\mu)y} + \frac{3A_2}{2}\right]^{-2/3} - y^2\right]^{1/2}. \tag{9}$$

$$x = 1 - \mu \pm \left[\left[1 - \frac{nW_1}{\mu y}\left(1 - \frac{5}{2}A_2\right)\right]^{-2/3} - y^2\right]^{1/2} \tag{10}$$

Using (9, 10) the position of L_1, L_2, L_3 is presented graphically in Figs. 4, 5 when $A_2 = 0.0024, 0 \le q_1 \le 1$ & $\mu = 0.00003$. We have seen that the equilibrium points are no-longer collinear with the primaries. The values of y are positive. We observe that the coordinates of L_3 are the functions of q_1 and A_2, corresponding different curves labeled by (1–4) are presented as in Fig. 8. Now when $1-\mu \le x$, there exists an equilibrium point L_2 this can be seen in Figs. 6, 7. This equilibrium point is also found away from the Ox axis. It is clear from the figure that L_2 is a function of q_1 and A_2. When $-\mu < x < 1-\mu$, there are almost three

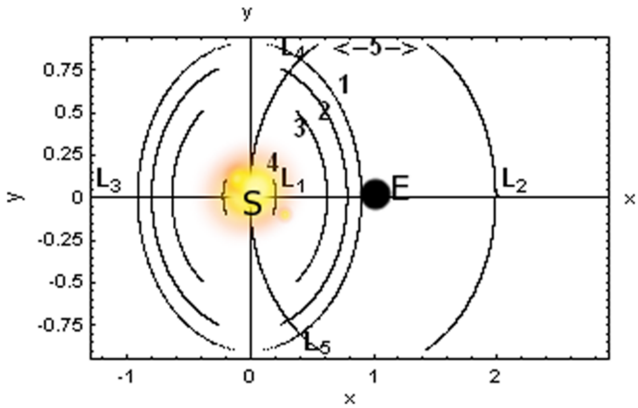

Fig. 5 The position of L_1, L_2, L_3, L_4 and L_5 for (1) $q_1 = 1$, (2) $q_1 = 0.75$, (3) $q_1 = 0.25$, (4) $q_1 = 0$, (5) $0 \le q_1 \le 1$, when $A_2 = 0.0024$ & $\mu = 0.00003$

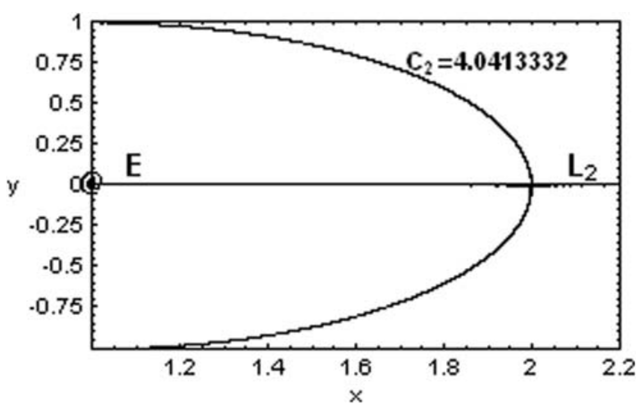

Fig. 6 Position of the Lagrangian equilibrium point L_2 when $A_2 = 0.0024$ & $\mu = 0.00003$, $0 \le q_1 \le 1$

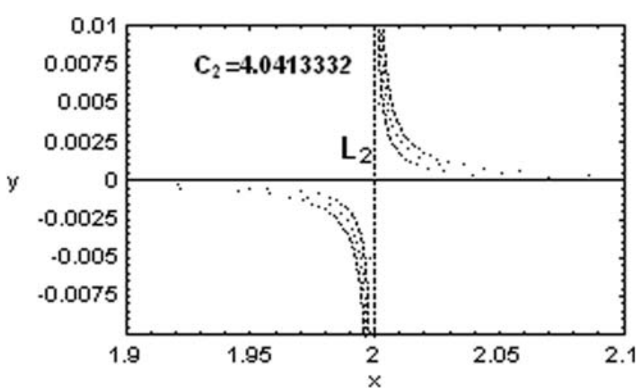

Fig. 7 Figure views the magnified region of L_2, when $A_2 = 0.0024$ & $\mu = 0.00003$, $0 \le q_1 \le 1$

equilibrium points they are L_1, L_4 and L_5 as in Fig. 9. It is clear from the figure, that the L_1 is not collinear any more. L_4 is positioned above the Ox axis while L_5 lies below it. All these results are similar with Szebehely (1967), Ragos and Zafiropoulos (1995), Papadakis and Kanavos (2007) and others.

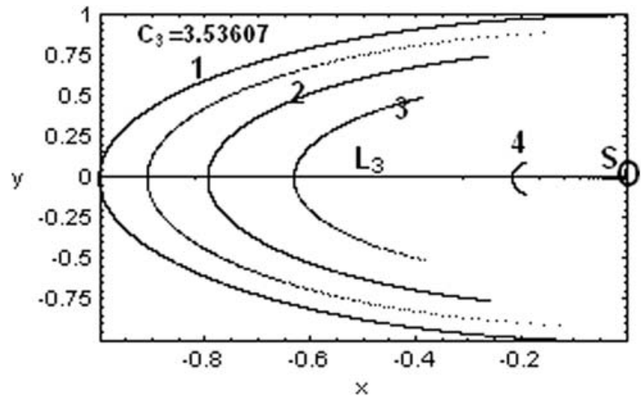

Fig. 8 Position of the Lagrangian equilibrium points L_3 when $A_2 = 0.0024$ & $\mu = 0.00003$ for (1) $q_1 = 1$, (2) $q_1 = 0.75$, (3) $q_1 = 0.25$, (4) $q_1 = 0$

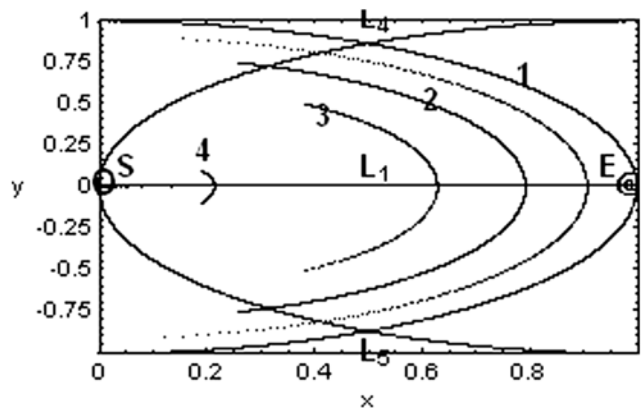

Fig. 9 Position of the Lagrangian equilibrium points L_1, L_4, L_5 when $A_2 = 0.0024$ & $\mu = 0.00003$, for (1) $q_1 = 1$, (2) $q_1 = 0.75$, (3) $q_1 = 0.25$, (4) $q_1 = 0$

3.2 Triangular equilibrium points

For the triangular equilibrium points $y \neq 0$, $U_x = U_y = 0$, then from (3) and (4) we get as follows:

$$n^2 x - \frac{(1-\mu)q_1(x+\mu)}{r_1^3} - \frac{\mu(x+\mu-1)}{r_2^3} - \frac{3}{2}\frac{\mu A_2(x+\mu-1)}{r_2^5} + \frac{W_1 n y}{r_1^2} = 0, \tag{11}$$

$$n^2 y - \frac{(1-\mu)q_1 y}{r_1^3} - \frac{\mu y}{r_2^3} - \frac{3}{2}\frac{\mu A_2 y}{r_2^5} - \frac{W_1}{r_1^2} n(x+\mu) = 0. \tag{12}$$

$$\left\{n^2 - \frac{(1-\mu)q_1}{r_1^3} - \frac{\mu}{r_2^3}\left(1 + \frac{3}{2}\frac{A_2}{r_2^2}\right)\right\}(x+\mu) + \frac{n W_1}{r_1^2} y = \mu\left\{n^2 - \frac{1}{r_2^3}\left(1 + \frac{3A_2}{2r_2^2}\right)\right\},$$

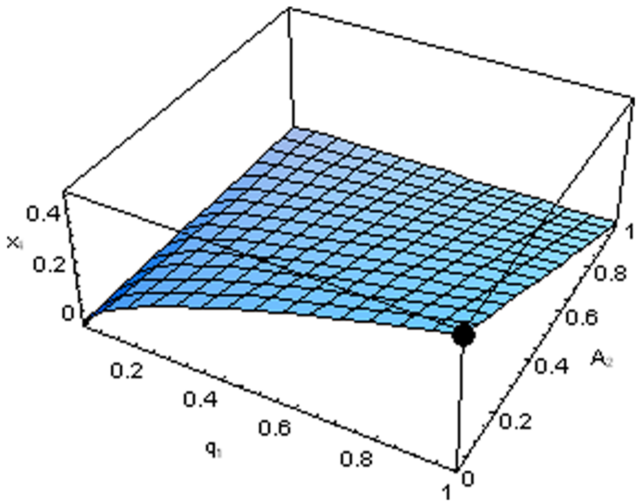

Fig. 10 The x_4 coordinate of equilibrium point when $0 \le A_2 \le 1, 0 \le q_1 \le 1$ & $\mu = 0.00003$, *black dot* on figure indicates the value of x_4 in classical case

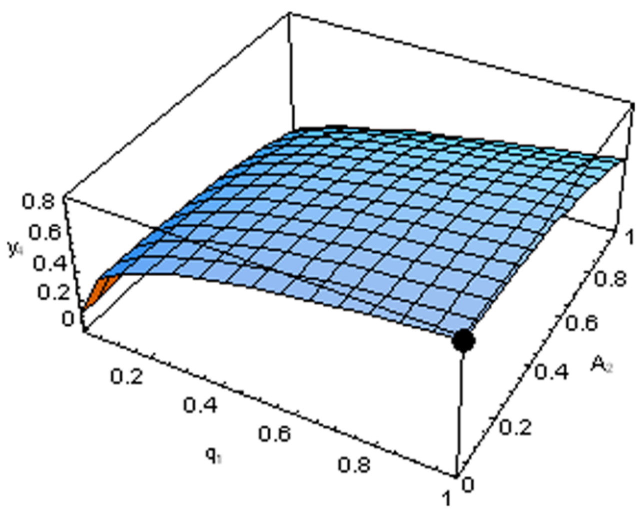

Fig. 11 The y_4 coordinate of equilibrium point when $0 \le A_2 \le 1, 0 \le q_1 \le 1$ & $\mu = 0.00003$, *black dot* on figure indicates the value of y_4 in classical case

$$\left\{n^2 - \frac{(1-\mu)q_1}{r_1^3} - \frac{\mu}{r_2^3}\left(1 + \frac{3A_2}{2r_2^2}\right)\right\}y - \frac{nW_1}{r_1^2}(x+\mu) = 0$$

From (11) and (12), we obtained as follows:

$$\left\{n^2 - \frac{1}{r_2^3}\left(1 + \frac{3A_2}{2r_2^2}\right)\right\}\mu y = nW_1. \tag{13}$$

In the case of photogravitational restricted three body problem we have $r_{1_0} = q_1^{1/3} = \delta$ and $r_{2_0} = 1$. We suppose that due to P-R drag and oblateness, perturbations in r_{1_0}, r_{2_0} are ϵ_1, ϵ_2 respectively, where ϵ_i's are very small quantities, then

$$r_1 = q_1^{1/3}(1+\epsilon_1), \qquad r_2 = 1 + \epsilon_2 \tag{14}$$

Putting these values in (12) and neglecting higher order terms of small quantities, we get

$$\left[n^2 - 1 + 3\epsilon_2 - \frac{3}{2}A_2 + \frac{15}{2}A_2\epsilon_2\right]\mu y = nW_1,$$

$$3\epsilon_2\left(1 + \frac{5}{2}A_2\right)\mu y = nW_1,$$

$$\epsilon_2 = \frac{nW_1(1 - \frac{5}{2}A_2)}{3\mu y_0},$$

where y_0 is y-coordinate of triangular equilibrium points in photogravitational restricted three body problem case. Using values r_1, r_2 with (12), we get following:

$$\left\{n^2 - \frac{(1-\mu)q_1}{r_1^3} - \frac{\mu}{r_2^3}\left(1 + \frac{3A_2}{2r_2^2}\right)\right\}y - \frac{nW_1}{r_1^2}(x+\mu) = 0,$$

$$\left[\left\{n^2 - (1-\mu)(1-\epsilon_1) - \mu(1-\epsilon_2)\left(1 + \frac{3A_2(1-2\epsilon_2)}{2}\right)\right\}\right]y - \frac{nW_1(r_1^2 - r_2^2 + 1)}{r_1^2} = 0,$$

$$\left[n^2 - (1-\mu) + 3(1-\mu)\epsilon_1 - \mu + 3\epsilon_2 - \mu(1-5\epsilon_2)\frac{3}{2}A_2\right]y = \frac{nW_1}{2}$$

we obtained $\epsilon_1 = -\frac{nW_1}{6(1-\mu)y_0} - \frac{A_2}{2}$, $r_1 = \delta\{1 - \frac{nW_1}{6(1-\mu)y_0} - \frac{A_2}{2}\}$, $r_2 = 1 + \frac{nW_1}{3\mu y_0}(1 - \frac{5}{2}A_2)$.

Now we have $x + \mu = \frac{r_1^2 - r_2^2 + 1}{2}$, $y^2 = r_1^2 - (x+\mu)^2$ therefore,

$$(x+\mu) = \frac{1}{2}\left[\delta^2\left\{1 - \frac{nW_1}{6(1-\mu)y_0} - \frac{A_2}{2}\right\}^2 - \left\{1 + \frac{nW_1}{3\mu y_0(1+\frac{5}{2}A_2)}\right\}^2 + 1\right]$$

or

$$x_4 = x_0\left[1 - \frac{\delta^2}{2}\frac{A_2}{x_0} - \frac{nW_1[(1-\mu)(1+\frac{5}{2}A_2) + \mu(1-\frac{A_2}{2})\frac{\delta^2}{2}]}{3\mu(1-\mu)y_0x_0}\right] \tag{15}$$

$$y^2 = r_1^2 - (x+\mu)^2 = \delta^2\left[1 - \frac{nW_1}{6(1-\mu)y_0}\right]^2 - \left[\frac{\delta^2}{2}\left(1 - \frac{\delta^2}{2}A_2\right)\right.$$

Table 2 x co-ordinate of L_4

A_2	$x_4 : q_1 = 1$	$x_4 : q_1 = 0.75$	$x_4 : q_1 = 0.5$	$x_4 : q_1 = 0.25$	$x_4 : q_1 = 0$
0.0	**0.499949**	0.412692	0.314934	0.198383	Complex Infinity
0.25	0.374929	0.30949	0.236175	0.148765	Complex Infinity
0.5	0.249906	0.206284	0.157412	0.099146	Complex Infinity
0.75	0.12488	0.103076	0.0786479	0.0495247	Complex Infinity
1	−0.000148407	−0.000134579	−0.000118667	−0.0000980564	Complex Infinity

Table 3 y co-ordinate of L_4

A_2	$y_4 : q_1 = 1$	$y_4 : q_1 = 0.75$	$y_4 : q_1 = 0.5$	$y_4 : q_1 = 0.25$	$y_4 : q_1 = 0$
0.0	**0.86605**	0.809425	0.728552	0.597927	Indeterminate
0.25	0.847795	0.790465	0.709777	0.581035	Indeterminate
0.5	0.790543	0.73068	0.650249	0.527174	Indeterminate
0.75	0.684579	0.61834	0.536576	0.422435	Indeterminate
1	0.499831	0.412542	0.314727	0.198036	Indeterminate

$$- \frac{nW_1[(1-\mu)(1+\frac{5}{2}A_2)+\mu(1-\frac{A_2}{2})\frac{\delta^2}{2}]}{3\mu(1-\mu)y_0}\Bigg]^2$$

or

$$y_4 = y_0\Bigg[1 - \frac{\delta^2(1-\frac{\delta^2}{2})A_2}{y_0^2} - \frac{nW_1\delta^2[2\mu-1-\mu(1-\frac{3A_2}{2})\frac{\delta^2}{2}+7(1-\mu)\frac{A_2}{2}]}{3\mu(1-\mu)y_0^3}\Bigg]^{1/2} \tag{16}$$

where (x_0, y_0) are coordinates of L_4, L_5 in photogravitational restricted three body problem case as:

$$x_0 = \frac{\delta^2}{2} - \mu, \qquad y_0 = \pm\delta\left(1 - \frac{\delta^2}{4}\right)^{1/2}, \qquad \delta = q_1^{1/3}$$

The position of $L_{4(5)}$ is given by (15), (16) which are valid for $W_1 \ll 1$, $A_2 \ll 1$. For simplicity we suppose $\gamma = 1-2\mu$, $q_1 = 1-\epsilon$, with $|\epsilon| \ll 1$ then the coordinates (x_4, y_4) of $L_{4(5)}$ can be written as follows:

$$x_4 = \frac{\gamma}{2} - \frac{\epsilon}{3} - \frac{A_2}{2} + \frac{A_2\epsilon}{3} - \frac{(9+\gamma)}{6\sqrt{3}}nW_1 - \frac{4\gamma\epsilon}{27\sqrt{3}}nW_1 \tag{17}$$

$$y_4 = \frac{\sqrt{3}}{2}\Bigg\{1 - \frac{2\epsilon}{9} - \frac{A_2}{3} - \frac{2A_2\epsilon}{9} + \frac{(1+\gamma)}{9\sqrt{3}}nW_1 - \frac{4\gamma\epsilon}{27\sqrt{3}}nW_1\Bigg\} \tag{18}$$

Figures 10, 11 and Tables 2, 3 show how the coordinates (x_4, y_4) points are decreasing functions of A_2, q_1, W_1. Black dots on the figures indicate the $(x_4, y_4) = (\frac{1}{2}-\mu, \frac{\sqrt{3}}{2})$.

4 Comments on the linear stability

In order to study the linear stability of any Lagrangian equilibrium point L_i $(i = 1-5)$ the origin of the coordinate system to its position (x_i, y_i) by means of $x = x_i + \alpha$, $y = y_i + \beta$, where $\alpha = \xi e^{\lambda t}$, $\beta = \eta e^{\lambda t}$ are the small displacements ξ, η, λ these parameters, have to be determined. Therefore the equations of perturbed motion corresponding to the system of (3), (4) may be written as follows:

$$\ddot{\alpha} - 2n\dot{\beta} = \alpha U^i_{xx} + \beta U^i_{xy} + \dot{\alpha}U^i_{x\dot{x}} + \dot{\beta}U^i_{x\dot{y}} \tag{19}$$

$$\ddot{\beta} + 2n\dot{\alpha} = \alpha U^i_{yx} + \beta U^i_{yy} + \dot{\alpha}U^i_{y\dot{x}} + \dot{\beta}U^i_{y\dot{y}} \tag{20}$$

where superfix i is corresponding to the L_i $(i = 1–5)$

$$(\lambda^2 - \lambda U^i_{x\dot{x}} - U^i_{xx})\xi + [-(2n + U^i_{x\dot{y}})\lambda - U^i_{xy}]\eta = 0 \tag{21}$$

$$[(2n - U^i_{y\dot{x}})\lambda - U^i_{yx}]\xi + (\lambda^2 - \lambda U^i_{y\dot{y}} - U^i_{yy})\eta = 0 \tag{22}$$

Now above system has singular solution if,

$$\begin{vmatrix} \lambda^2 - \lambda U^i_{x\dot{x}} - U^i_{xx} & -(2n + U^i_{x\dot{y}})\lambda - U^i_{xy} \\ (2n - U^i_{y\dot{x}})\lambda - U^i_{yx} & \lambda^2 - \lambda U^i_{y\dot{y}} - U^i_{yy} \end{vmatrix} = 0$$

$$\Rightarrow \quad \lambda^4 + a\lambda^3 + b\lambda^2 + c\lambda + d = 0 \tag{23}$$

At the equilibrium points (3), (4) give us the following:

$$a = 3\frac{W_1}{r^2_{1i}}, \qquad b = 2n^2 - f_i - \frac{3\mu A_2}{r^5_{2i}} + \frac{2W_1^2}{r^4_{1i}}$$

$$c = -a(1+e),$$

$$e = \frac{\mu}{r^5_{2i}}A_2 + \frac{\mu}{r^2_{1i}r^5_{2i}}\left(1 + \frac{5A_2}{2r^2_{2i}}\right)y_i^2,$$

$$d = (n^2 - f_i)\left[n^2 + 2f_i - \frac{3\mu A_2}{r^5_{2i}}\right]$$

$$+\frac{9\mu(1-\mu)q_1}{r_{1i}^5 r_{2i}^5}\left(1+\frac{5A_2}{2r_{2i}^2}\right)y_i^2$$

$$-\frac{6nW_1}{r_{1i}^4 r_{2i}^5}\left(1+\frac{5A_2}{2r_{2i}^2}\right)\left\{(x_i+\mu)(x_i+\mu-1)+y_i^2\right\},$$

$$f_i=\frac{(1-\mu)q_1}{r_{1_i}^3}+\frac{\mu}{r_{2i}^3}\left(1+\frac{3}{2}\frac{A_2}{r_{2i}^2}\right)$$

The points L_1, L_2, L_3 no longer lie along the line joining the primaries, since the condition is not satisfied for them, so taking $y\to 0$, $\frac{W_1}{y}\to 0$ because $y\gg W_1, x\gg W_1$, from (7) we have $r_1\approx\frac{q_1^{1/3}}{n^2}$. In this case $f_i>1$ for $i=1,2,3$, so that for each L_1, L_2, L_3 characteristic equation (23) has at least one positive root, this implies that these points are unstable in the sense of Lyapunov.

4.1 Linear stability without P-R effect

Now we consider the problem when P-R effect is not included ($W_1=0$), then $r_1=q_1^{1/3}(1-\frac{A_2}{2})$, $r_2=1$. The coordinates of triangular points $L_{4(5)}$ are as:

$$x=\left(\frac{q_1^{1/3}}{2}-\mu\right)-\frac{q_1^{1/3}A_2}{2} \tag{24}$$

$$y=q_1^{1/3}\left[\left(1-\frac{q_1^{2/3}}{4}\right)^{1/2}-\left(1-\frac{q_1^{2/3}}{2}\right)\left(1+\frac{q_1^{2/3}}{4}\right)A_2\right] \tag{25}$$

$$a=0,\qquad c=0,\qquad f_i=n^2,\quad (i=4,5)$$

$$b=n^2-3\mu A_2,\qquad d=9\mu(1-\mu)g$$

where $g=(1-A_2)[1-\frac{q_1^{2/3}(1-A_2)}{4}]$. From characteristic equation (23) we obtained,

$$\lambda^2=\frac{-b\pm(b^2-4d)^{1/2}}{2} \tag{26}$$

For stable motion $0<4d<b^2$, i.e.

$$\left(n^2-3\mu A_2\right)^2>36\mu(1-\mu)g$$

In classical case $A_2=0$, $q_1=1$, $W_1=0$, $n=1$, we have following: $1>27\mu(1-\mu)\ \Rightarrow\ \mu<0.0385201$. The possible roots of (26) are given in Table 6, we see that all the roots are purely imaginary quantities. Hence the triangular equilibrium points are stable in the sense of Lyapunov stability provided $\mu<\mu_{Routh}=0.0385201$.

Equation (26) has imaginary roots $\lambda_{1,2}=\pm\mathbf{i}\omega_1$, $\lambda_{3,4}=\pm\mathbf{i}\omega_2$, $\mathbf{i}=\sqrt{-1}$, this gives us:

$$\omega_{1,2}=\left(\frac{-b\pm(b^2-4d)^{1/2}}{2}\right)^{1/2} \tag{27}$$

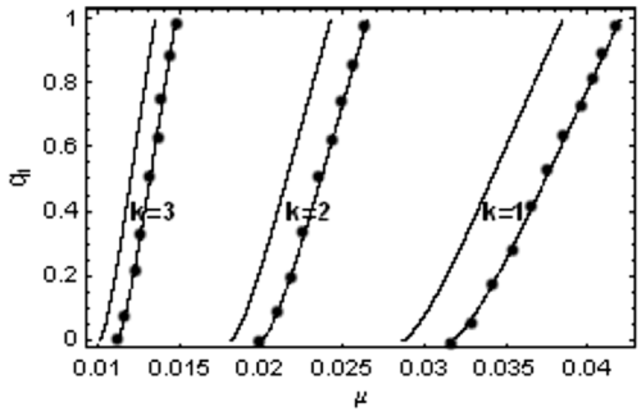

Fig. 12 The linear stability region for μ–q_1 parameter space and resonance curves $\omega_1-k\omega_2=0, k=1,2,3$ at $A_2=0, 0.02$ (*doted lines*) & $\mu=0.00003$, $0\le q_1\le 1$

There are three main cases of resonances as:

$$\omega_1-k\omega_2=0,\qquad k=1,2,3 \tag{28}$$

For $k=1$ we have positive stable resonance and for $k=2,3$ we have unstable resonances. Using (27) and (28) we obtained a root of mass parameter.

$$\mu_k=\frac{3g+2KA_2+3KA_2^2-\sqrt{g}\sqrt{9g-4K+9KA_2^2}}{6(g+KA_2^2)} \tag{29}$$

where $K=\frac{k^2}{(k^2+1)^2}$. Now we suppose $q_1=1-\epsilon$, with $|\epsilon|\ll 1$, neglecting higher order terms, we obtained the critical mass parameter values corresponding to $k=1,2,3$ as:

$$\mu_1=0.0385208965+0.6755841373A_2-0.0089174706\epsilon \tag{30}$$

$$\mu_2=0.0242938971+0.4322031625A_2-0.0055364958\epsilon \tag{31}$$

$$\mu_3=0.0135160160+0.2430452832A_2-0.0030452832\epsilon \tag{32}$$

The linear stability region and corresponding main resonance curves in μ–q_1 parameter space are shown in Fig. 12 the doted lines are corresponding to $A_2=0.02$, the curve corresponding to $k=1$, $(q_1=1, A_2=0, \mu_1=\mu_{Routh}=0.038521)$ is actual boundary of the stability region these results are similar to Markellos et al. (1996) and others. The critical values of mass parameter μ are given in the Tables 4, 5 for various values of q_1, A_2. The classical critical values of μ are similar to Deprit and Deprit-Bartholome (1967). We observe that the effect of radiation pressure reduces the linear stability zones, these are also affected by oblateness of second primary.

Table 4 $A_2 = 0$

k	$\mu_k : q_1 = 1$	$\mu_k : q_1 = 0.75$	$\mu_k : q_1 = 0.5$	$\mu_k : q_1 = 0.25$	$\mu_k : q_1 = 0$
1	0.0385209	0.0363201	0.0341355	0.0318518	0.0285955
2	0.0242939	0.0229262	0.0215661	0.0201415	0.0181056
3	0.013516	0.0127632	0.0120136	0.0112275	0.0101021
4	0.00827037	0.0078121	0.00735548	0.00687629	0.00618979
5	0.0055092	0.00520474	0.004901287	0.00490128	0.00412616

Table 5 $A_2 = 0.02$

k	$\mu_k : q_1 = 1$	$\mu_k : q_1 = 0.75$	$\mu_k : q_1 = 0.5$	$\mu_k : q_1 = 0.25$	$\mu_k : q_1 = 0$
1	0.0413469	0.0390477	0.0367581	0.0343567	0.0309186
2	0.026094	0.0246633	0.0232361	0.0217366	0.019585
3	0.0145252	0.0137369	0.0129496	0.0121214	0.0109312
4	0.00889015	0.00841007	0.00793029	0.00742525	0.00669897
5	0.00592287	0.00560384	0.00528491	0.00494909	0.00446598

4.2 Linear stability with P-R effect

Now consider the problem when P-R effect is included i.e., $q_1 \neq 1$, $W_1 \neq 0$, $A_2 \neq 0$. Using Ferrari's theorem the roots of characteristic equation (23) are given by:

$$\lambda_i = -\frac{(a+A)}{4} \pm \sqrt{\left(\frac{a+A}{4}\right)^2 - B} \tag{33}$$

where $A = \pm\sqrt{8l - 4b + a^2}$, $B = l(1 + \frac{a}{A}) - \frac{c}{A}$, $i = 1, 2, 3, 4$ and l is any real root of the equation

$$8l^3 - 4bl^2 + (2ac - 8d)l + d(4b - a^2) - c^2 = 0$$

This can be written as:

$$2l^2 - bl^2 - 2dl + db = \frac{a^2}{4}\left\{(1+e)^2 - 2(1+e)l + d\right\} \tag{34}$$

This equation has an exact real root $l = \frac{b}{2}$ for $a = 0$. When $a \neq 0$ the roots of characteristic equation (23) will be obtained in form of the rapidly convergent series

$$l = \frac{b}{2} + \sum_{j=1}^{\infty} \alpha_j a^{2j} \tag{35}$$

Using (34, 35), taking the coefficients of a^2 only, we get $A = a \pm \sqrt{1 + 8\alpha_1}$,

$$B = \left(\frac{b}{2} + \alpha_1 a^2\right)\left(1 \pm \sqrt{1 + 8\alpha_1}\right) \mp \frac{1+e}{\sqrt{1+8\alpha_1}},$$

$$\alpha_1 = \frac{(1+e)(1+e^2-b)+d}{2(b^2-4d)} > 0 \tag{36}$$

We obtained the characteristic roots

$$\lambda_{1,2} = -\frac{a\left(1+\sqrt{1+8\alpha_1}\right)}{4} \pm \sqrt{\frac{a^2\left(1+\sqrt{1+8\alpha_1}\right)}{16} - B_1} \tag{37}$$

$$\lambda_{3,4} = -\frac{a\left(1-\sqrt{1+8\alpha_1}\right)}{4} \pm \sqrt{\frac{a^2\left(1-\sqrt{1+8\alpha_1}\right)}{16} - B_2} \tag{38}$$

where

$$B_1 = \left(\frac{b}{2} + \alpha_1 a^2\right)\left(1 + \sqrt{1+8\alpha_1}\right) - \frac{1+e}{\sqrt{1+8\alpha_1}},$$

$$B_2 = \left(\frac{b}{2} + \alpha_1 a^2\right)\left(1 - \sqrt{1+8\alpha_1}\right) + \frac{1+e}{\sqrt{1+8\alpha_1}}$$

Using above equations (37, 38), we obtained the roots of characteristic equation (23) which are presented in Table 7 for various values of q_1, A_2. We see that at least one of the roots λ_i $(i = 1, 2, 3, 4)$ have a positive real part due to P-R effect as in Chernikov (1970). Hence the triangular equilibrium points are unstable in the sense of Lyapunov stability.

5 Conclusion

The distances of L_1, L_2 from the second primary are monotonically increasing with μ. The Jacobi's constant at the L_1, L_3 increases monotonically with μ. If $C < C_1$, the particles can leave the system while if $C_1 < C < C_2$, i.e. the third body is not confined to its motion around the Sun but it is allowed to become a satellite of the Earth. This does not mean that it will become permanently a satellite of the Earth since its Jacobi's constant is such that it is not confined to the zero velocity oval around Earth. The particle with $C > C_2$ then it can not change position from the vicinity of Sun to

Table 6 Roots of characteristic equation when P-R-effect is not considered

q_1	$\lambda_{1,2}: A_2=0$	$\lambda_{3,4}: A_2=0$	$\lambda_{1,2}: A_2=0.02$	$\lambda_{3,4}: A_2=0.02$	$\lambda_{1,2}: A_2=0.04$	$\lambda_{3,4}: A_2=0.04$	$\lambda_{1,2}: A_2=0.06$	$\lambda_{3,4}: A_2=0.06$
1	$\pm0.0100624\mathbf{i}$	$\pm1.414178\mathbf{i}$	$\pm0.0103471\mathbf{i}$	$\pm0.0103471\mathbf{i}$	$\pm1.435232\mathbf{i}$	$\pm1.45598\mathbf{i}$	$\pm0.0109142\mathbf{i}$	$\pm1.476440\mathbf{i}$
0.8	$\pm0.0102916\mathbf{i}$	$\pm1.414176\mathbf{i}$	$\pm0.010577\mathbf{i}$	$\pm1.435230\mathbf{i}$	$\pm0.0108624\mathbf{i}$	$\pm1.45598\mathbf{i}$	$\pm0.0111478\mathbf{i}$	$\pm1.476438\mathbf{i}$
0.6	$\pm0.0105354\mathbf{i}$	$\pm1.414174\mathbf{i}$	±0.0108217	$\pm1.435229\mathbf{i}$	$\pm0.0111086\mathbf{i}$	$\pm1.455978\mathbf{i}$	$\pm0.0113964\mathbf{i}$	$\pm1.476436\mathbf{i}$
0.4	$\pm0.0108019\mathbf{i}$	$\pm1.414172\mathbf{i}$	$\pm0.0110893\mathbf{i}$	$\pm1.435226\mathbf{i}$	$\pm.3704\mathbf{i}$	$\pm1.455976\mathbf{i}$	$\pm0.0116684\mathbf{i}$	$\pm1.476434\mathbf{i}$
0.2	$\pm0.0347\mathbf{i}$	$\pm1.414170\mathbf{i}$	$\pm0.0114002\mathbf{i}$	$\pm1.435224\mathbf{i}$	$\pm0.011378\mathbf{i}$	$\pm1.455973\mathbf{i}$	$\pm0.0119841\mathbf{i}$	$\pm1.476432\mathbf{i}$
0.0		$\mathbf{i}$	$1.435269\mathbf{i}$	0	$1.456020\mathbf{i}$	0	$\pm1.476480\mathbf{i}$	0

Note. Table 6 presents the roots of characteristic equation (23) for $\mu = 0.00003$, when P-R effect is not considered

Table 7 Roots of characteristic equation when P-R effect is considered

q_1	$\lambda_{1,2}: A_2=0$	$\lambda_{3,4}: A_2=0$	$\lambda_{1,2}: A_2=0.02$	$\lambda_{3,4}: A_2=0.02$	$\lambda_{1,2}: A_2=0.04$	$\lambda_{3,4}: A_2=0.04$
1	$0.0 \pm 0.0242\mathbf{i}$	$0.0 \pm 0.9998\mathbf{i}$	±0.2522	$0.0 \pm 1.0455\mathbf{i}$	±0.3692	$0.0 \pm 1.0926\mathbf{i}$
0.8	$-1.1614 \times 10^{-9} \pm 0.0254\mathbf{i}$	$2.5627 \times 10^{-13} \pm 0.9997\mathbf{i}$	±0.2523	$-3.4918 \times 10^{-11} \pm 1.0455\mathbf{i}$	±0.3696	$-7.0611 \times 10^{-11} \pm 1.0927\mathbf{i}$
0.6	$-2.8140 \times 10^{-9} \pm 0.0272\mathbf{i}$	$6.9284 \times 10^{-13} \pm 0.9996\mathbf{i}$	±0.2523	$-8.4597 \times 10^{-11} \pm 1.0455\mathbf{i}$	±0.36700	$-1.7126 \times 10^{-10} \pm 1.0928\mathbf{i}$
0.4	$-5.5311 \times 10^{-9} \pm 0.0298\mathbf{i}$	$1.5839 \times 10^{-12} \pm 0.9996\mathbf{i}$	±0.2522	$-1.6621 \times 10^{-10} \pm 1.0454\mathbf{i}$	$\pm.3704$	$-3.3695 \times 10^{-10} \pm 1.0929\mathbf{i}$
0.2	$-1.1707 \times 10^{-8} \pm 0.0347\mathbf{i}$	$4.3264 \times 10^{-12} \pm 0.9995\mathbf{i}$	±0.2519	$-3.5125 \times 10^{-10} \pm 1.0454\mathbf{i}$	±0.3707	$-7.1369 \times 10^{-10} \pm 1.0930\mathbf{i}$
0.0	Indeterminate	Indeterminate	$0 \pm 0.0242\mathbf{i}$	Indeterminate	Indeterminate	Indeterminate

Note. Table 7 is presents the roots of characteristic equation (23) for $\mu = 0.00003$ when P-R effect is included that of Earth. The particles with very high C values have low relative energy levels and they either move around one of the primaries or move for outside of the system. Finally we have conclude that, if one the primary is exerts the light radiation pressure and second primary is an oblate spheroid, then the gravitational radiation force and oblateness influence the existence, location of equilibrium points. In classical case when q_1, $A_2 = 0$, we have the collinear points (L_1, L_2, L_3) and two triangular equilibrium points L_4, L_5. We have seen that L_1, L_2, L_3 are linearly unstable, while L_4, L_5 are conditionally stable in the sense of Lyapunov when P-R effect is not considered. The effect of radiation pressure reduces the linear stability zones while P-R effect induces an instability in the sense of Lyapunov.

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