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INVITED REVIEW

The effect of solar radiation pressure on the Lagrangian points in the elliptic restricted three-body problem

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Abstract In this paper the effect of solar radiation pressure on the location and stability of the five Lagrangian points is studied, within the frame of elliptic restricted three-body problem, where the primaries are the Sun and Jupiter acting on a particle of negligible mass. We found that the radiation pressure plays the rule of slightly reducing the effective mass of the Sun and changes the location of the Lagrangian points. New formulas for the location of the collinear libration points were derived. For large values of the force ratio β, we found that at $\beta = 0.12$, the collinear point L_3 is stable and some families of periodic orbits can be drawn around it.

Keywords Elliptic restricted three-body problem · Radiation pressure · Libration points · Stability

1 Introduction

In spite of the large amount of analytical and numerical work in the circular restricted three-body problem (CRTBP), there are relatively few analytical results on the subject of elliptic restricted three-body problem (ERTBP) (Cors et al. 2001) and show the existence of a new class of periodic orbits in the three dimensional (ERTBP) in the case of equal masses of the primaries. Arenstorf (1966) showed the existence of second kind of periodic orbits (i.e. some which come from analytic continuation of unperturbed elliptic Keplerian orbits, with arbitrary eccentricity) in the planar (RTBP) in the neighbourhood of one of the primaries, irrespective of the mass ratio. Jefferys (1965) showed that there exist doubly symmetric, almost circular periodic solutions if one of the primaries is sufficiently small. He showed as well (Jefferys 1966) the existence of families of elliptic orbits for any value of the eccentricity and critical inclination. The elliptic restricted three-body problem (ERTBP) has not been fully explored (planar or spatial), although a number of papers have been devoted to it: Katsiaris (1972) computed some periodic orbits when the eccentricity, e, of the primaries lies in the range [0, 0.4]; Olle and Pacha (1991) computed families of periodic solutions, very far from the primaries, which were assumed to be of equal masses and in the rectilinear collision orbit. Danby (1964) studied the (ERTBP) and used numerical integration to determine the linear stability of the elliptic Lagrange orbits. Using the traditional mass value μ and the eccentricity e as parameters, he obtained a stability diagram in the μ–e plane and noted that there are cases where the elliptic orbits appear to be linearly stable even though the circular ones are not. Roberts (2002) studied the linear stability of the periodic orbits of Lagrange in the (ERTBP), using perturbation technique, he proved that for some mass values; the elliptic orbits were linearly stable.

Şelaru and Cucu-Dumitrescu (1994) performed an analytical investigation concerning the structure of asymptotic perturbative approximations for small amplitude motions of the (infinitesimal) third point mass in the neighbourhood of a Lagrangian equilateral libration position in the planar, elliptic restricted problem of three bodies. After a sequence of canonical transformations, they formulated the Hamiltonian governing the motion of the negligible mass body, using the eccentric anomaly of the primaries' elliptic Keplerian orbit as the independent variable, they then studied the linearized system of differential equations of motion obtained from expanding the Hamiltonian around a Lagrangian solution. Also they developed their theory and calculations of an

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asymptotic solution up to the first order in the orbital eccentricity of the primaries taken as the perturbation parameter: terms up to the first power. At later stage, Şelaru and Cucu-Dumitrescu (1995) presented an extension of these considerations to a similar second order theory. Floria (2004) undertakes an approximate integration of the (ERTBP) by means of a perturbation technique based on Lie series developments, which leads to an approximate solution to the differential system of canonical equations of motion derived from the chosen Hamiltonian function (expanded in powers of the numerical eccentricity a of the elliptic orbit of the two primaries, which plays the role of the small parameter of the perturbation).

In the present work we studied the combined effects of gravitational forces of the primaries rotating in an elliptic orbit around their center of mass and the solar radiation pressure emanating from the Sun on the particle. We find analytically the locations of the five equilibrium points in terms of the mass ratio of the primaries together with the force ratio, β of the radiation force and the gravitational force of the Sun. We also studied the linear stability of the libration points under the gravitational and radiation forces.

Finally, we applied our formulae to the Sun–Jupiter system to calculate the corresponding solutions for both the triangular and collinear points. Also we plotted all the solutions with respect to the true anomaly

Zero velocity curves around the triangular point L_4 were plotted in the case of the presence of SRP and without it.

2 Formulation of the problem

The ERTBP models the motion of a test particle having an infinitesimal mass, m, and moving under the influence of the gravitational field of two massive bodies of masses m_1 and m_2 that revolve around their center of mass in an elliptic orbit. The motion of the finite masses are non sensibly affected by the gravitational attraction due to the infinitesimal particle, and their motion can be considered as a Keplerian motion of pure unperturbed two body problem, and can be considered as completely known and given in advance. In what follows, we concentrate our attention on the motion of the infinitesimal mass, m, assuming that this motion takes place in space "spatial elliptic restricted three body problem" about the center of mass of the primaries. Consequently, the relative orbit of one of the primaries with respect to the other is also an ellipse. All these ellipses share a common numerical value of the eccentricity, e.

Introducing the elliptic orbit for the primaries generalizes the original (CRTBP), and improves its applicability, while some outstanding and useful properties of the circular model still hold true or can be adapted to the elliptic case.

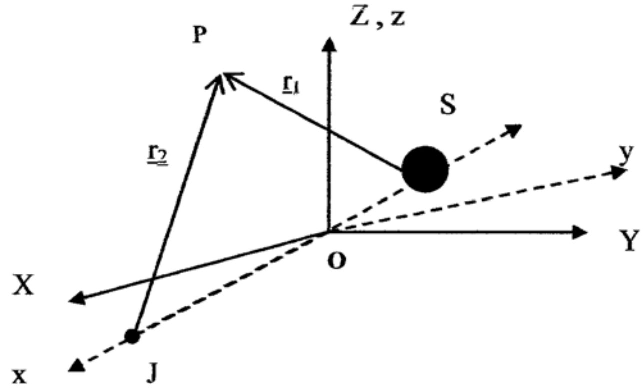

Fig. 1 The location of the Sun (S), Jupiter (J) and the Particle (P) in the rotating frame

According to the usual practice and without loss of generality we chose a system of units as: the gravitational constant and the sum of the finite masses equals to the unity, i.e. $m_1 + m_2 = 1$, and we choose

$$\frac{m_2}{m_1+m_2} = \mu, \qquad \frac{m_1}{m_1+m_2} = 1-\mu, \quad 0 < \mu < \frac{1}{2}. \tag{1}$$

If in addition to this, the value of the orbital angular momentum of the relative motion of the primaries is unity, then the semi-latus rectum of the elliptic relative orbit will be equals one, and the polar equation of this ellipse will take the form

$$r = \frac{1}{1 + e\cos f}, \tag{2}$$

where f is the true anomaly of the smaller primary m_2 and e is the eccentricity of the elliptic orbit of either primaries.

To describe the motion of a particle of negligible mass P under the action of the gravitational field of a two primaries, Sun (S) and Jupiter (J), taking into account the effect of solar radiation pressure (SRP), we shall use a set of rotating axes $Oxyz$ centered at the center of mass of the two primaries, which are rotating around their center of mass in elliptic orbits with variable angular velocity $\underline{\dot{f}} = (0, 0, \dot{f})$, with respect to an inertial frame XYZ, and coincides with it at $t = 0$.

Assume that the primaries are located on the rotating x-axis at the points $S(x_1, 0, 0)$ and $J(x_2, 0, 0)$ as in Fig. 1.

Apart from the gravitational acceleration of the Sun on the particle, $F_{gr} = m_1/r_1^2$, the solar radiation pressure exerts a small acceleration on the particle, m, in the opposite direction to the gravitational acceleration which may be written as (Montenbruck and Gill 2005)

$$F_{rad} = \frac{1}{4\pi}\frac{(A/m)}{r_1^2}\frac{L_0 Q_{pr}}{c}, \tag{3}$$

where A is the geometric cross–section of the particle, c is the speed of light, Q_{pr} is the radiation pressure coefficient, depending on the properties of the particle (density, shape, size, etc.), L_0 is the luminosity of the Sun.

Define the ratio of the two forces by a dimensionless quantity β as:

$$\beta = \frac{F_{\text{rad}}}{F_{\text{gr}}} = \frac{(A/m) Q_{\text{pr}} L_0}{4\pi c m_1}. \tag{4}$$

Note that the ratio β is independent of the distance r_1 between the Sun and the particle. The total action from the Sun on the particle can be expressed by the acceleration:

$$F_{\text{total}} = F_{\text{gr}} - F_{\text{rad}} = (1-\beta) F_{\text{gr}} = -\frac{(1-\beta) m_1}{r_1^2}.$$

3 Equations of motion

The equations of motion of the small particle $P(x, y, z)$ under the action of the gravitational forces of the primaries and the solar radiation pressure taken into account can be formulated in a Cartesian barycentric system of coordinates *Oxyz*, in which the two perpendicular axes *Ox* and *Oy* rotate with a variable angular velocity $\dot{f}$, in the orbital plane around the Z-axes with respect to a fixed reference frame *OXYZ*, with origin placed at the center of mass of the two primaries, where f is the true anomaly of the finite mass m_2 in its elliptic motion, and may be written as:

$$\begin{aligned}
&\ddot{x} - x\dot{f}^2 - 2\dot{y}\dot{f} - y\ddot{f} = -(1-\beta)\frac{(1-\mu)(x-x_1)}{r_1^3} - \frac{\mu(x-x_2)}{r_2^3},\\
&\ddot{y} - y\dot{f}^2 + 2\dot{x}\dot{f} + x\ddot{f} = -(1-\beta)\frac{(1-\mu)y}{r_1^3} - \frac{\mu y}{r_2^3},\\
&\ddot{z} = -(1-\beta)\frac{(1-\mu)z}{r_1^3} - \frac{\mu z}{r_2^3},
\end{aligned} \tag{5}$$

where

$$r_i^2 = (x - x_i)^2 + y^2 + z^2, \quad i = 1, 2 \tag{6}$$

and

$$x_1 = -\mu r, \qquad x_2 = (1-\mu) r.$$

Where the dots indicate differentiation with respect to time.

In the system of equations (5) the separation between the primaries is variable, and given by (2). In order to maintain the primaries in fixed positions, we transform now to a rotating pulsating coordinate system, in which, the unit of length may be chosen as the instantaneous distance of the primaries, namely r defined by (2), the independent variable can be chosen so as to be the true anomaly f of the Keplerian motion described by the smaller primary. So in the system of equations (5) we put

$$\begin{aligned}
&x = r\tilde{x}, \qquad y = r\tilde{y}, \qquad z = r\tilde{z},\\
&r_1 = r\tilde{r}_1 \quad \text{and} \quad r_2 = r\tilde{r}_2.
\end{aligned} \tag{7}$$

Accordingly, the separation between the primaries will be constant and equal to one, and the position of m_1 and m_2 will be determined by the coordinate

$$(\tilde{x}_1, \tilde{y}_1, \tilde{z}_1) = (-\mu, 0, 0), \qquad (\tilde{x}_2, \tilde{y}_2, \tilde{z}_2) = (1-\mu, 0, 0).$$

The equations of motion of the particle $P(\tilde{x}, \tilde{y}, \tilde{z})$ in this pulsating system, when we take the true anomaly f, as the new independent variable rather than the time, t, using the relation $df/dt = 1/r^2$ can be written in the form:

$$\begin{aligned}
&\tilde{x}'' - 2\tilde{y}' = \frac{1}{(1+e\cos f)}\left[\tilde{x} - \frac{(1-\beta)(1-\mu)}{\tilde{r}_1^3}(\tilde{x}+\mu) - \frac{\mu}{\tilde{r}_2^3}(\tilde{x}+\mu-1)\right],\\
&\tilde{y}'' + 2\tilde{x}' = \frac{1}{(1+e\cos f)}\left[1 - \frac{(1-\beta)(1-\mu)}{\tilde{r}_1^3} - \frac{\mu}{\tilde{r}_2^3}\right]\tilde{y},\\
&\tilde{z}'' + \tilde{z} = \frac{1}{(1+e\cos f)}\left[1 - \frac{(1-\beta)(1-\mu)}{\tilde{r}_1^3} - \frac{\mu}{\tilde{r}_2^3}\right]\tilde{z}.
\end{aligned} \tag{8}$$

These equations can be reformulated as:

$$\begin{aligned}
&\tilde{x}'' - 2\tilde{y}' = G(e, f)\frac{\partial U}{\partial \tilde{x}},\\
&\tilde{y}'' + 2\tilde{x}' = G(e, f)\frac{\partial U}{\partial \tilde{y}},\\
&\tilde{z}'' + \tilde{z} = G(e, f)\frac{\partial U}{\partial \tilde{z}},
\end{aligned} \tag{9}$$

where

$$U = (1-\mu)\left[\frac{1}{2}\tilde{r}_1^2 + \frac{(1-\beta)}{r_1}\right] + \mu\left[\frac{1}{2}\tilde{r}_2^2 + \frac{1}{\tilde{r}_2}\right] - \frac{1}{2}\frac{\tilde{z}^2}{r}, \tag{10}$$

$$\begin{aligned}
&\tilde{r}_1^2 = (\tilde{x}+\mu)^2 + \tilde{y}^2 + \tilde{z}^2,\\
&\tilde{r}_2^2 = (\tilde{x}+\mu-1)^2 + \tilde{y}^2 + \tilde{z}^2,
\end{aligned} \tag{11}$$

$$G(e, f) = (1 + e\cos f)^{-1}. \tag{12}$$

It may be noted at this stage that U itself is not dependent explicitly on the true anomaly f which enters only through

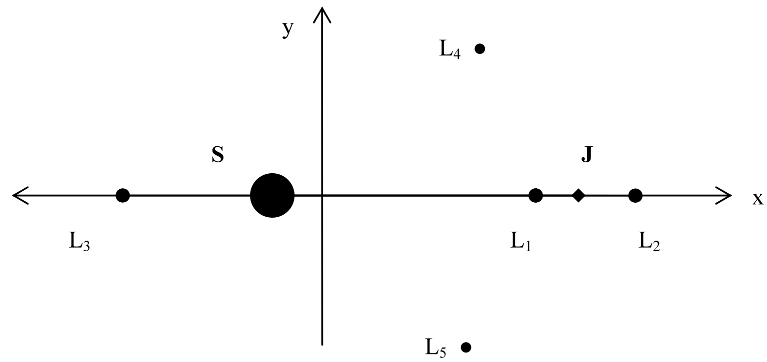

Fig. 2 The location of the five equilibrium points ($L_i, i = 1, 2, \ldots, 5$)

the common term $1/(1+e\cos f)$ outside the partial derivatives on the right hand side of (8), (9) nor it is dependent explicitly on the time. Where the differentiation with respect to f in (8), (9) was denoted by a dash.

4 Lagrangian equilibrium points

The critical values of the potential function

$$\Omega(\tilde{x}, \tilde{y}, \tilde{z}, f) = U(1+e\cos f)^{-1}, \tag{13}$$

where U is defined by (10) are called the Lagrangian (libration) points. They determine the equilibrium points of the differential equations (8) in the rotating pulsating coordinates. It is well known that there exist five relative equilibrium solutions (libration points), where in our case the gravitational forces, the force due to the solar radiation pressure and the centrifugal forces balance each other; three of them are collinear, denoted by (L_1, L_2 and L_3) and the other two are triangular, denoted by (L_4 and L_5) shown in Fig. 2.

The Lagrangian points that appear in the restricted problem of the three bodies are very important for astronautic applications, they are very good points to locate a space station, since they require a small amount of fuel for station keeping. In order to search for the location of the equilibrium points, i.e. the points where the particle has zero velocity and zero acceleration in the rotating pulsating frame, we write the equation of motion again for convenience.

$$\begin{aligned} \tilde{x}'' - 2\tilde{y}' &= \frac{\partial \Omega}{\partial \tilde{x}}, \\ \tilde{y}'' + 2\tilde{x}' &= \frac{\partial \Omega}{\partial \tilde{y}}, \\ \tilde{z}'' + \tilde{z} &= \frac{\partial \Omega}{\partial \tilde{z}}, \end{aligned} \tag{14}$$

where on using (8) we obtain

$$\Omega_{\tilde{x}} = \frac{1}{(1+e\cos f)}\left[(1-A)\,\tilde{x} - \mu\,(1-\mu)\left(\frac{1-\beta}{\tilde{r}_1^3} - \frac{1}{\tilde{r}_2^3}\right)\right], \tag{15}$$

$$\Omega_{\tilde{y}} = \frac{1}{(1+e\cos f)}\,(1-A)\,\tilde{y}, \tag{16}$$

$$\Omega_{\tilde{z}} = \frac{1}{(1+e\cos f)}(1-A)\tilde{z}, \tag{17}$$

where

$$A = \frac{(1-\beta)\,(1-\mu)}{\tilde{r}_1^3} + \frac{\mu}{\tilde{r}_2^3}. \tag{18}$$

To obtain the location of the equilibrium points, we put $\tilde{x}'' = \tilde{x}' = \tilde{y}'' = \tilde{y}' = \tilde{z}'' = 0$, into (14). From the third equation of the system (14), the condition $\Omega_{\tilde{z}} = 0$ implies that $\tilde{z} = 0$. That is all the critical points are planar, and no equilibrium points can be found outside the $\tilde{x}\tilde{y}$-plane (Szebehely 1967).

4.1 The triangular equilibrium points

In the $\tilde{x}\tilde{y}$ plane, let the coordinate of the equilibrium points be $(\tilde{x}_0, \tilde{y}_0)$. In the case when $\tilde{y} \neq 0$, the equilibrium points can be obtained from the conditions:

$$\partial U/\partial \tilde{r}_1 = \partial U/\tilde{r}_2 = 0. \tag{19}$$

So, when using (10) the conditions $\partial U/\partial \tilde{r}_1 = \partial U/\tilde{r}_2 = 0$ give:

$$\begin{aligned} \frac{\partial U}{\partial \tilde{r}_1} &= (1-\mu)\left(\tilde{r}_1 - \frac{(1-\beta)}{\tilde{r}_1^2}\right) = 0, \\ \frac{\partial U}{\partial \tilde{r}_2} &= \mu\left(\tilde{r}_2 - \frac{1}{\tilde{r}_2^2}\right) = 0 \end{aligned} \tag{20}$$

from which, we get

$$\tilde{r}_1 = (1-\beta)^{1/3}, \qquad \tilde{r}_2 = 1. \tag{21}$$

Substituting (21) into (11) we obtain the coordinate of the triangular points (L_4 and L_5) in the form

$$(\tilde{x}_0, \tilde{y}_0) = \left(-\mu + \frac{1}{2}(1-\beta)^{2/3}, \pm (1-\beta)^{1/3}\sqrt{1 - \frac{1}{4}(1-\beta)^{2/3}}\right), \tag{22}$$

where by convention the leading triangular points is taken to be L_4 and the point L_5.

4.2 The collinear equilibrium points

Consider now the case when $\tilde{y}=0$, i.e. when the equilibrium points lie along the $\tilde{x}$-axis. These points are found from the condition $\partial U/\partial\tilde{x}=0$. Note that

$$\frac{\partial U}{\partial \tilde{x}}=\frac{\partial U}{\partial \tilde{r}_1}\frac{\partial \tilde{r}_1}{\partial \tilde{x}}+\frac{\partial U}{\partial \tilde{r}_2}\frac{\partial \tilde{r}_2}{\partial \tilde{x}}=0.$$

Using (10) and (11) we obtain the condition in the form

$$(1-\mu)\left[\tilde{r}_1-\frac{(1-\beta)}{\tilde{r}_1^2}\right]\frac{\tilde{x}+\mu}{\tilde{r}_1}+\mu\left[\tilde{r}_2-\frac{1}{\tilde{r}_2^2}\right]\frac{\tilde{x}+\mu-1}{\tilde{r}_2}=0. \tag{23}$$

The collinear equilibrium points will be denoted by L_1, L_2 and L_3, and are shown in Fig. 2. They will be defined as:

(i) L_1 lies between S and J: $-\mu<\tilde{x}<1-\mu$,
(ii) L_2 is to the right of J: $\tilde{x}>1-\mu$,
(iii) L_3 is to the left of S: $\tilde{x}<-\mu$.

4.2.1 *Location of* L_1 $(-\mu<\tilde{x}<1-\mu)$

At the point L_1, we have $\tilde{r}_1+\tilde{r}_2=1, \tilde{r}_1=\tilde{x}+\mu, \tilde{r}_2=1-\mu-\tilde{x}$.

So let , $\tilde{r}_2=\rho, \tilde{r}_1=1-\rho$, then substitution into (23) and after rearrangement we obtain:

$$\frac{\mu}{3(1-\mu)}=\frac{\rho^3(1-\rho+\frac{\rho^2}{3}-\frac{\beta}{3\rho})}{(1-\rho)^2(1-\rho^3)}. \tag{24}$$

Because of the occurrence of the term, $\beta/3\rho$, in the numerator of (24) we replace ρ by $[1-(1-\rho)]$, and expand $(1/\rho)$ up to $O[\rho^4]$. So, we can write (24) up t $O[\rho^4]$ as:

$$\begin{aligned}\alpha=\rho&+\frac{1}{3}\left(\frac{1-2\beta/3}{1-4\beta/3}\right)\rho^2\\&+\frac{1}{3}\left(\frac{1-8\beta/3+44\beta^2/27}{(1-4\beta/3)^2}\right)\rho^3\\&+\frac{1431-5697\beta+7596\beta^2}{2187(1-4\beta/3)^3}\rho^4+O[\rho^5].\end{aligned} \tag{25}$$

Where α is defined as:

$$\alpha=\left(\frac{\mu}{3(1-4\beta/3)(1-\mu)}\right)^{1/3}. \tag{26}$$

This equation can be solved using Lagrange inversion formula (Murray 1999) to obtain ρ in terms of α. Thus up to $O[\alpha^4]$, we have

$$\begin{aligned}\rho=\alpha&-\frac{1}{3}\left(\frac{1-2\beta/3}{1-4\beta/3}\right)\alpha^2\\&-\frac{1}{9}\left(\frac{1-16\beta/3+4\beta^2}{(1-4\beta/3)^2}\right)\alpha^3\\&-\frac{1}{3}\left(\frac{23-91\beta+148\beta^2}{(1-4\beta/3)^3}\right)\alpha^4+O[\alpha^5].\end{aligned} \tag{27}$$

4.2.2 *Location of* L_2 $(\tilde{x}>1-\mu)$

At the point L_2, we have $\tilde{r}_1-\tilde{r}_2=1, \tilde{r}_1=\tilde{x}+\mu, \tilde{r}_2=\tilde{x}+\mu-1$.

So let , $\tilde{r}_2=\rho, \tilde{r}_1=1+\rho$, then substitution into (23) gives:

$$\frac{\mu}{3(1-\mu)}=\frac{\rho^3(1+\rho+\frac{\rho^2}{3}+\frac{\beta}{3\rho})}{(1+\rho)^2(1-\rho^3)}. \tag{28}$$

After expansion up to $O[\rho^4]$, we get

$$\begin{aligned}\gamma=\rho&-\frac{1}{3}\left(\frac{1+14\beta/3}{1+4\beta/3}\right)\rho^2\\&+\frac{1}{3}\left(\frac{1+8\beta/3+140\beta^2/27}{(1+4\beta/3)^2}\right)\rho^3\\&+\frac{1+157\beta+(412/3)\beta^2}{81(1+4\beta/3)^3}\rho^4+O[\rho^5].\end{aligned} \tag{29}$$

Where

$$\gamma=\left(\frac{\mu}{3(1+4\beta/3)(1-\mu)}\right)^{1/3}. \tag{30}$$

Inverting (29) to obtain ρ in terms of γ, we obtain:

$$\begin{aligned}\rho=\gamma&+\frac{1}{3}\frac{1+14\beta/3}{1+4\beta/3}\gamma^2-\frac{1}{9}\frac{1-32\beta/3-28\beta^2}{(1+4\beta/3)^2}\gamma^3\\&-\frac{1}{81}\frac{31+227\beta-158\beta^2/3}{(1+4\beta/3)^3}\gamma^4+O[\gamma^5].\end{aligned} \tag{31}$$

4.2.3 *Location of* L_3 $(\tilde{x}<-\mu)$

At the point L_3, we have $\tilde{r}_2-\tilde{r}_1=1, \tilde{r}_1=-\tilde{x}-\mu, \tilde{r}_2=-\tilde{x}-\mu+1$. So let , $\tilde{r}_1=\rho, \tilde{r}_2=1+\rho$, then substitution into (23) gives:

$$\frac{\mu}{1-\mu}=\frac{(1-\beta-\rho^3)(1+\rho)^2}{\rho^3(3+3\rho+\rho^2)}. \tag{32}$$

Because of the occurrence of the term, ρ^3, in the denominator of (32) we replace ρ by $(1+u)$, and expand $1/\rho$ up

Table 1 Location of collinear points for different values of β

Equilibrium point	$\beta=0.0$	$\beta=0.03$	$\beta=0.05$	$\beta=0.07$	$\beta=0.1$
L_1	(0.93245, 0)	(0.93248, 0)	(0.93251, 0)	(0.93253, 0)	(0.93257, 0)
L_2	(1.06883, 0)	(1.06816, 0)	(1.06772, 0)	(1.06728, 0)	(1.06663, 0)
L_3	(−0.99932, 0)	(−0.98933, 0)	(−0.98267, 0)	(−0.97601, 0)	(−0.96602, 0)

to $O[\rho^3]$. Thus we can write (32) up to $O[u^3]$ as:

$$\frac{\mu}{1-\mu}=-\frac{4\beta}{7}+\left(1-\frac{19}{21}\beta\right)\left(-\frac{12}{7}u\right)+\left(1-\frac{989}{1008}\beta\right)\left(-\frac{12}{7}u\right)^2+\left(\frac{1567}{1728}-\frac{5465}{6048}\beta\right)\left(-\frac{12}{7}u\right)^3+O[u^4]. \quad (33)$$

Using the method of successive approximations up to $O[(\mu/1-\mu)^3]$ and retaining only the linear terms of β, we obtain

$$\rho=1+u=1-\frac{\beta}{3}-\frac{12}{7}\left(1-\frac{5}{21}\beta\right)\left(\frac{\mu}{1-\mu}\right)+\frac{12}{7}\left(1-\frac{1054}{1008}\beta\right)\left(\frac{\mu}{1-\mu}\right)^2-\left(\frac{13223}{20736}+\frac{27820}{62208}\beta\right)\left(\frac{\mu}{1-\mu}\right)^3+O\left[\frac{\mu}{1-\mu}\right]^4. \quad (34)$$

In the case where there is no radiation pressure, i.e. if $\beta=0$, we obtain from (22), (27), (31), and (34) the same results as in the classical (RTBP) (Floria 2004) for both the triangular and collinear points.

For numerical calculations, we consider our model of the (ERTBP) to be composed of the Sun and Jupiter as primaries and the particle is a space craft. Due to our previous choice of units, we define the following quantities:

* the mass parameter $\mu=\frac{m_2}{m_1+m_2}=9.5359\times10^{-4}$,
* the eccentricity of the elliptic orbit of the primaries, $e=4.59\times10^{-2}$.

We can obtain the location of the equilibrium (Lagrangian) points for different values of the force ratio $\beta\in[0,0.1]$ from Tables 1 and 2.

5 Stability of the equilibrium points

For the stability analysis let $\tilde{x}=\tilde{x}_{0i}$ and $\tilde{y}=\tilde{y}_{0i}$ be the coordinates of the Lagrangian points L_i; $i=1,2,3,4,5$. Suppose that the particle receive a small displacement from the equilibrium position $(\tilde{x}_0,\tilde{y}_0,0)$ to the position $(\tilde{x}_0+u,\tilde{y}_0+v,w)$ where (u,v,w) are small quantities. To obtain the variation equations, we substitute the coordinates of the displayed point into the equations of motion (9), and expanding in a Taylor's series about the equilibrium points. Considering only the linear terms of the coordinates, we obtain the linearized variation equations

Table 2 Location of the triangular points for different values of β

Equilibrium point	β	(x, y)
$L_{4,5}$	0.0	(0.499046, ±0.866025)
	0.03	(0.488996, ±0.860144)
	0.05	(0.482238, ±0.856101)
	0.07	(0.475432, ±0.851955)
	0.10	(0.465131, ±0.845538)

$$\begin{aligned} u''-2v'&=G(e,f)[(U_{\tilde{x}\tilde{x}})_0u+(U_{\tilde{x}\tilde{y}})_0v+(U_{\tilde{x}\tilde{z}})_0w],\\ v''+2u'&=G(e,f)[(U_{\tilde{y}\tilde{x}})_0u+(U_{\tilde{y}\tilde{y}})_0v+(U_{\tilde{y}\tilde{z}})_0w],\\ w''+w&=G(e,f)[(U_{\tilde{z}\tilde{x}})_0u+(U_{\tilde{z}\tilde{y}})_0v+(U_{\tilde{z}\tilde{z}})_0w], \end{aligned} \quad (35)$$

where the subscript (0) indicates that the second partial derivatives are to be evaluated at the equilibrium points $(\tilde{x}_0,\tilde{y}_0,0)$, and $U_{ij}=\partial^2U/\partial i\partial j$.

Since there is no stability outside the $\tilde{x}\tilde{y}$-plane, we deal with the last equation in the system (35) separately.

5.1 Stability of the equilibrium points in the $\tilde{x}\tilde{y}$-plane

In the $\tilde{x}\tilde{y}$-plane, we have $U_{\tilde{x}\tilde{z}}=U_{\tilde{y}\tilde{z}}=0$. So, the variation equations will take the form

$$\begin{aligned} u''-2v'&=G(e,f)[(U_{\tilde{x}\tilde{x}})_0u+(U_{\tilde{x}\tilde{y}})_0v],\\ v''+2u'&=G(e,f)[(U_{\tilde{y}\tilde{x}})_0u+(U_{\tilde{y}\tilde{y}})_0v]. \end{aligned} \quad (36)$$

Calculating the required partial derivatives included in the system (36), we get

$$U_{\tilde{x}\tilde{x}}=1-(1-\beta)(1-\mu)\left[\frac{\rho_1^2-3(\tilde{x}+\mu)^2}{\rho_1^5}\right]-\mu\left[\frac{\rho_2^2-3(\tilde{x}+\mu-1)^2}{\rho_2^5}\right], \quad (37)$$

$$U_{\tilde{y}\tilde{y}} = 1 - (1-\beta)(1-\mu)\left[\frac{\rho_1^2 - 3\tilde{y}^2}{\rho_1^5}\right] - \mu\left[\frac{\rho_2^2 - 3\tilde{y}^2}{\rho_2^5}\right], \tag{38}$$

$$U_{\tilde{x}\tilde{y}} = (1-\beta)(1-\mu)\left[\frac{3(\tilde{x}+\mu)\tilde{y}}{\rho_1^5}\right] + \mu\left[\frac{3(\tilde{x}+\mu-1)\tilde{y}}{\rho_2^5}\right]. \tag{39}$$

5.2 Stability of the triangular points

Substituting the coordinates of the equilibrium points L_4 and L_5 from (22) into (37), (38) and (39), we obtain

$$\begin{aligned} (U_{\tilde{x}\tilde{x}})_0 &= \frac{3}{4}[Q + \mu_2(4-Q)(1-Q)], \\ (U_{\tilde{y}\tilde{y}})_0 &= \frac{3}{4}[4 - Q - \mu_2(4-Q)(1-Q)], \\ (U_{\tilde{x}\tilde{y}})_0 &= \pm\frac{3}{4}\sqrt{Q}\sqrt{4-Q}(1+\mu_2 Q - 3\mu_2), \end{aligned} \tag{40}$$

where the (+ve) sign in the third equation of (40) is for the point L_4, and the (−ve) sign is for the point L_5, and

$$Q = (1-\beta)^{2/3}. \tag{41}$$

Assuming the solution of the system (36) to be of the form

$$u = A\exp(\lambda f), \qquad v = B\exp(\lambda f). \tag{42}$$

Substituting into (36), we obtain

$$\begin{aligned} (\lambda^2 - C_{xx})A - (2\lambda + C_{xy})B &= 0, \\ (2\lambda - C_{xy})A + (\lambda^2 - C_{yy})B &= 0, \end{aligned} \tag{43}$$

where

$$\begin{aligned} C_{xx} &= G(e,f)(U_{\tilde{x}\tilde{x}})_0, \qquad C_{xy} = G(e,f)(U_{\tilde{x}\tilde{y}})_0, \\ C_{yy} &= G(e,f)(U_{\tilde{y}\tilde{y}})_0. \end{aligned} \tag{44}$$

The characteristic equation of the system (43) can be obtained as the bi-quadratic equation

$$\lambda^4 + (4 - C_{xx} - C_{yy})\lambda^2 + (C_{xx}C_{yy} - C_{xy}^2) = 0. \tag{45}$$

This equation may be considered as a quadratic equation in λ^2 whose roots may be obtained as

$$\lambda_{1,2}^2 = \frac{1}{2}\left[-(4 - C_{xx} - C_{yy}) \mp \sqrt{(4 - C_{xx} - C_{yy})^2 - 4(C_{xx}C_{yy} - C_{xy}^2)}\right] \tag{46}$$

Table 3 The values of μ against $\beta \in [0.0, 0.12]$ for $e = 0$ and e = 0.04839

β	$e = 0.0$	$e = 0.04839$
0.0	$\mu = 0.038520896$	$\mu = 0.0558856$
0.01	$\mu = 0.0384319$	$\mu = 0.0557539$
0.03	$\mu = 0.0382552$	$\mu = 0.0554924$
0.05	$\mu = 0.0380801$	$\mu = 0.0552333$
0.07	$\mu = 0.0379066$	$\mu = 0.0549767$
0.09	$\mu = 0.0377347$	$\mu = 0.0547224$
0.10	$\mu = 0.037649358$	$\mu = 0.0545962$
0.12	$\mu = 0.0374798$	$\mu = 0.0543454$

We have from (40), (44) the following relations:

$$\begin{aligned} C_{xx} + C_{yy} &= 3G(e,f), \\ C_{xx}C_{yy} - C_{xy}^2 &= \tfrac{9}{4}G^2(e,f)(4-Q)\mu(1-\mu). \end{aligned} \tag{47}$$

Thus using (47) and writing $G(e,f)$ in its explicit form we obtain the characteristic roots of (46) as:

$$\lambda_{1,2}^2 = \frac{1}{2}\frac{1+4e\cos f}{1+e\cos f}\left[-1 \mp \sqrt{1 - \frac{9(4-Q)\mu(1-\mu)}{(1+4e\cos f)^2}}\right]. \tag{48}$$

The stability of the triangular points requires that λ^2 roots must be negative to obtain pure imaginary roots, i.e. the condition for stability requires that

$$9(4-Q)\mu(1-\mu) \le (1+4e\cos f)^2 \tag{49}$$

which up to the linear term in the force ratio β, this condition may be rewritten in the form:

$$(27+6\beta)\mu(1-\mu) \le (1+4e\cos f)^2 \tag{50}$$

when there is no radiation pressure, i.e. when $\beta = 0$, and the primaries were moving in circular orbits, $e = 0$, (50) will reduce to the well known condition for stability of the triangular libration points L_4 and L_5 in the (CRTBP), that is

$$27\mu(1-\mu) \le 1. \tag{51}$$

For sufficiently small values of μ determined by the condition (50) all the libration points will be stable. The maximum value of μ, depends on the value of β and e, can be found from the relation

$$\mu = \frac{1}{2} - \frac{1}{2}\sqrt{1 - \frac{4+32e(1+2e)}{27+6\beta}}. \tag{52}$$

Table 3 gives the values for μ against $\beta \in [0.0, 0.12]$ in the circular case $e = 0$, and the eccentric case of Sun–Jupiter as primaries and the relation (52) is drawn in Fig 3 between

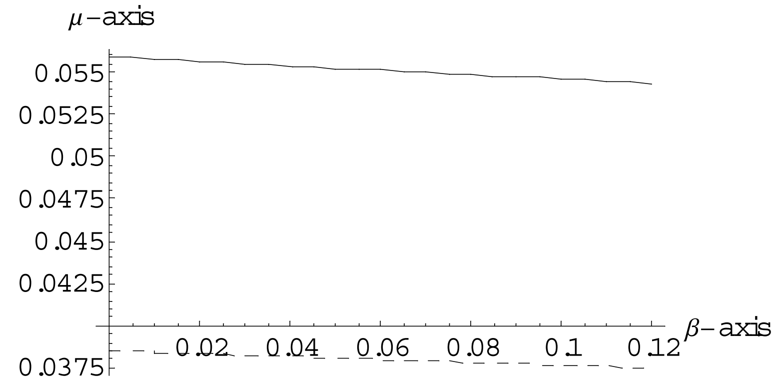

Fig. 3 The relation between μ and $\beta \in [0.0, 0.12]$ for $e = 0$ (shown *dashed*) and $e = 0.04839$ (shown *solid*)

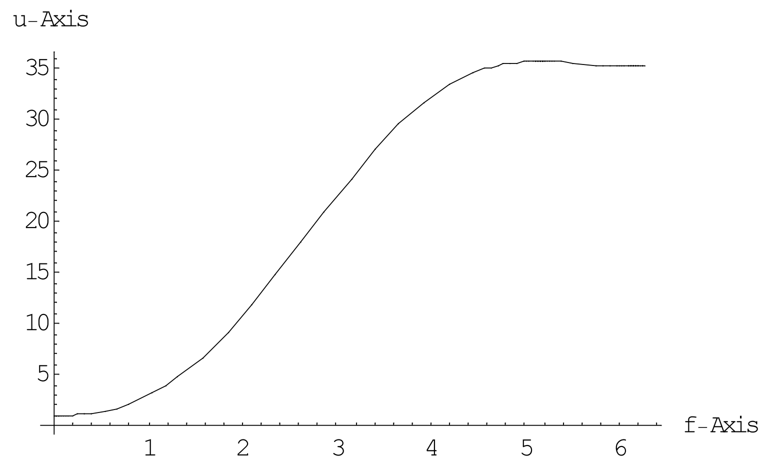

Fig. 4 The $u - f$ solution in the vicinity of the L_4 point

μ and $\beta \in [0.0, 0.12]$ for both circular (shown dashed) and elliptic orbits (shown solid).

From the Table 3, we conclude that the permissible mass parameter μ increases with the (S.R.P.), also in the eccentric case this value is larger than that in the circular case.

Consider as a specific example the perturbed motion around L_4 with initial conditions

$$u_0 = v_0 = 10^{-5} \quad \text{and} \quad u'_0 = v'_0 = 0.$$

In this case when using the values in Table 3, with $\beta = 0.1$. Equations (47) with $f = 0$ give

$$c_{11} = 0.666998, \qquad c_{22} = 2.19452, \qquad c_{12} = 1.20738.$$

Substituting into (48), we find the roots of (45) as:

$$\lambda_{1,2} = \pm i(1.06452), \qquad \lambda_{3,4} = \pm i(0.0726614) \tag{53}$$

i.e. all the roots are purely imaginary. Thus we conclude that the equilibrium motion around the L_4(or L_5) point is stable. The solutions for the perturbed motion around L_4 may be written as

$$u(f) = \sum_{j=1}^{4} A_j e^{\lambda_j f}, \qquad v(f) = \sum_{j=1}^{4} B_j e^{\lambda_j f}. \tag{54}$$

The constants A_j and B_j are not independent; they are related by (43). So we can deduce these relations as

$$B_j = \frac{\lambda_j^2 - c_{11}}{2\lambda_j + c_{12}} A_j; \quad j = 1, 2, 3, 4.$$

Applying the initial condition to the solution, we may write the solutions as

$$\begin{aligned} u(f) &= 2.66648 \times 10^{-5} \cos(0.07266 f) \\ &\quad - 1.66648 \times 10^{-5} \cos(1.0645 f) \\ &\quad + 8.30177 \times 10^{-4} \sin(0.07266 f) \\ &\quad - 5.66659 \times 10^{-5} \sin(1.0645 f), \end{aligned} \tag{55}$$

$$\begin{aligned} v(f) &= 4.02077 \times 10^{-5} \cos(0.07266 f) \\ &\quad - 3.02077 \times 10^{-5} \cos(1.0645 f) \\ &\quad + 4.5741 \times 10^{-4} \sin(0.07266 f) \\ &\quad - 3.12217 \times 10^{-5} \sin(1.0645 f). \end{aligned} \tag{56}$$

Therefore the motion of the particle at the triangular points is of an oscillatory type with periods $(2\pi/1.0645)$ and

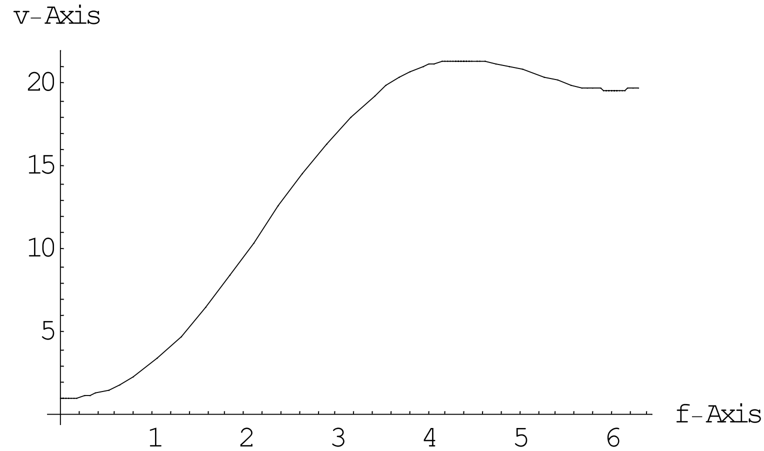

Fig. 5 The $v-f$ solution in the vicinity of the L_4 point

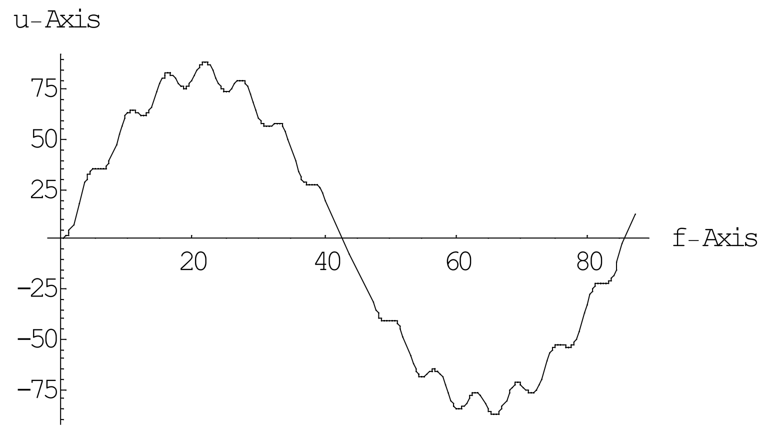

Fig. 6 The graph of the functions $u(f)$ for $f \in [0, 27.75\pi]$

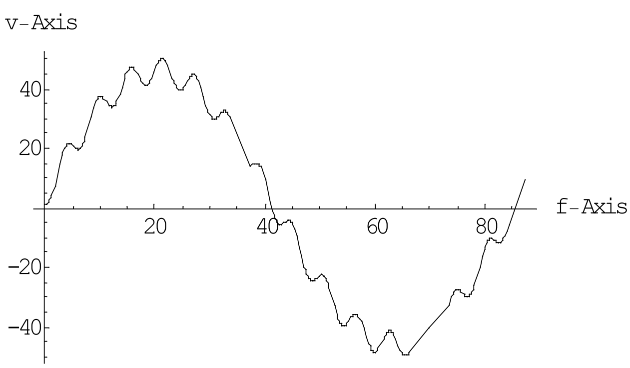

Fig. 7 The graph of the functions $v(f)$ for $f \in [0, 27.75\pi]$

$(2\pi/0.07266)$. The solutions equations (55), (56) are shown in Figs. 4, 5 respectively for $f \in [0, 2\pi]$.

The existence of these two frequencies in the solution to the perturbed motion near to the L_4 and L_5 points can be interpreted as follow:

The resulting motion of the particle is composed of two different types:

(i) A short-period motion with frequency $\approx 2\pi$ (i.e. a period that is close to the orbital period of the smaller primary μ).
(ii) A superimposed long-period with period $\approx 13.76(2\pi)$ known as a libration around the equilibrium point L_4.

The amplitudes of these motions are determined by the constant A_j and B_j.

The graph of the solutions $u(f)$ and $v(f)$ are shown in Figs. 6, 7 for $f \in [0, 27.75\pi]$. The short-period motion (shown dashed) of the epicenter around the equilibrium point with the particle simultaneously executing a long-period motion around the epicenter is shown in Fig. 8 for $f \in [0, 27.75\pi]$.

5.3 Stability of the collinear points

We shall now study the stability of the collinear points. Substituting $\tilde{y} = 0$, into (37), (38), (39) we have $(U_{\tilde{x}\tilde{y}})_0 = 0$. Thus the characteristic equation (46) will be reduced to

$$\lambda^4 + (4 - \tilde{C}_{xx} - \tilde{C}_{yy})\lambda^2 + \tilde{C}_{xx}\tilde{C}_{yy} = 0, \quad (57)$$

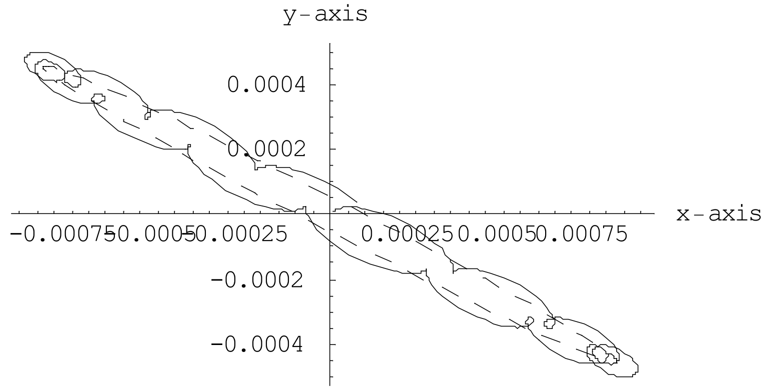

Fig. 8 The graph of trajectory in the vicinity of L_4 point for $f \in [0, 27.75\pi]$

where

$$(U_{\tilde{x}\tilde{x}})_0 = 1 + 2(1-\beta)\frac{(1-\mu)}{\tilde{r}_1^3} + 2\frac{\mu}{\tilde{r}_2^3}, \tag{58}$$

$$(U_{\tilde{y}\tilde{y}})_0 = 1 - (1-\beta)\frac{1-\mu}{\tilde{r}_1^3} - \frac{\mu}{\tilde{r}_2^3}. \tag{59}$$

We use the quantity A defined in (18) to rewrite (58) and (59) as:

$$(U_{\tilde{x}\tilde{x}})_0 = 1 + 2A, \qquad (U_{\tilde{y}\tilde{y}})_0 = 1 - A. \tag{60}$$

Using (44), (60) we can write the characteristic equation (57) in the form

$$\lambda^4 + [4 - (2+A)G(e,f)]\lambda^2 + G^2(e,f)(1+2A)(1-A) = 0. \tag{61}$$

Since the product of the two square roots of this polynomial must equal to the constant term, we conclude that

$$\lambda_1^2\lambda_2^2 = G^2(e,f)(1+2A)(1-A),$$

where λ_1^2 and λ_2^2 are the two square roots of the polynomial (61). The stability condition requires that the two square roots must be negative, i.e.

$$\lambda_1^2 < 0, \qquad \lambda_2^2 < 0.$$

In other words, we must have

$$(1+2A)(1-A) > 0. \tag{62}$$

In addition to this the sum of the roots of the polynomial (61) must be equal to the coefficient of λ^2 with inverse sign, i.e.

$$\lambda_1^2 + \lambda_2^2 = -[4 - (2+A)G(e,f)]$$

and since we require that both λ_1^2 and λ_2^2 are negative, then we must have

$$A < 2 + 4e\cos f.$$

The maximum value for the right hand side of this inequality is $(2+4e)$. So, we may put the extra condition for stability as

$$A < 2 + 4e. \tag{63}$$

Inequality (62) is satisfied if both the parentheses are negative or positive simultaneously. So, if both parentheses are negative, we obtain the conditions of stability are

$$A < -0.5 \quad \text{and} \quad A > 1$$

which will give a contradiction for the value of A.

If both parentheses are positive, we obtain the conditions of stability as

$$-\frac{1}{2} < A < 1 \tag{64}$$

which is in agreement with (63). We shall deal with each of the collinear points separately.

5.3.1 *The collinear point L_1*

At the point L_1, we have

$$\tilde{r}_1 + \tilde{r}_2 = 1, \qquad \tilde{r}_1 = \tilde{x} + \mu, \qquad \tilde{r}_2 = 1 - \mu - \tilde{x}. \tag{65}$$

Using (27) and (61), with $\tilde{r}_2 = \rho$. We obtain the quantity A up to the second power of β, and the first power of the mass ratio $\mu/(1-\mu)$ as

$$A = (1-\mu)\Bigg[\Bigg\{4 + 4.16\left(\frac{\mu}{1-\mu}\right)^{1/3} + 3.85\left(\frac{\mu}{1-\mu}\right)^{2/3} + 2.7\left(\frac{\mu}{1-\mu}\right)\Bigg\} - \Bigg\{5 + 1.62\left(\frac{\mu}{1-\mu}\right)^{1/3} + 1.23\left(\frac{\mu}{1-\mu}\right)^{2/3} + 0.11\left(\frac{\mu}{1-\mu}\right)\Bigg\}\beta$$

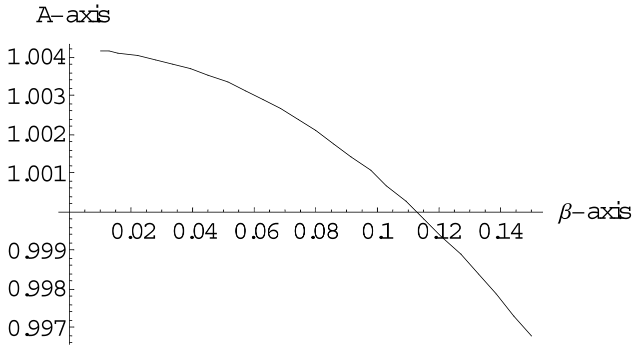

Fig. 9 The graph of the quantity A corresponding to L_3 against β for $\beta \in [0.001, 0.15]$

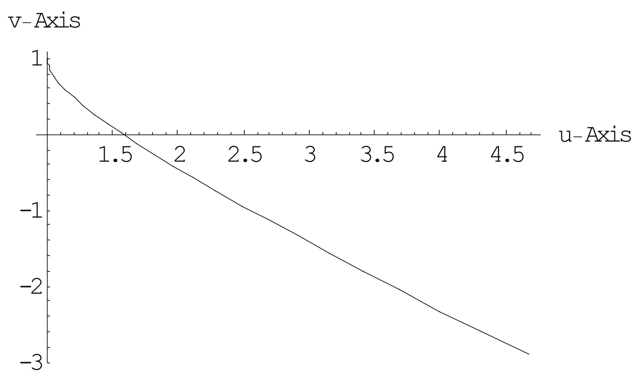

Fig. 10 The graph of the trajectory in the vicinity of L_1 point for $f \in [0, 0.5\pi]$

$$+\left\{1.62\left(\frac{\mu}{1-\mu}\right)^{1/3} + 3.9\left(\frac{\mu}{1-\mu}\right)^{2/3} + 8.44\left(\frac{\mu}{1-\mu}\right)\right\}\beta^2\Bigg]. \quad (66)$$

For $\mu = 9.5359 \times 10^{-4}$, it is clear that $A > 1$, and the stability conditions (64) fails at the point L_1. The graph of the quantity A against β for $\beta \in [0.01, 0.15]$ is shown in Fig. 9.

With the same initial conditions as in the triangular case we can obtain the solutions in the vicinity of L_1 in the form

$$\begin{aligned} u(f) = {} & -3.5738 \times 10^{-6}\cos(2.12286 f) \\ & + 2.61298 \times 10^{-6}\sin(2.12286 f) \\ & + 5.70443 \times 10^{-6} e^{2.5622 f} \\ & + 7.8694 \times 10^{-6} e^{-2.5622 f}, \end{aligned} \quad (67)$$

$$\begin{aligned} v(f) = {} & 8.4692 \times 10^{-6}\cos(2.12286 f) \\ & + 1.1584 \times 10^{-6}\sin(2.12286 f) \\ & - 4.03336 \times 10^{6} e^{2.56217 f} \\ & + 5.5641 \times 10^{-6} e^{-2.56217 f}. \end{aligned} \quad (68)$$

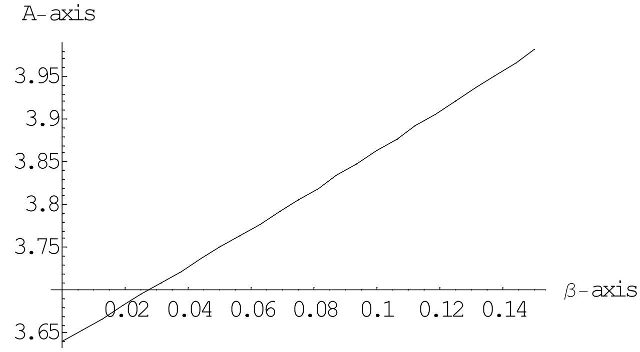

Fig. 11 The graph of the quantity A corresponding to L_2 against β for $\beta \in [0.01, 0.15]$

We notice that the third term of these equations is the dominant term and leads to exponential growth in both u and v which leads to instability of this collinear point. The graphs of the trajectory is shown in Fig. 10 for $f \in [0, 0.5\pi]$.

Now we shall examine the stability of the exterior points L_2 and L_3.

5.3.2 The equilibrium point L_2

At the point L_2, we have

$$\tilde{r}_1 - \tilde{r}_2 = 1, \qquad \tilde{r}_1 = \tilde{x} + \mu, \qquad \tilde{r}_2 = \tilde{x} + \mu - 1. \quad (69)$$

Following the same procedures as in the case of L_1, we get

$$\begin{aligned} A = (1-\mu)\Bigg[& \left\{4 - 3.78\left(\frac{\mu}{1-\mu}\right)^{1/3} + 1.59\left(\frac{\mu}{1-\mu}\right)^{2/3} - 2.7\left(\frac{\mu}{1-\mu}\right)\right\} \\ & + \left\{3 - 5.98\left(\frac{\mu}{1-\mu}\right)^{1/3} - 32.44\left(\frac{\mu}{1-\mu}\right)^{2/3} + 43.22\left(\frac{\mu}{1-\mu}\right)\right\}\beta \\ & + \left\{1.26\left(\frac{\mu}{1-\mu}\right)^{1/3} + 65.67\left(\frac{\mu}{1-\mu}\right)^{2/3} + 58.52\left(\frac{\mu}{1-\mu}\right)\right\}\beta^2\Bigg]. \end{aligned} \quad (70)$$

Again, for $\mu = 9.5359 \times 10^{-4}$, it is clear that $A > 1$, and the stability conditions (64) fails at the point L_2. The graph of the quantity A corresponding to L_2 against β for $\beta \in [0.01, 0.15]$ is shown in Fig 11.

With the same initial conditions as in L_1 case we can obtain the solutions in the vicinity of L_2 in the form

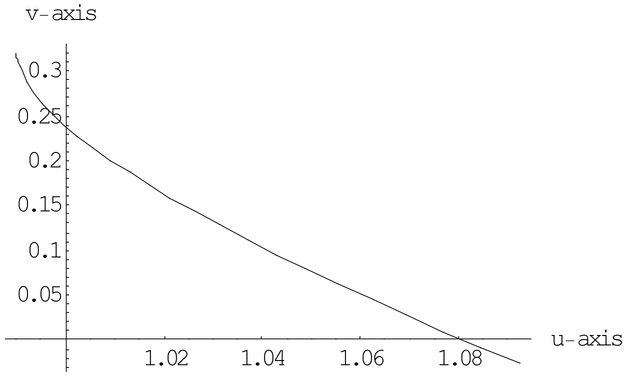

Fig. 12 The graph of the trajectory around L_2 scaled to 10^5 for $f \in [0, 0.05\pi]$

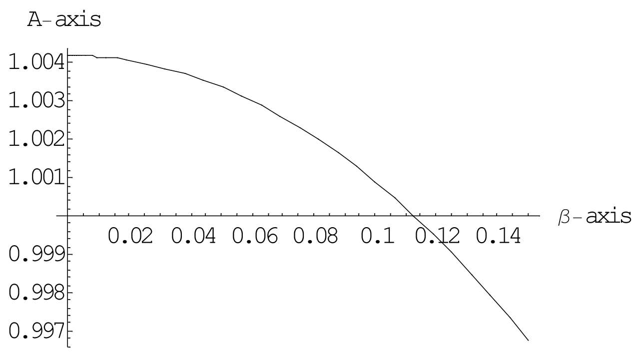

Fig. 13 The graph of the quantity A corresponding to L_3 against β for $\beta \in [0.01, 0.15]$

$$u(f) = -2.8157 \times 10^{-6}\cos(2.001f) + 2.8184 \times 10^{-6}\sin(2.001f) + 5.2123 \times 10^{-6}e^{2.3586f} + 7.5034 \times 10^{-6}e^{-2.3586f}, \quad (71)$$

$$v(f) = 8.6346 \times 10^{-6}\cos(2.0096f) + 8.6265 \times 10^{-6}\sin(2.0096f) - 0.9765 \times 10^{-6}e^{2.3586f} + 4.342 \times 10^{-6}e^{-2.3586f}. \quad (72)$$

The graph of the trajectory around L_2 is shown in Fig. 12 for $f \in [0, 0.05\pi]$.

5.3.3 The collinear point L_3

$$\tilde{r}_2 - \tilde{r}_1 = 1, \qquad \tilde{r}_1 = -\tilde{x} - \mu, \qquad \tilde{r}_2 = -\tilde{x} - \mu + 1. \quad (73)$$

With similar procedure as the L_1 and L_2 cases we obtain up to the second power in β

$$A = 1 + \frac{123}{28}\mu + \frac{337}{588}\mu\beta - \left[\frac{1}{3} + \frac{2815}{2352}\mu\right]\beta^2. \quad (74)$$

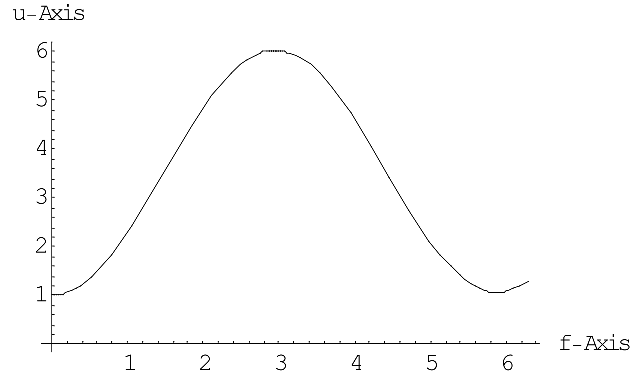

Fig. 14 The graph of the solutions of (78) for $f \in [0, 2\pi]$

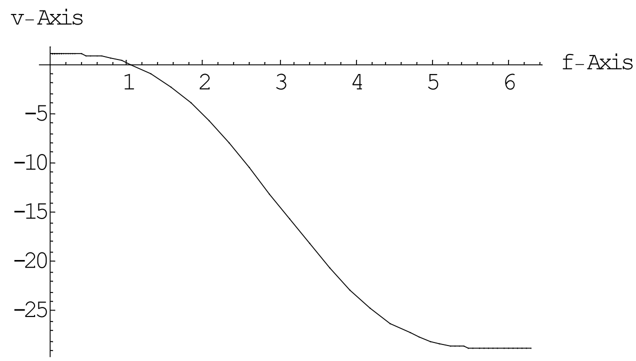

Fig. 15 The graph of the solutions of (79) for $f \in [0, 2\pi]$

The graph of the quantity A corresponding to L_3 against β for $\beta \in [0.01, 0.15]$ is shown in Fig. 13.

With $\beta = 0.1$, we have $A = 1.0009 > 1$, and we conclude that the point L_3 is unstable.

With the same initial conditions as in L_1 case we can obtain the solutions in the vicinity of L_3 in the form

$$u(f) = -2.5092 \times 10^{-5}\cos(1.673f) + 7.6216 \times 10^{-9}\sin(1.673f) + 1.7426 \times 10^{-5}e^{0.03414f} + 1.7655 \times 10^{-5}e^{-0.03414f}, \quad (75)$$

$$v(f) = 1.4288 \times 10^{-8}\cos(1.0673f) + 4.7037 \times 10^{-5}\sin(1.0673f) - 7.303 \times 10^{-4}e^{0.03414f} + 7.403 \times 10^{-4}e^{-0.03414f}. \quad (76)$$

These solutions of (75) and (76) are shown in Figs. 14 and 15 for $f \in [0, 2\pi]$. Also the graph of trajectory around L_3 is shown in Fig. 16 for $f \in [0, 2\pi]$.

If we increase the force ratio β to the value 0.12, we obtain a different configuration of the solution. Using (74) we obtain for $\beta = 0.12$, the quantity $A = 0.999438 < 1$. Thus we conclude that for this value of β the motion in the vicinity of L_3 is stable. For $\beta = 0.12$ the solution can be written as:

$$u(f) = 3.52137 \times 10^{-5} \cos(0.054216 f) + 3.80221 \times 10^{-7} \sin(0.054216 f) - 2.52137 \times 10^{-5} \cos(1.06616 f) - 1.93347 \times 10^{-8} \sin(1.06616 f), \quad (77)$$

$$v(f) = 1.00362 \times 10^{-5} \cos(0.054216 f) - 9.29494 \times 10^{-5} \sin(0.054216 f) - 3.62325 \times 10^{-8} \cos(1.06616 f) + 4.724953 \times 10^{-5} \sin(1.06616 f). \quad (78)$$

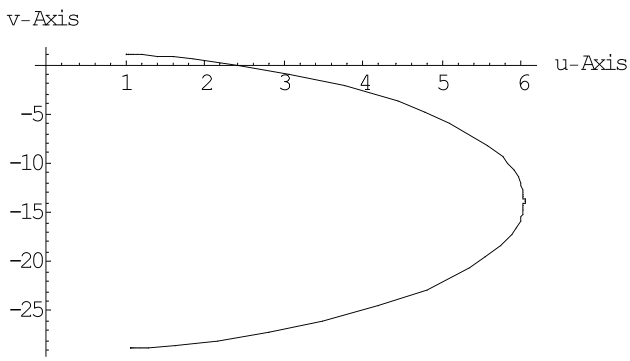

Fig. 16 The graph of trajectory around L_3 scaled to 10^5 for $f \in [0, 2\pi]$

The graph of the solutions (77) and (78) are shown in Figs. 17, 18 for $f \in [0, 37.5\pi]$. Also the graph of the trajectory around L_3 is shown in Fig. 19 for $f \in [0, 37.5\pi]$.

As it is clear from Fig. 9, for all values of $\beta \geq 0.115$, the value of $A < 0$ and the motion in the vicinity of L_3 is stable. We must note that these values for β are very large in realistic situation at least till now.

5.3.4 *The z-equation*

Consider now the third equation of the system (36). In the $\tilde{x}\tilde{y}$-plane, we have

$$U_{\tilde{x}\tilde{z}} = U_{\tilde{y}\tilde{z}} = 0,$$

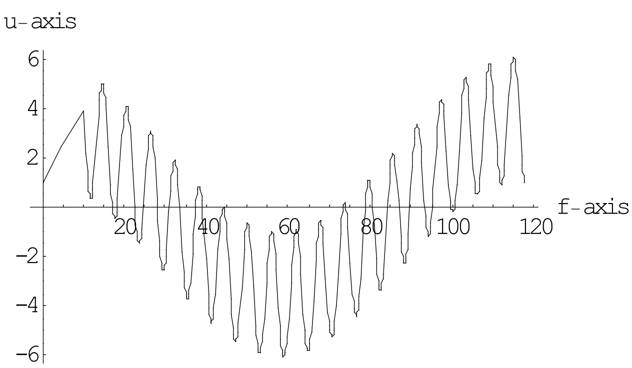

Fig. 17 The graph of the solution (80) for $f \in [0, 37.5\pi]$

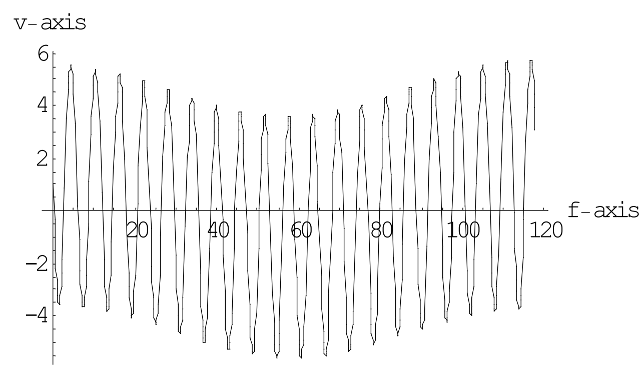

Fig. 18 The graph of the solution (81) for $f \in [0, 37.5\pi]$

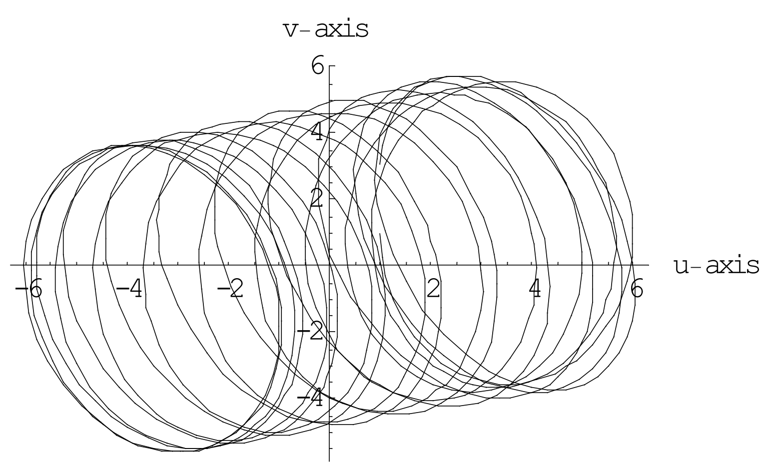

Fig. 19 The graph of the trajectory around L_3 for $\beta = 0.12$ and $f \in [0, 37.5\pi]$

$$U_{\tilde{z}\tilde{z}} = 1 - \frac{(1-\beta)(1-\mu)}{\rho_1^3} - \frac{\mu}{\rho_2^3}.$$

(i) At the triangular points, we have

$$\rho_1 = (1-\beta)^{1/3}, \qquad \rho_2 = 1.$$

Thus, we obtain $U_{\tilde{z}\tilde{z}} = 0$, and the third equation of the system (35), reduces to

$$w'' + w = 0 \tag{79}$$

which represents a simple harmonic motion with period equals 2π.

(ii) At the collinear points we have

$$\Omega_{\tilde{z}\tilde{z}} = \frac{1-A}{1+e\cos f}. \tag{80}$$

The third equation of the system (35), reduces to

$$w'' = -\left(\frac{A+e\cos f}{1+e\cos f}\right) w. \tag{81}$$

Since at all the collinear points, the values of $A > 1$ (except the special values for $\beta \geq 0.115$ at the point L_3), then we conclude that (81) represents a simple harmonic motion with different periods at each collinear point according to the value of the quantity A.

6 Zero velocity curves

In the circular problem of three-bodies, where $e = 0$, $f =$ constant, $r = 1$, due to our choice of units. The system (8) or (9) would be equivalent to the usual equations of motion referred to axes rotating with constant angular velocity. The system (9) reduce to the classical (CRTBP), and can be integrated by multiplying in turn by $\tilde{x}'$, $\tilde{y}'$, $\tilde{z}'$ and added, we obtain

$$(\tilde{x}'^2 + \tilde{y}'^2 + \tilde{z}'^2) = -\tilde{z}^2 + 2U - C_J, \tag{82}$$

where C_J is clearly the Jacobi integral in the scaled variables. Although the (CRTBP) is not integrable, there exists a first integral of the system, the so called Jacobi integral. In a rotating reference frame this problem has the property that its Hamiltonian does not possesses an integral (the Jacobi-integral). The existence of this integral helps to establish the regions of possible motions for given sets of initial conditions, to regularize the problem or to analyze the stability of motions.

When $e \neq 0$, on the other hand, if we multiply the three equations (9) by $\tilde{x}'$, $\tilde{y}'$, $\tilde{z}'$ in turn and added, we obtain

$$\frac{1}{2}\frac{d}{df}(\tilde{x}'^2 + \tilde{y}'^2 + \tilde{z}'^2)$$
$$= -\frac{1}{2}\frac{d\tilde{z}^2}{df} + r\left(\frac{\partial U}{\partial \tilde{x}}\frac{d\tilde{x}}{df} + \frac{\partial U}{\partial \tilde{y}}\frac{d\tilde{y}}{df} + \frac{\partial U}{\partial \tilde{z}}\frac{d\tilde{z}}{df}\right)$$
$$= -\frac{1}{2}\frac{d\tilde{z}^2}{df} + r\frac{dU}{df}. \tag{83}$$

Since U does not contain the time (true anomaly) explicitly. Therefore (83) can be formally integrated to give

$$\tilde{x}'^2 + \tilde{y}'^2 + \tilde{z}'^2 = -\tilde{z}^2 + 2\int_0^f \frac{dU}{1+e\cos f}. \tag{84}$$

Due to the presence of $(1+e\cos f)$ in the denominator of the integrand of (84), this equation is not possible in reality to integrate to any corresponding form. Thus, as far as the ERTBP is concerned, it does not admit this integral (Jacobi integral) of the circular problem, at least not in its usual sense.

In the elliptic problem, the energy along any orbit is a time-dependent quantity. A consequence of this fact is, once again, the non-existence of the Jacobi integral (Floria 2004). The elliptic problem is thus fundamentally different from the circular restricted problem. On the other hand, and in spite of this intrinsic difficulty, certain adequate changes of variables (like the use of the rotating pulsating system) bring the analytical form of the equations for the orbit of the particle into formula which are similar to those governing the (CRTBP), and then important conclusions can be drawn. As we mentioned before, no exact, complete and general solution to the (ERTBP) can be obtained in finite term, but this mathematical inconveniences is usually overcome or, at least softened—by concentrating on the through investigation of significant features and properties of certain special cases of the problem based on simplifying hypotheses concerning the mathematical model under discussion. Now if we define the potential function as:

$$\Omega(\tilde{x}, \tilde{y}, \tilde{z}; f) = \frac{U(\tilde{x}, \tilde{y}, \tilde{z})}{1+e\cos f}, \tag{85}$$

where Ω depends not only on the position coordinates of the particle but also on the independent variable, f, we select the start point, say, $f = 0$, and we consider only a part of the trajectory between $f = 0$ and $f = \varepsilon$ where ε is sufficiently small positive quantity.

This restriction amounts to considering a sufficiently small time interval, during which the primaries describe sufficiently small arcs (Szebehely 1967) with this restriction, we may define a Jacobi-constant in the elliptic case as:

$$C_J = V^2 - \frac{1}{(1+e\cos f)}(\tilde{x}^2 + \tilde{y}^2 - \tilde{z}^2 e\cos f)$$
$$- \frac{2}{(1+e\cos f)}\left[\frac{(1-\beta)(1-\mu)}{\tilde{r}_1} + \frac{\mu}{\tilde{r}_2}\right] \tag{86}$$

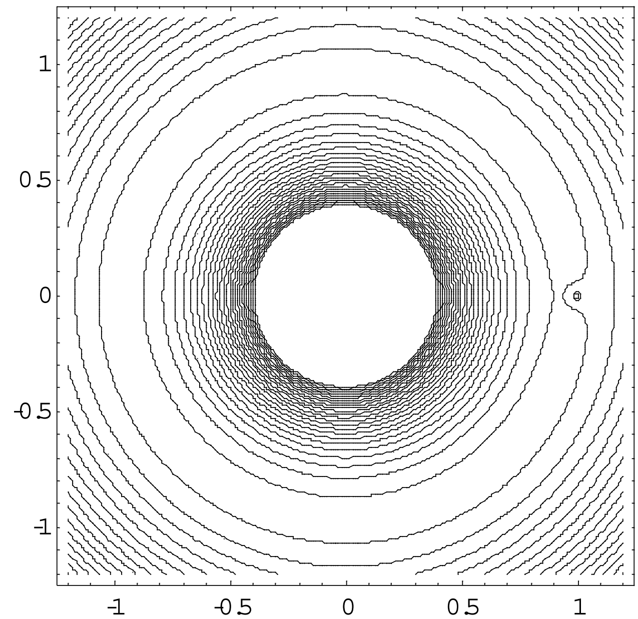

Fig. 20a ZVCs about L_4 at $\beta = 0.1$, $e = 0.04839$, $C_J = 3.07$

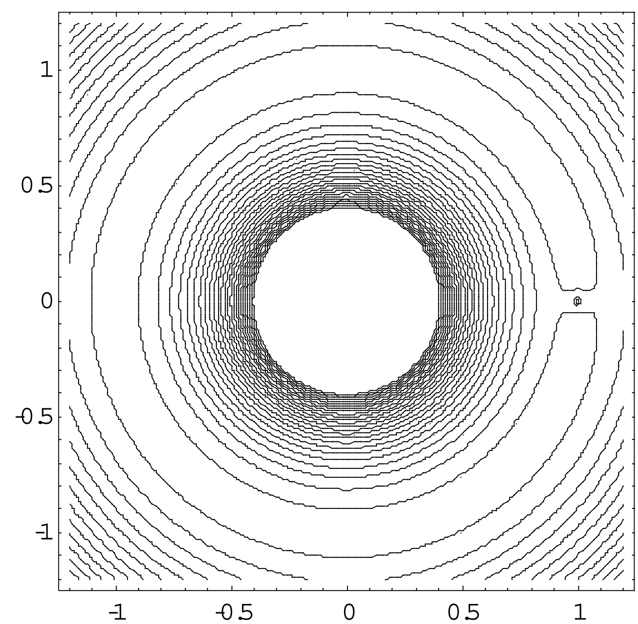

Fig. 20b ZVCs about L_4 at $\beta = 0.0$, $e = 0.04839$, $C_J = 3.07$

which allows us to talk about such concepts as the energy manifold or the zero velocity surface (curves), at each given instant of time. The zero velocity surfaces (curves) are now pulsating with the frequency of the nominal elliptic motion. Therefore at the planer ($\tilde{z} = 0$) (ERTBP), the zero velocity curves are obtained from the equation

$$\tilde{x}^2 + \tilde{y}^2 + \frac{2(1-\beta)(1-\mu)}{\tilde{r}_1} + \frac{2\mu}{\tilde{r}_2} = -C^*, \quad (87)$$

where

$$C^* = C(1 + e\cos f). \quad (88)$$

Geometrically, this means that at every time (or any value of the true anomaly, f) different sets of zero velocity curves are to be constructed at any instant (Szebehely 1967), the redrawn curves of zero velocity will govern the motion by establishing forbidden regions. Figures 20a, 20b represent the zero velocity curves around the point L_4 in the case of the presence of SRP and without it respectively, where we consider $C = 3.07$, $e = 0.04839$.

7 Conclusion

The three dimensional elliptic restricted three-body problem is described, and the effect of solar radiation pressure on the location and stability of the Lagrangian points is studied. We adopted a set of rotating pulsating axes centered at the center of mass of the two primaries m_S (Sun), and m_J (Jupiter). We found that the radiation pressure plays the rule of reducing the effective mass of the Sun and slightly changes the location of the Lagrangian points.

The force ratio, $\beta = F_{rad.}/F_{grav.}$ is used to derive new formulas for the locations of the collinear equilibrium points. It is shown that all the triangular librations points will be stable for any value of $\mu \le 0.5$, satisfying the condition $(27+6\beta)\mu(1-\mu) \le (1+4e\cos f)^2$, where e and f are the eccentricity and the true anomaly of either of the primaries. Also, all the collinear points L_1, L_2 and L_3 are unstable for values of the radiation ratio $\beta \le 0.1$.

On examining the stability of the collinear point L_3 for larger value of the radiation ratio β we found that at the values of $\beta \ge 0.115$, (which are hypothesis values) the collinear point L_3 is stable and we can draw a family of periodic orbits around it.

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