Astrophys Space Sci (2008) 314: 73–77
DOI 10.1007/s10509-008-9739-1

ORIGINAL ARTICLE

Exact Bianchi type II, VIII and IX string cosmological models in Saez-Ballester theory of gravitation

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Received: 3 September 2007 / Accepted: 5 January 2008 / Published online: 6 February 2008

Abstract Exact Bianchi type II, VIII and IX String cosmological models are obtained in the Saez-Ballester theory of gravitation. Some physical and geometrical properties of the models are studied.

Keywords Bianchi type-II · VIII and IX · Saez-Ballester theory · Cosmic strings

1 Introduction

Einstein's general theory of relativity (1916) has provided a sophisticated theory of gravitation. It has been very successful in describing gravitational phenomena. It has also served as a basis for models of the universe. The homogeneous isotropic expanding model based on general relativity appears to provide a grand approximation to the observed large scale properties of the universe. However, since Einstein first published his theory of gravitation a no. of modifications have been proposed from time to time which seek to incorporate into the theory certain desirable features lacking in the original theory. For example, Einstein himself pointed out that general relativity does not account satisfactorily for the inertial properties of matter, i.e. Mach's principle is not substantiated by general relativity. So, in recent years, there have been some interesting attempts to generalize the general theory of relativity by incorporating Mach's principle and other desired features which are lacking in the original theory.

Brans and Dicke (1961) scalar-tensor theory of gravitation introduces an additional scalar field ϕ beside the metric tensor g_{ij} and a dimensionless value coupling constant ω. This theory tends to general relativity for large value of the coupling constant ($\omega > 500$). Saez and Ballester (1986) have formulated a scalar tensor theory of gravitation in which the metric is coupled with a dimensionless scalar field in a simple manner. This coupling gives a satisfactory description of the weak field. In spite of the dimensionless character of the scalar field an antigravity regime appears. This theory also suggests a possible way to solve missing matter problem in non-flat FRW cosmologies.

Singh and Rai (1983) gives a detailed discussion of Brans and Dicke cosmological models while Singh and Agarwal (1991), Shri Ram and Tiwari (1998), Reddy and Venkateswara Rao (2001), Rao et al. (2007) have investigated several cosmological models of Saez and Ballester (1986) scalar-tensor theory with a perfect fluid source.

In recent years, there has been a considerable interest in cosmological models in Einstein's theory and in several alternative theories of gravitation with cosmic string source. Cosmic strings and domain walls are the topological defects associated with spontaneous symmetry breaking whose plausible production site is cosmological phase transitions in the early universe (Kibble 1976). The gravitational effects of cosmic strings have been extensively discussed by Vilenkin (1981), Gott (1985), Letelier (1983) and Satchel (1980) in general relativity. Relativistic string models in the context of Bianchi space times have been obtained by Krori et al. (1990), Banerjee et al. (1990), Tikekar and Patel (1990) and Bhattacharjee and Baruah (2001). String cosmological models in scalar-tensor theories of gravitation have been investigated by Sen (2000), Barros et al. (2001), Sen et al. (1997), Gundalach and Ortiz (1990), Barros and Romero (1995), Rahaman et al. (2003) and others.

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In particular, Reddy (2003), Reddy et al. (2006) and Reddy and Naidu (2007) have discussed some string cosmological models in Saez-Ballester scalar-tensor theory of gravitation.

Bianchi type space-times play a vital role in understanding and description of the early stages of evolution of the universe. In particular, the study of Bianchi type-II, VIII and IX universes are important because familiar solutions like FRW universe with positive curvature, the de sitter universe, the Taub-Nut solutions etc correspond of Bianchi type-II, VIII and IX space-times, Chakraborty (1991), Raj Bali and Dave (2001), Raj Bali and Yadav (2005), Bianchi type IX string as well as viscous fluid models in general relativity. Reddy et al. (1993) studied Bianchi type-II, VIII and IX models in scale covariant theory of gravitation. Shanthi and Rao (1991) studied Bianchi type-VIII and IX models in Lyttleton-Bondi Universe. Also Rao and Sanyasi Raju (1992) and Sanyasi Raju and Rao (1992) have studied Bianchi type-VIII and IX in Zero mass scalar fields and self creation cosmology. Rahaman et al. (2003) have investigated Bianchi type-IX string cosmological model in a scalar-tensor theory formulated by Sen (1957) based on Lyra (1951) manifold. Recently Venkateswarlu et al. (2007) have investigated Bianchi type-I, II, VIII and IX string cosmological solutions in self creation theory of gravitation.

In this paper we discuss spatially homogeneous Bianchi type-II, VIII and IX string cosmological models in the scalar-tensor theory proposed by Saez and Ballester.

2 Metric and field equations

We consider a spatially homogeneous Bianchi type-II, VIII and IX metrics of the form

$$ds^2 = -dt^2 + R^2[d\theta^2 + f^2(y)d\phi^2] + S^2[d\varphi + h(y)d\phi]^2 \tag{1}$$

where (θ, ϕ, φ) are the Eulerian angles, R and S are functions of t only. It represents

Bianchi-type II if $f(y) = 1$ and $h(y) = y$

Bianchi-type VIII if $f(y) = \cosh y$ and $h(y) = \sinh y$

Bianchi-type IX if $f(y) = \sin y$ and $h(y) = \cos y$

The field equations given by Saez and Ballester (1986) for the combined scalar and tensor fields are

$$G_{ij} - \omega\phi^n\left(\phi_{,i}\phi_{,j} - \frac{1}{2}g_{ij}\phi_{,k}\phi^{,k}\right) = -T_{ij} \tag{2}$$

and the scalar field ϕ satisfies the equation

$$2\phi^n\phi^{,i}_{;i} + n\phi^{n-1}\phi_{,k}\phi^{,k} = 0 \tag{3}$$

where $G_{ij} = R_{ij} - \frac{1}{2}Rg_{ij}$ is an Einstein tensor, T_{ij} is the stress energy tensor of the matter, ω is the dimensionless constant, n is also a constant.

The equation of motion

$$T^{ij}_{;j} = 0 \tag{4}$$

is a consequence of the field equations (2) and (3).

The energy momentum tensor for cosmic string is

$$T_{ij} = \rho u_i u_j - \lambda x_i x_j \tag{5}$$

where ρ is the rest energy density of cloud of strings with particles attached to them, λ is tension density of strings, u^i is cloud four-velocity, x^i is direction of anisotropy.

Orthonormalisation of u^i and x^i are given as

$$u^i u_i = -x^i x_i = -1 \quad \text{and} \quad u^i x_i = 0 \tag{6}$$

Also we have

$$\rho = \rho_p + \lambda$$

where ρ_p is the rest energy density of particles and x^i to be along Y-axis, so that

$$x^i = (0, S^{-1}, 0, 0)$$

In the commoving coordinate system, we have from (5) and (6)

$$\begin{aligned} &T^1_1 = T^3_3 = 0, \quad T^2_2 = -\lambda, \quad T^4_4 = -\rho \quad \text{and} \\ &T^i_j = 0 \quad \text{for } i \neq j \end{aligned} \tag{7}$$

The quantities ρ, λ and the scalar field ϕ in the theory depend on t only.

The field (2), (3) and (4) for the metric (1) with the help of (5), (6) and (7) can be written as

$$\frac{\ddot{R}}{R} + \frac{\ddot{S}}{S} + \frac{\dot{R}\dot{S}}{RS} + \frac{S^2}{4R^4} - \frac{1}{2}\omega\phi^n\dot{\phi}^2 = 0 \tag{8}$$

$$\frac{2\ddot{R}}{R} + \frac{\dot{R}^2 + \delta}{R^2} - \frac{3S^2}{4R^4} - \frac{1}{2}\omega\phi^n\dot{\phi}^2 = \lambda \tag{9}$$

$$\frac{2\dot{R}\dot{S}}{RS} - \frac{S^2}{4R^4} + \frac{\dot{R}^2 + \delta}{R^2} + \frac{1}{2}\omega\phi^n\dot{\phi}^2 = \rho \tag{10}$$

$$\ddot{\phi} + \dot{\phi}\left(\frac{2\dot{R}}{R} + \frac{\dot{S}}{S}\right) + \frac{n}{2\phi}\dot{\phi}^2 = 0 \tag{11}$$

$$\dot{\rho} + \frac{\dot{S}}{S}(\rho - \lambda) + 2\rho\frac{\dot{R}}{R} = 0 \tag{12}$$

where '.' denotes differentiation with respect to 't'. Also for $\delta = 0, -1$ and $+1$, the field equations (8)–(12) correspond to the Bianchi type-II, VIII and IX universes respectively.

Using the transformation $R = e^{\alpha}$, $S = e^{\beta}$, $dt = R^2 S dT$ the (8) to (12) reduces to

$$\alpha'' + \beta'' - \alpha'^2 - 2\alpha'\beta' + \frac{e^{4\beta}}{4} - \frac{\omega}{2}\phi^n\phi'^2 = 0 \tag{13}$$

$$2\alpha'' - \alpha'^2 - 2\alpha'\beta' + \delta e^{(2\alpha+2\beta)} - \frac{3}{4}e^{4\beta} - \frac{\omega}{2}\phi^n\phi'^2 = \lambda e^{(4\alpha+2\beta)} \tag{14}$$

$$2\alpha'\beta' + \alpha'^2 + \delta e^{(2\alpha+2\beta)} - \frac{1}{4}e^{4\beta} + \frac{\omega}{2}\phi^n\phi'^2 = \rho e^{(4\alpha+2\beta)} \tag{15}$$

$$\phi'' + \frac{n}{2\phi}\phi'^2 = 0 \tag{16}$$

$$\rho' + 2\rho\alpha' + (\rho - \lambda)\beta' = 0 \tag{17}$$

where '$\prime$' denotes differentiation with respect to 'T'.

Here we consider the geometric string or Nambu string model with $\rho = \lambda$. In this case from (17), we get

$$\rho R^2 = c_1 \tag{18}$$

where c_1 is an integration constant. From (13) and (15), we have

$$\alpha'' + \beta'' + \delta e^{(2\alpha+2\beta)} = \rho e^{(4\alpha+2\beta)} \tag{19}$$

From (13) and (14), we have

$$\alpha'' - \beta'' + \delta e^{(2\alpha+2\beta)} - e^{4\beta} = \lambda e^{(4\alpha+2\beta)} \tag{20}$$

From (19) and (20), we get

$$2\beta'' + e^{4\beta} = 0 \tag{21}$$

which on integration, gives

$$S^2 = e^{2\beta} = 2m_1 \sec h 2m_1(T + n_1) \tag{22}$$

Now, from (18) and (19), we have

$$\alpha'' + \beta'' + \delta(1 - c_1)e^{(2\alpha+2\beta)} = 0 \tag{23}$$

Case (1): $(1 - c_1) \neq 0$, then let us take $(1 - c_1) = c_2^2$ (where c_2 is a constant), therefore the (23) can be written as

$$\alpha'' + \beta'' + \delta c_2^2 e^{(2\alpha+2\beta)} = 0 \tag{24}$$

Now the above equation can be written as

$$[\log(RS)]'' + \delta c_2^2 (RS)^2 = 0 \tag{25}$$

Which on integration, yields

$$RS = e^{m_2 T + n_2}, \quad \text{if } \delta = 0 \tag{26}$$

$$RS = \frac{m_2}{c_2}\cos ech[-m_2(T + n_2)], \quad \text{if } \delta = -1 \tag{27}$$

$$RS = \frac{m_2}{c_2}\sec h[m_2(T + n_2)], \quad \text{if } \delta = +1 \tag{28}$$

where m_2 and n_2 are constants of integration.

From (22), (26), (27) and (28) we get

$$R^2 = \frac{e^{2(m_2T+n_2)}}{2m_1}\cosh[2m_1(T + n_1)], \quad \text{if } \delta = 0$$

$$R^2 = e^{2\alpha} = \frac{m_2^2}{2c_2^2 m_1}\cos ech^2[m_2(T + n_2)] \times \cosh[2m_1(T + n_1)], \quad \text{if } \delta = -1$$

$$R^2 = e^{2\alpha} = \frac{m_2^2}{2c_2^2 m_1}\sec h^2[m_2(T + n_2)] \times \cosh[2m_1(T + n_1)], \quad \text{if } \delta = +1$$

The exact solution for the Bianchi-type II metric is given by

$$S^2 = e^{2\beta} = 2m_1 \sec h 2m_1(T + n_1) \tag{29}$$

$$R^2 = \frac{e^{2(m_2T+n_2)}}{2m_1}\cosh[2m_1(T + n_1)] \tag{30}$$

$$\phi = \left[(cT + d)\left(\frac{n}{2} + 1\right)\right]^{\frac{2}{n+2}} \tag{31}$$

$$\rho = \lambda = c_1\left[\frac{e^{2(m_2T+n_2)}}{2m_1}\cosh[2m_1(T + n_1)]\right]^{-1} \tag{32}$$

where m_1, m_2, n_1 and n_2 are constants of integration.

The corresponding model can be written as

$$ds^2 = -\frac{e^{4x_2}}{2m_1}\cosh x_1 dT^2 + \frac{e^{2x_2}}{2m_1}\cosh x_1[d\theta^2 + d\phi^2] + 2m_1 \sec h x_1[d\varphi + y d\phi]^2 \tag{33a}$$

where $x_1 = 2m_1(T + n_1)$, $x_2 = m_2(T + n_2)$.

The exact solution for the Bianchi-type VIII metric is given by

$$S^2 = e^{2\beta} = 2m_1 \sec h 2m_1(T + n_1) \tag{34}$$

$$R^2 = e^{2\alpha} = \frac{m_2^2}{2c_2^2 m_1}\cos ech^2[m_2(T + n_2)]\cosh[2m_1(T + n_1)] \tag{35}$$

$$\phi = \left[(cT+d)\left(\frac{n}{2}+1\right)\right]^{\frac{2}{n+2}} \tag{36}$$

$$\begin{aligned}\rho &= \lambda \\ &= c_1\left[\frac{m_2^2}{2c_2^2 m_1}\cos ech^2[m_2(T+n_2)]\right. \\ &\quad \left.\times \cosh[2m_1(T+n_1)]\right]^{-1}\end{aligned} \tag{37}$$

where m_1, m_2, n_1 and n_2 are constants of integration.

The corresponding model can be written as

$$\begin{aligned}ds^2 &= -\frac{m_2^4}{2c_2^4 m_1}\cos ech^4 x_2 \cosh x_1 dT^2 \\ &\quad + \frac{m_2^2}{2c_2^2 m_1}\cos ech^2 x_2 \cosh x_1[d\theta^2+\cosh^2 y d\phi^2] \\ &\quad + 2m_1 \sec h x_1[d\varphi + \sinh^2 y d\phi]^2\end{aligned} \tag{33b}$$

where $x_1 = 2m_1(T+n_1), x_2 = m_2(T+n_2)$.

The exact solution for the Bianchi-type IX metric is given by

$$s^2 = e^{2\beta} = 2m_1 \sec h 2m_1(T+n_1) \tag{38}$$

$$R^2 = e^{2\alpha} = \frac{m_2^2}{2c_2^2 m_1}\sec h^2[m_2(T+n_2)]\cosh[2m_1(T+n_1)] \tag{39}$$

$$\phi = \left[(cT+d)\left(\frac{n}{2}+1\right)\right]^{\frac{2}{n+2}} \tag{40}$$

$$\begin{aligned}\rho &= \lambda \\ &= c_1\left[\frac{m_2^2}{2c_2^2 m_1}\sec h^2[m_2(T+n_2)]\cosh[2m_1(T+n_1)]\right]^{-1}\end{aligned} \tag{41}$$

where m_1, m_2, n_1 and n_2 are constants of integration.

The corresponding model can be written as

$$\begin{aligned}ds^2 &= -\frac{m_2^4}{2c_2^4 m_1}\sec h^4 x_2 \cosh x_1 dT^2 \\ &\quad + \frac{m_2^2}{2c_2^2 m_1}\sec h^2 x_2 \cosh x_1[d\theta^2+\sin^2 y d\phi^2] \\ &\quad + 2m_1 \sec h x_1[d\varphi + \cos^2 y d\phi]^2\end{aligned} \tag{33c}$$

where $x_1 = 2m_1(T+n_1), x_2 = m_2(T+n_2)$.

Case (2): If $(1-c_1)=0$, we get the same set of field equations as in the case of $\delta = 0$.

So, in this case we get only Bianchi type II model (33a) along with the scalar field given by the (31).

3 Physical and geometrical properties

The models represented by the (33a), (33b) and (33c) represents spatially homogeneous Bianchi type-II, VIII and IX cosmological models in the scalar-tensor theory of gravitation proposed by Saez and Ballester (1986). The scalar field ϕ is given by the (31), (36) and (40). The energy density and the tension density of the models are given by (32), (37) and (41) respectively.

The volume element V, expansion θ and shear σ for the models (33a), (33b) and (33c) are given by

$$V = (-g)^{\frac{1}{2}} = \frac{e^{4(m_2T+n_2)}}{2m_1}\cosh 2m_1(T+n_1)$$

$$\theta = \frac{1}{3}[4m_2 + 2m_1 \tanh 2m_1(T+n_1)]$$

$$\sigma = -\frac{4}{3}m_1^2 sceh^2 2m_1(T+n_1)$$

for the Bianchi-type II universe,

$$V = (-g)^{\frac{1}{2}} = \frac{m_2^4}{2c_2^4 m_1}\cos ech^4 x_2 \cosh x_1 \cosh y$$

$$\theta = \left[m_1 \tanh 2m_1(T+n_1) - \frac{m_2}{c_2}\coth m_2(T+n_2)\right]$$

$$\begin{aligned}\sigma &= -\frac{10}{3}\left[m_1^2 sceh^2 2m_1(T+n_1)\right. \\ &\quad \left.+ \frac{m_2^2}{c_2^2}\cos ech^2 m_2(T+n_2)\right]\end{aligned}$$

for the Bianchi-type VIII universe and

$$V = (-g)^{\frac{1}{2}} = \frac{m_2^4}{2c_2^4 m_1}\sec h^4 x_2 \cosh x_1 \sin y$$

$$\theta = \left[m_1 \tanh 2m_1(T+n_1) + \frac{m_2}{c_2}\tanh m_2(T+n_2)\right]$$

$$\sigma = -\frac{10}{3}\left[m_1^2 sceh^2 2m_1(T+n_1) + \frac{m_2^2}{c_2^2}\sec h^2 m_2(T+n_2)\right]$$

for the Bianchi-type IX universe.

4 Conclusions

It is well known that Bianchi type exact models form a unique class in the set of all solutions to the Einstein's equations. Spatially homogeneous and anisotropic Bianchi type II, VIII and IX solutions in the general theory of relativity have been first given by Ruban (1976; 1978) and Maarteus and Nel (1978). Bianchi type II, VIII and IX string cosmological models in general theory of relativity have been obtained by Krori et al. (1990). Here we have presented the exact string cosmological models for the Bianchi type II, VIII and IX space times in Saez-Ballester scalar tensor theory of gravitation. Our solutions represent a generalization of the general relativistic Bianchi type II, VIII and IX string cosmological models with the dimensionless scalar field introduced by the Saez-Ballester. If ϕ is constant (i.e., when the integration constant $c = 0$ in (31)) the cosmological models (33a) with (32), (33b) with (37) and (33c) with (41) respectively represent exact Bianchi type II, VIII and IX string cosmological models in Einstein's general theory of relativity, which are quite different from the models obtained by Krori et al. (1990). It is observed that the cosmological models presented in this paper exhibit anisotropic expansion with time and have no singularities at $T = 0$.

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