Astrophys Space Sci (2007) 312: 279–293
DOI 10.1007/s10509-007-9688-0

ORIGINAL ARTICLE

Nonlinear stability in the generalised photogravitational restricted three body problem with Poynting-Robertson drag

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Received: 19 August 2007 / Accepted: 4 October 2007 / Published online: 9 November 2007

Abstract The nonlinear stability of triangular equilibrium points has been discussed in the generalised photogravitational restricted three body problem with Poynting-Robertson drag. The problem is generalised in the sense that smaller primary is supposed to be an oblate spheroid. The bigger primary is considered as radiating. We have performed first and second order normalization of the Hamiltonian of the problem. We have applied KAM theorem to examine the condition of non-linear stability. We have found three critical mass ratios. Finally we conclude that triangular points are stable in the nonlinear sense except three critical mass ratios at which KAM theorem fails.

Keywords Nonlinear stability · Triangular points · Generalised photogravitational · RTBP · P-R drag

1 Introduction

The simplest form of the three-body problem is called the restricted three-body problem (RTBP), in which a particle of infinitesimal mass moves in the gravitational field of two massive bodies orbiting according to the exact solution of the two-body problem. In the circular problem, the two finite masses are fixed in a coordinate system rotating at the orbital angular velocity, with the origin (axis of rotation) at the centre of mass of the two bodies. Lagrange showed that in this rotating frame there are five stationary points at which the massless particle would remain fixed if placed there. There are three such points lying on the line connecting the two finite masses: one between the masses and one outside each of the masses. The other two stationary points, called the triangular points, are located equidistant from the two finite masses at a distance equal to the finite mass separation they are stable in classical case. The two masses and the triangular stationary points are thus located at the vertices of equilateral triangles in the plane of the circular orbit. There is a group of enthusiasts who want to setup a colony at L_5 point of the Earth-Moon system. As already noted, because L_4 and L_5 are the stable points of equilibrium, they have been proposed for sites of large self-contained "Space colonies", an idea developed and advocated by the late O'Neill (1974). The three body problem have an interesting application for artificial satellites and future space colonization. Triangular points of the Sun-Jupiter or Sun-Earth system would be convenient sites to locate future space colonies. Application of results to realistic actual problem is obvious.

The classical restricted three body problem is generalized to include the force of radiation pressure, the Poynting-Robertson (P-R) effect and oblateness effect. The photogravitational restricted three body problem arises from the classical problem when at least one of the interacting bodies exerts radiation pressure, for example, binary star systems (both primaries radiating). The photogravitational restricted three body problem under different aspects was studied by Radzievskii (1950), Chernikov (1970), Bhatnagar

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and Chawla (1979), Schuerman (1980), Ishwar and Kushvah (2006), Kushvah et al. (2007a).

The Poynting-Robertson drag named after John Henry Poynting and Howard Percy Robertson, is a process by which solar radiation causes dust grains in a solar system to slowly spiral inward. Poynting (1903) considered the effect of the absorption and subsequent re-emission of sunlight by small isolated particles in the solar system. His work was later modified by Robertson (1937) who used precise relativistic treatments of the first order in the ratio of the velocity of the particle to that of light.

The location and stability of the five Lagrangian equilibrium points in the planar, circular restricted three-body problem was investigated by Murray (1994) when the third body is acted on by a variety of drag forces. The approximate locations of the displaced equilibrium points are calculated for small mass ratios and a simple criterion for their linear stability is derived. They showed if a_1 and a_3 denote the coefficients of the linear and cubic terms in the characteristic equation derived from a linear stability analysis, then an equilibrium point is asymptotically stable provided $0 < a_1 < a_3$. In cases where a_1 is approximately equal to 0 or a_1 is approximately equal to a_3 the point is unstable but there is a difference in the e-folding time scales of the shifted L_4 and L_5 points such that the L_4 point, if it exists, is less unstable than the L_5 point. The results are applied to a number of general and specific drag forces. They have shown that, contrary to intuition, certain drag forces produce asymptotic stability of the displaced triangular equilibrium points, L_4 and L_5. Ishwar and Kushvah (2006) examined the linear stability of triangular equilibrium points in the generalised photogravitational restricted three body problem with Poynting-Robertson drag and conclude that the triangular equilibrium points are unstable due to Poynting-Robertson drag. Kushvah et al. (2007b) performed higher order normalizations in the generalized photogravitational restricted three body problem with Poynting-Robertson drag.

Deprit and Deprit-Bartholome (1967) investigated the nonlinear stability of triangular points by applying Moser's modified version of Arnold's theorem (1961). Bhatnagar and Hallan (1983) studied the effect of perturbations on the nonlinear stability of triangular points. Ishwar (1997) studied nonlinear stability in the generalized restricted three body problem. His problem is generalized in the sense that the infinitesimal body and one of the primaries have been taken as oblate spheroid. Subba Rao and Krishan Sharma (1997) examined effect of oblateness on the non-linear stability of L_4 in the restricted three body problem. Hence we aim to study nonlinear stability of triangular points in our problem.

To examine the nonlinear stability of triangular points we used the KAM theorem (the work of Kolmogorov (1957) extended by Arnold (1961) and Moser (1962). Moser's conditions are utilised in this study by employing the iterative scheme of Henrard for transforming the Hamiltonian to the Birkhoff's normal form with the help of double D'Alembert's series. We have found the second order coefficients in the frequencies. For this we have obtained the partial differential equations which are satisfied by the third order homogeneous components of the fourth order part of Hamiltonian H_4 and second order polynomials in the frequencies. We have found the coefficients of sine and cosine in the homogeneous components of order three. They are critical terms. We have eliminated these critical terms by choosing properly the coefficients in the polynomials. Then we have obtained the values of the coefficients A, B, C occurring in the fourth order part of the normalized Hamiltonian in KAM theorem. We have applied KAM theorem to examine the conditions of nonlinear stability. Using the first condition of the theorem, we have found two critical mass ratios μ_{c1}, μ_{c2} where this condition fails. By taking the second order coefficients, we have calculated the determinant D occurring in the second condition of the theorem. From this, we have found the third critical mass ratio μ_{c3} where the second condition of the theorem fails. We conclude that triangular points are stable for all mass ratios in the range of stability except three critical mass ratios where KAM theorem fails. The stability conditions are different from classical case and others, due to radiation pressure, oblateness and P-R drag.

2 First order normalization

We used Whittaker (1965) method for the transformation of H_2 into the normal form.

Equations of motion are as in Ishwar and Kushvah (2006) and given by

$$\ddot{x} - 2n\dot{y} = U_x, \quad \text{where } U_x = \frac{\partial U_1}{\partial x} - \frac{W_1 n_1}{r_1^2}, \tag{1}$$

$$\ddot{y} + 2n\dot{x} = U_y, \quad U_y = \frac{\partial U_1}{\partial y} - \frac{W_1 n_2}{r_1^2}, \tag{2}$$

$$\begin{aligned} U_1 &= \frac{n^2(x^2+y^2)}{2} + \frac{(1-\mu)q_1}{r_1} + \frac{\mu}{r_2} + \frac{\mu A_2}{2r_2^3}, \\ r_1^2 &= (x+\mu)^2 + y^2, \qquad r_2^2 = (x+\mu-1)^2 + y^2, \\ n^2 &= 1 + \frac{3}{2}A_2, \\ n_1 &= \frac{(x+\mu)[(x+\mu)\dot{x} + y\dot{y}]}{r_1^2} + \dot{x} - ny, \\ n_2 &= \frac{y[(x+\mu)\dot{x} + y\dot{y}]}{r_1^2} + \dot{y} + n(x+\mu). \end{aligned} \tag{3}$$

$W_1 = \frac{(1-\mu)(1-q_1)}{c_d}$, $\mu = \frac{m_2}{m_1+m_2} \leq \frac{1}{2}$, m_1, m_2 be the masses of the primaries, $A_2 = \frac{r_e^2 - r_p^2}{5r^2}$ be the oblateness coefficient,

r_e and r_p be the equatorial and polar radii respectively, r be the distance between primaries, $c_d = 299792458$ be the dimensionless velocity of light, $q_1 = (1 - \frac{F_p}{F_g})$ be the mass reduction factor expressed in terms of the particle's radius a, density ρ and radiation pressure efficiency factor χ (in the C.G.S. system) i.e., $q_1 = 1 - \frac{5.6\times10^{-5}\chi}{a\rho}$. Assumption $q_1 = constant$ is equivalent to neglecting fluctuation in the beam of solar radiation, the effect of the planet's shadow, obviously $q_1 \le 1$. Triangular equilibrium points are given by $U_x = 0$, $U_y = 0$, $y \neq 0$, then we have

$$x_* = x_0\left\{1 - \frac{nW_1[(1-\mu)(1+\frac{5}{2}A_2) + \mu(1-\frac{A_2}{2})\frac{\delta^2}{2}]}{3\mu(1-\mu)y_0x_0} - \frac{\delta^2}{2}\frac{A_2}{x_0}\right\}, \tag{4}$$

$$y_* = y_0\left\{1 - \frac{nW_1\delta^2[2\mu - 1 - \mu(1-\frac{3A_2}{2})\frac{\delta^2}{2} + 7(1-\mu)\frac{A_2}{2}]}{3\mu(1-\mu)y_0^3} - \frac{\delta^2(1-\frac{\delta^2}{2})A_2}{y_0^2}\right\}^{1/2}, \tag{5}$$

where $x_0 = \frac{\delta^2}{2} - \mu$, $y_0 = \pm\delta(1 - \frac{\delta^2}{4})^{1/2}$ and $\delta = q_1^{1/3}$, as in Ishwar and Kushvah (2006)

The Lagrangian function of the problem can be written as

$$L = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + n(x\dot{y} - \dot{x}y) + \frac{n^2}{2}(x^2 + y^2) + \frac{(1-\mu)q_1}{r_1} + \frac{\mu}{r_2} + \frac{\mu A_2}{2r_2^3} + W_1\left\{\frac{(x+\mu)\dot{x} + y\dot{y}}{2r_1^2} - n\arctan\frac{y}{(x+\mu)}\right\} \tag{6}$$

and the Hamiltonian is $H = -L + p_x\dot{x} + p_y\dot{y}$, where p_x, p_y are the momenta coordinates given by

$$p_x = \frac{\partial L}{\partial \dot{x}} = \dot{x} - ny + \frac{W_1}{2r_1^2}(x+\mu),$$

$$p_y = \frac{\partial L}{\partial \dot{y}} = \dot{y} + nx + \frac{W_1}{2r_1^2}y.$$

For simplicity we suppose $q_1 = 1 - \epsilon$, with $|\epsilon| \ll 1$ then coordinates of triangular equilibrium point L_4 can be written in the form

$$x = \frac{\gamma}{2} - \frac{\epsilon}{3} - \frac{A_2}{2} + \frac{A_2\epsilon}{3} - \frac{(9+\gamma)}{6\sqrt{3}}W_1 - \frac{4\gamma\epsilon}{27\sqrt{3}}W_1, \tag{7}$$

$$y = \frac{\sqrt{3}}{2}\left\{1 - \frac{2\epsilon}{9} - \frac{A_2}{3} - \frac{2A_2\epsilon}{9} + \frac{(1+\gamma)}{9\sqrt{3}}W_1 - \frac{4\gamma\epsilon}{27\sqrt{3}}W_1\right\}, \tag{8}$$

where $\gamma = 1 - 2\mu$. We shift the origin to L_4. For that, we change $x \to x_* + x$ and $y \to y_* + y$. Let $a = x_* + \mu$, $b = y_*$ so that

$$a = \frac{1}{2}\left\{1 - \frac{2\epsilon}{3} - A_2 + \frac{2A_2\epsilon}{3} - \frac{(9+\gamma)}{3\sqrt{3}}W_1 - \frac{8\gamma\epsilon}{27\sqrt{3}}W_1\right\}, \tag{9}$$

$$b = \frac{\sqrt{3}}{2}\left\{1 - \frac{2\epsilon}{9} - \frac{A_2}{3} - \frac{2A_2\epsilon}{9} + \frac{(1+\gamma)}{9\sqrt{3}}W_1 - \frac{4\gamma\epsilon}{27\sqrt{3}}W_1\right\}. \tag{10}$$

Expanding L in power series of x and y, we get

$$L = L_0 + L_1 + L_2 + L_3 + \cdots \tag{11}$$

$$H = H_0 + H_1 + H_2 + H_3 + \cdots = -L + p_x\dot{x} + p_y\dot{y}, \tag{12}$$

where $L_0, L_1, L_2, L_3, \ldots$ are

$$L_0 = \frac{3}{2} - \frac{2\epsilon}{3} - \frac{\gamma\epsilon}{3} + \frac{3\gamma A_2}{4} - \frac{3A_2\epsilon}{2} - \gamma A_2 - \frac{\sqrt{3}W_1}{4} + \frac{2\gamma}{3\sqrt{3}}W_1 - \frac{\epsilon W_1}{3\sqrt{3}} - \frac{23\epsilon W_1}{54\sqrt{3}} - n\arctan\frac{b}{a}, \tag{13}$$

$$\begin{aligned} L_1 = {} & \dot{x}\left\{-\frac{\sqrt{3}}{2} - \frac{5A_2}{8\sqrt{3}} + \frac{7\epsilon A_2}{12\sqrt{3}} + \frac{4W_1}{9} - \frac{1\gamma W_1}{18}\right\} \\ & + \dot{y}\left\{\frac{1}{2} - \frac{\epsilon}{3} - \frac{A_2}{8} + \frac{\epsilon A_2}{12\sqrt{3}} - \frac{W_1}{6\sqrt{3}} + \frac{2\epsilon W_1}{3\sqrt{3}}\right\} \\ & - x\left\{-\frac{1}{2} + \frac{\gamma}{2} + \frac{9A_2}{8} + \frac{15\gamma A_2}{8} - \frac{35\epsilon A_2}{12}\right. \\ & \left. - \frac{29\gamma\epsilon A_2}{12} + \frac{3\sqrt{3}W_1}{8} - \frac{2\gamma}{3\sqrt{3}}W_1 - \frac{5\epsilon W_1}{12\sqrt{3}}\right\} \\ & - y\left\{\frac{15\sqrt{3}A_2}{2} + \frac{9\sqrt{3}\gamma A_2}{8} - 2\sqrt{3}\epsilon A_2 - 2\sqrt{3}\gamma\epsilon A_2\right. \\ & \left. - \frac{W_1}{8} + \gamma W_1 - \frac{43\epsilon}{36}W_1\right\}, \end{aligned} \tag{14}$$

$$L_2 = \frac{(\dot{x}^2 + \dot{y}^2)}{2} + n(x\dot{y} - \dot{x}y) + \frac{n^2}{2}(x^2 + y^2) - Ex^2 - Fy^2 - Gxy, \tag{15}$$

$$L_3 = -\frac{1}{3!}\{x^3T_1 + 3x^2yT_2 + 3xy^2T_3 + y^3T_4 + 6T_5\}, \tag{16}$$

$$L_4 = -\frac{1}{4!}\{N_1x^4 + 4N_2x^3y + 6N_3x^2y^2 + 4N_4xy^3 + 24N_6\}, \tag{17}$$

where

$$E=\frac{1}{16}\left[2-6\epsilon-3A_2-\frac{31A_2\epsilon}{2}-\frac{69W_1}{6\sqrt{3}}\right.$$
$$\left.+\gamma\left\{2\epsilon+12A_2+\frac{A_2\epsilon}{3}+\frac{199W_1}{6\sqrt{3}}\right\}\right], \tag{18}$$

$$F=\frac{-1}{16}\left[10-2\epsilon+21A_2-\frac{717A_2\epsilon}{18}-\frac{67W_1}{6\sqrt{3}}\right.$$
$$\left.+\gamma\left\{6\epsilon-\frac{293A_2\epsilon}{18}+\frac{187W_1}{6\sqrt{3}}\right\}\right], \tag{19}$$

$$G=\frac{\sqrt{3}}{8}\left[2\epsilon+6A_2-\frac{37A_2\epsilon}{2}-\frac{13W_1}{2\sqrt{3}}\right.$$
$$\left.-\gamma\left\{6\epsilon-\frac{\epsilon}{3}+13A_2-\frac{33A_2\epsilon}{2}+\frac{11W_1}{2\sqrt{3}}\right\}\right]. \tag{20}$$

T_i, N_j $(i=1,\dots,5,\ j=1,\dots,6)$ are as in Kushvah et al. (2007b).

The second order part H_2 of the corresponding Hamiltonian takes the form

$$H_2=\frac{p_x^2+p_y^2}{2}+n(yp_x-xp_y)+Ex^2+Fy^2+Gxy. \tag{21}$$

To investigate the stability of the motion, as in (Whittaker 1965), we consider the following set of linear equations in the variables x, y:

$$-\lambda p_x=\frac{\partial H_2}{\partial x},\qquad \lambda x=\frac{\partial H_2}{\partial p_x},$$
$$-\lambda p_y=\frac{\partial H_2}{\partial y},\qquad \lambda y=\frac{\partial H_2}{\partial p_y}$$

i.e.

$$AX=\mathbf{0}, \tag{22}$$

where

$$X=\begin{bmatrix}x\\ y\\ p_x\\ p_y\end{bmatrix},\qquad A=\begin{bmatrix}2E & G & \lambda & -n\\ G & 2F & n & \lambda\\ -\lambda & n & 1 & 0\\ -n & -\lambda & 0 & 1\end{bmatrix}. \tag{23}$$

Clearly $|A|=0$, implies that the characteristic equation corresponding to Hamiltonian H_2 is given by

$$\lambda^4+2(E+F+n^2)\lambda^2+4EF-G^2+n^4$$
$$-2n^2(E+F)=0. \tag{24}$$

This is characteristic equation whose discriminant is

$$D=4(E+F+n^2)^2-4\{4EF-G^2+n^4-2n^2(E+F)\}. \tag{25}$$

Stability is assured only when $D>0$, i.e.

$$\mu<\mu_{c_0}-0.221896\epsilon+2.103887A_2+0.493433\epsilon A_2$$
$$+0.704139W_1+0.401154\epsilon W_1, \tag{26}$$

where $\mu_{c_0}=0.038521$ (Routh's critical mass ratio). When $D>0$ the root $\pm i\omega_1$ and $\pm i\omega_2$ (ω_1, ω_2 being the long/short-periodic frequencies) are related to each other as

$$\omega_1^2+\omega_2^2=1-\frac{\gamma\epsilon}{2}+\frac{3\gamma A_2}{2}+\frac{83\epsilon A_2}{12}-\frac{W_1}{24\sqrt{3}}, \tag{27}$$

$$\omega_1^2\omega_2^2=\frac{27}{16}-\frac{27\gamma^2}{16}+\frac{9\epsilon}{8}+\frac{9\gamma\epsilon}{8}+\frac{117\gamma A_2}{16}$$
$$-\frac{241\epsilon A_2}{32}+\frac{35W_1}{16\sqrt{3}}-\frac{55\sqrt{3}\gamma W_1}{16} \tag{28}$$
$$\left(0<\omega_2<\frac{1}{\sqrt{2}}<\omega_1<1\right).$$

From (27) and (28) it may be noted that ω_j $(j=1,2)$ satisfy

$$\gamma^2=1+\frac{4\epsilon}{9}-\frac{107\epsilon A_2}{27}+\frac{2\gamma\epsilon}{3}-\frac{25W_1}{27\sqrt{3}}$$
$$+\left(-\frac{16}{27}+\frac{32\epsilon}{243}+\frac{208A_2}{81}\right.$$
$$\left.-\frac{8\gamma A_2}{27}-\frac{4868\epsilon A_2}{729}+\frac{296W_1}{243\sqrt{3}}\right)\omega_j^2$$
$$+\left(\frac{16}{27}-\frac{32\epsilon}{243}-\frac{208A_2}{81}\right.$$
$$\left.-\frac{1880\epsilon A_2}{729}-\frac{2720W_1}{2187\sqrt{3}}\right)\omega_j^4. \tag{29}$$

Alternatively, it can also be seen that if $u=\omega_1\omega_2$, then (28) gives

$$\gamma^2=1+\frac{4\epsilon}{9}-\frac{107\epsilon A_2}{27}-\frac{25W_1}{27\sqrt{3}}$$
$$+\gamma\left(\frac{2\epsilon}{3}+\frac{1579\epsilon A_2}{324}-\frac{55\gamma W_1}{9\sqrt{3}}\right)$$
$$+\left(-\frac{16}{27}+\frac{32\epsilon}{243}+\frac{208A_2}{81}\right.$$
$$\left.-\frac{1880\epsilon A_2}{729}+\frac{320W_1}{243\sqrt{3}}\right)u^2. \tag{30}$$

Following the method for reducing H_2 to the normal form, as in Whittaker (1965), use the transformation

$$X=JT, \tag{31}$$

where

$$X = \begin{bmatrix} x \\ y \\ p_x \\ p_y \end{bmatrix}, \qquad J = [J_{ij}]_{1\le i,j\le 4}, \qquad T = \begin{bmatrix} Q_1 \\ Q_2 \\ P_1 \\ P_2 \end{bmatrix},$$

where J_{ij} are as in Kushvah et al. (2007b), $P_i = (2I_i\omega_i)^{1/2} \times \cos\phi_i$, $Q_i = (\frac{2I_i}{\omega_i})^{1/2}\sin\phi_i$ $(i = 1, 2)$.

The transformation changes the second order part of the Hamiltonian into the normal form

$$H_2 = \omega_1 I_1 - \omega_2 I_2. \tag{32}$$

The general solution of the corresponding equations of motion are

$$I_i = \text{const.} \quad \phi_i = \pm\omega_i + \text{const.} \quad (i = 1, 2). \tag{33}$$

If the oscillations about L_4 are exactly linear, the (33) represent the integrals of motion and the corresponding orbits will be given by

$$x = J_{13}\sqrt{2\omega_1 I_1}\cos\phi_1 + J_{14}\sqrt{2\omega_2 I_2}\cos\phi_2, \tag{34}$$

$$y = J_{21}\sqrt{\frac{2I_1}{\omega_1}}\sin\phi_1 + J_{22}\sqrt{\frac{2I_2}{\omega_2}}\sin\phi_2 + J_{23}\sqrt{2I_1}\omega_1\cos\phi_1 + J_{24}\sqrt{2I_2}\omega_2\sin\phi_2. \tag{35}$$

3 Second order normalization

In order to perform Birkhoff's normalization, we use Henrard's method (Deprit and Deprit-Bartholome 1967) for which the coordinates (x, y) of infinitesimal body, to be expanded in double D'Alembert series $x = \sum_{n\ge1} B_n^{1,0}$, $y = \sum_{n\ge1} B_n^{0,1}$ where the homogeneous components $B_n^{1,0}$ and $B_n^{0,1}$ of degree n are of the form

$$\sum_{0\le m\le n} I_1^{\frac{n-m}{2}} I_2^{\frac{m}{2}} \sum_{(p,q)} [C_{n-m,m,p,q}\cos(p\phi_1 + q\phi_2) + S_{n-m,m,p,q}\sin(p\phi_1 + q\phi_2)]. \tag{36}$$

The conditions in double summation are (i) p runs over those integers in the interval $0 \le p \le n - m$ that have the same parity as $n - m$, (ii) q runs over those integers in the interval $-m \le q \le m$ that have the same parity as m. Here I_1, I_2 are the action momenta coordinates which are to be taken as constants of integer, ϕ_1, ϕ_2 are angle coordinates to be determined as linear functions of time in such a way that $\dot\phi_1 = \omega_1 + \sum_{n\ge1} f_{2n}(I_1, I_2)$, $\dot\phi_2 = -\omega_2 + \sum_{n\ge1} g_{2n}(I_1, I_2)$ where ω_1, ω_2 are the basic frequencies, f_{2n} and g_{2n} are of the form

$$f_{2n} = \sum_{0\le m\le n} f'_{2(n-m),2m} I_1^{n-m} I_2^m, \tag{37}$$

$$g_{2n} = \sum_{0\le m\le n} g'_{2(n-m),2m} I_1^{n-m} I_2^m. \tag{38}$$

The first order components $B_1^{1,0}$ and $B_1^{0,1}$ are the values of x and y given by (34, 35). In order to find out the second order components $B_2^{1,0}$, $B_2^{0,1}$ we consider Lagrange's equations of motion

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot x}\right) - \frac{\partial L}{\partial x} = 0, \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot y}\right) - \frac{\partial L}{\partial y} = 0, \tag{39}$$

i.e.

$$\begin{cases} \ddot x - 2n\dot y + (2E - n^2)x + Gy = \dfrac{\partial L_3}{\partial x} + \dfrac{\partial L_4}{\partial x}, \\ \ddot x + 2n\dot x + (2F - n^2)y + Gx = \dfrac{\partial L_3}{\partial y} + \dfrac{\partial L_4}{\partial y}. \end{cases} \tag{40}$$

Since x and y are double D'Alembert series, $x^j x^k (j \ge 0, k \ge 0, j + k \ge 0)$ and the time derivatives $\dot x, \dot y, \ddot x, \ddot y$ are also double D'Alembert series. We can write

$$\dot x = \sum_{n\ge1} \dot x_n, \quad \dot y = \sum_{n\ge1} \dot y_n, \quad \ddot x = \sum_{n\ge1} \ddot x_n, \quad \ddot y = \sum_{n\ge1} \ddot y_n,$$

where $\dot x, \dot y, \ddot x, \ddot y$ are homogeneous components of degree n in $I_1^{1/2}, I_2^{1/2}$ i.e.

$$\dot x = \frac{d}{dt}\sum_{n\ge1} B_n^{1,0} = \sum_{n\ge1}\left[\frac{\partial B_n^{1,0}}{\partial\phi_1}(\omega_1 + f_2 + f_4 + \cdots) + \frac{\partial B_n^{1,0}}{\partial\phi_2}(-\omega_2 + g_2 + g_4 + \cdots)\right]. \tag{41}$$

We write three components $\dot x_1, \dot x_2, \dot x_3$ of $\dot x$:

$$\dot x_1 = \omega_1\frac{\partial B_1^{1,0}}{\partial\phi_1} - \omega_2\frac{\partial B_1^{1,0}}{\partial\phi_2} = DB_1^{1,0}, \tag{42}$$

$$\dot x_2 = \omega_1\frac{\partial B_2^{1,0}}{\partial\phi_1} - \omega_2\frac{\partial B_2^{1,0}}{\partial\phi_2} = DB_2^{1,0}, \tag{43}$$

$$\dot x_3 = \omega_1\frac{\partial B_3^{1,0}}{\partial\phi_1} - \omega_2\frac{\partial B_3^{1,0}}{\partial\phi_2} + f_2\frac{\partial B_1^{1,0}}{\partial\phi_1} - g_2\frac{\partial B_1^{1,0}}{\partial\phi_2} = DB_2^{1,0} + f_2\frac{\partial B_1^{1,0}}{\partial\phi_1} - g_2\frac{\partial B_1^{1,0}}{\partial\phi_2}, \tag{44}$$

where

$$D \equiv \omega_1\frac{\partial}{\partial\phi_1} - \omega_2\frac{\partial}{\partial\phi_2}. \tag{45}$$

Similarly three components $\ddot{x}_1, \ddot{x}_2, \ddot{x}_3$ of $\ddot{x}$ are

$$\ddot{x}_1 = D^2 B_1^{1,0},\ \ddot{x}_2 = D^2 B_2^{1,0},$$

$$\ddot{x}_3 = D^2 B_3^{1,0} + 2\omega_1 f_2 \frac{\partial^2 B_1^{1,0}}{\partial \phi_1^2} - 2\omega_2 g_2 \frac{\partial^2 B_1^{1,0}}{\partial \phi_2^2}.$$

In similar manner we can write the components of $\dot{y}, \ddot{y}$. Putting the values of $x, y, \dot{x}, \dot{y}, \ddot{x}$ and $\ddot{y}$ in terms of double D'Alembert series in (40) we get

$$\left(D^2 + 2E - 1 - \frac{3}{2}A_2\right) B_2^{1,0} - \left\{2\left(1 + \frac{3}{4}A_2\right) D - G\right\} B_2^{0,1} = X_2, \tag{46}$$

$$\left\{2\left(1 + \frac{3}{4}A_2\right) D + G\right\} B_2^{1,0} + \left(D^2 + 2F - 1 - \frac{3}{2}A_2\right) B_2^{0,1} = Y_2, \tag{47}$$

where

$$X_2 = \left[\frac{\partial L_3}{\partial x}\right]_{x=B_1^{1,0}, y=B_1^{0,1}}, \qquad Y_2 = \left[\frac{\partial L_3}{\partial y}\right]_{x=B_1^{1,0}, y=B_1^{0,1}}.$$

These are two simultaneous partial differential equations in $B_2^{1,0}$ and $B_2^{0,1}$. We solve these equations to find the values of $B_2^{1,0}$ and $B_2^{0,1}$, from (46) and (47)

$$\triangle_1 \triangle_2 B_2^{1,0} = \Phi_2, \qquad \triangle_1 \triangle_2 B_2^{0,1} = -\Psi_2, \tag{48}$$

where

$$\triangle_1 = D^2 + \omega_1^2, \qquad \triangle_2 = D^2 + \omega_2^2$$

$$\Phi_2 = (D^2 + 2F - n^2) X_2 + (2nD - G) Y_2, \tag{49}$$

$$\Psi_2 = (2nD + G) X_2 - (D^2 + 2E - n^2) Y_2. \tag{50}$$

Equation (48) can be solved for $B_2^{1,0}$ and $B_2^{0,1}$ by putting the formula

$$\frac{1}{\triangle_1 \triangle_2} \left\{ \begin{array}{c} \cos(p\phi_1 + q\phi_2) \\ \text{or} \\ \sin(p\phi_1 + q\phi_2) \end{array} \right. = \frac{1}{\triangle_{p,q}} \left\{ \begin{array}{c} \cos(p\phi_1 + q\phi_2) \\ \text{or} \\ \sin(p\phi_1 + q\phi_2), \end{array} \right.$$

where

$$\triangle_{p,q} = [\omega_1^2 - (\omega_1 p - \omega_2 q)^2][\omega_2^2 - (\omega_1 p - \omega_2 q)^2]$$

provided $\triangle_{p,q} \neq 0$. Since $\triangle_{1,0} = 0, \triangle_{0,1} = 0$ the terms $\cos\phi_1, \sin\phi_1, \cos\phi_2, \sin\phi_2$ are the critical terms. Φ_2 and Ψ_2 are free from such terms. By condition (1) of Moser's theorem $k_1\omega_1 + k_2\omega_2 \neq 0$ for all pairs (k_1, k_2) of integers such that $|k_1| + |k_2| \leq 4$, therefore each of $\omega_1, \omega_2, \omega_1 \pm 2\omega_2, \omega_2 \pm 2\omega_1$ is different from zero and consequently none of the divisors $\triangle_{0,0}, \triangle_{0,2}, \triangle_{2,0}, \triangle_{1,1}, \triangle_{1,-1}$ is zero. The second order components $B_2^{1,0}$, $B_2^{0,1}$ are as follows:

$$\begin{aligned} B_2^{1,0} = {} & r_1 I_1 + r_2 I_2 + r_3 I_1 \cos 2\phi_1 \\ & + r_4 I_2 \cos 2\phi_2 + r_5 I_1^{1/2} I_2^{1/2} \cos(\phi_1 - \phi_2) \\ & + r_6 I_1^{1/2} I_2^{1/2} \cos(\phi_1 + \phi_2) + r_7 I_1 \sin 2\phi_1 \\ & + r_8 I_2 \sin 2\phi_2 + r_9 I_1^{1/2} I_2^{1/2} \sin(\phi_1 - \phi_2) \\ & + r_{10} I_1^{1/2} I_2^{1/2} \sin(\phi_1 + \phi_2), \end{aligned} \tag{51}$$

$$\begin{aligned} B_2^{0,1} = {} & -\{s_1 I_1 + s_2 I_2 + s_3 I_1 \cos 2\phi_1 \\ & + s_4 I_2 \cos 2\phi_2 + s_5 I_1^{1/2} I_2^{1/2} \cos(\phi_1 - \phi_2) \\ & + s_6 I_1^{1/2} I_2^{1/2} \cos(\phi_1 + \phi_2) + s_7 I_1 \sin 2\phi_1 \\ & + s_8 I_2 \sin 2\phi_2 + s_9 I_1^{1/2} I_2^{1/2} \sin(\phi_1 - \phi_2) \\ & + s_{10} I_1^{1/2} I_2^{1/2} \sin(\phi_1 + \phi_2)\}, \end{aligned} \tag{52}$$

where r_i, s_i $(i = 1, \ldots, 10)$ are as in Kushvah et al. (2007b). Using tranformation $x = B_1^{1,0} + B_2^{1,0}$ and $y = B_1^{0,1} + B_2^{0,1}$ the third order part H_3 of the Hamiltonian in $I_1^{1/2}, I_2^{1/2}$ is of the form

$$H_3 = A_{3,0} I_1^{3/2} + A_{2,1} I_1 I_2^{1/2} + A_{1,2} I_1^{1/2} I_2 + A_{0,3} I_2^{3/2}. \tag{53}$$

We can verify that in (53) $A_{3,0}$ vanishes independently as in Deprit and Deprit-Bartholome (1967). Similarly the other coefficients $A_{2,1}$, $A_{1,2}$, $A_{0,3}$ are also found to be zero independently.

4 Second order coefficients in the frequencies

In order to find out the second order coefficients $f_{2,0}$, $f_{0,2}$, $g_{2,0}, g_{0,2}$ in the polynomials f_2 and g_2 we have done as in Deprit and Deprit-Bartholome (1967). Proceeding as (48), we find

$$\triangle_1 \triangle_2 B_3^{1,0} = \Phi_3 - 2 f_2 P - 2 g_2 Q, \tag{54}$$

$$\triangle_1 \triangle_2 B_3^{0,1} = \Psi_3 - 2 f_2 U - 2 g_2 V, \tag{55}$$

where

$$\Phi_3 = [D^2 + 2F - n^2] X_3 + [(2nD - G)] Y_3, \tag{56}$$

$$\Psi_3 = -[2(nD + G)] X_3 + [D^2 + 2nE - n^2] Y_3, \tag{57}$$

$$\begin{aligned} P = {} & [D^2 + 2F - n^2]\left[\omega_1 \frac{\partial^2 B_1^{1,0}}{\partial \phi_1^2} - n \frac{\partial B_1^{0,1}}{\partial \phi_1}\right] \\ & + (2nD - G)\left[\omega_1 \frac{\partial^2 B_1^{0,1}}{\partial \phi_1^2} + n \frac{\partial B_1^{1,0}}{\partial \phi_1}\right], \end{aligned} \tag{58}$$

$$Q = -[D^2 + 2F - n^2]\left[\omega_2 \frac{\partial^2 B_1^{1,0}}{\partial \phi_2^2} - n\frac{\partial B_1^{0,1}}{\partial \phi_1}\right] - (2nD - G)\left[\omega_2 \frac{\partial^2 B_1^{0,1}}{\partial \phi_2^2} + n\frac{\partial B_1^{1,0}}{\partial \phi_2}\right], \quad (59)$$

$$U = -(2nD + G)\left[\omega_1 \frac{\partial^2 B_1^{1,0}}{\partial \phi_1^2} - n\frac{\partial B_1^{0,1}}{\partial \phi_1}\right] + [D^2 + 2E - n^2]\left[\omega_1 \frac{\partial^2 B_1^{0,1}}{\partial \phi_1^2} + n\frac{\partial B_1^{1,0}}{\partial \phi_1}\right], \quad (60)$$

$$V = (2nD + G)\left[\omega_2 \frac{\partial^2 B_1^{1,0}}{\partial \phi_2^2} - n\frac{\partial B_1^{0,1}}{\partial \phi_2}\right] - [D^2 + 2E - n^2]\left[\omega_2 \frac{\partial^2 B_1^{0,1}}{\partial \phi_2^2} - n\frac{\partial B_1^{1,0}}{\partial \phi_2}\right], \quad (61)$$

$$X_3 = \frac{\partial}{\partial x}(L_3 + L_4), \qquad Y_3 = \frac{\partial}{\partial y}(L_3 + L_4), \quad (62)$$

i.e.

$$X_3 = \frac{T_1}{2}x^2 + T_2 xy + \frac{T_3}{2}y^2 + \frac{N_1}{6}x^3 + \frac{N_2}{2}x^2 y + \frac{N_3}{2}xy^2 + \frac{N_4}{6}y^3 + \frac{\partial T_5}{\partial x} + \frac{\partial N_6}{\partial x}, \quad (63)$$

$$Y_3 = \frac{T_2}{2}x^2 + T_3 xy + \frac{T_4}{2}y^2 + \frac{N_2}{6}x^3 + \frac{N_3}{2}x^2 y + \frac{N_4}{2}xy^2 + \frac{N_5}{6}y^3 + \frac{\partial T_5}{\partial y} + \frac{\partial N_6}{\partial y}. \quad (64)$$

Equations (56) and (57) are the partial differential equations which are satisfied by the third order components $B_3^{1,0}$, $B_3^{0,1}$ and the second order polynomials f_2, g_2 in the frequencies. We do not require to find out the components $B_3^{1,0}$ and $B_3^{0,1}$. We find the coefficients of $\cos\phi_1, \sin\phi_1, \cos\phi_2$ and $\sin\phi_2$ in the right hand sides of (56, 57). They are the critical terms, since $\triangle_{1,0} = \triangle_{0,1} = 0$. We eliminate these terms by choosing properly the coefficients in the polynomials

$$f_2 = f_{2,0}I_1 + f_{0,2}I_2, \qquad g_2 = g_{2,0}I_1 + g_{0,2}I_2. \quad (65)$$

Further, we find that

$$f_{2,0} = \frac{(\text{coefficient of } \cos\phi_1 \text{ in } \Phi_3)}{2(\text{coefficient of } \cos\phi_1 \text{ in } P)} = A, \quad (66)$$

$$f_{0,2} = g_{2,0} = \frac{(\text{coefficient of } \cos\phi_2 \text{ in } \Phi_3)}{2(\text{coefficient of } \cos\phi_2 \text{ in } Q)} = B, \quad (67)$$

$$g_{0,2} = \frac{(\text{coefficient of } \cos\phi_2 \text{ in } \Psi_3)}{2(\text{coefficient of } \cos\phi_2 \text{ in } Q)} = C, \quad (68)$$

where

$$A = A_{1,1} + (A_{1,2} + A_{1,3}\gamma)\epsilon + (A_{1,4} + A_{1,5}\gamma)A_2 + (A_{1,6} + A_{1,7}\gamma)W_1, \quad (69)$$

$$B = B_{1,1} + (B_{1,2} + B_{1,3}\gamma)\epsilon + (B_{1,4} + B_{1,5}\gamma)A_2 + (B_{1,6} + B_{1,7}\gamma)W_1, \quad (70)$$

$$C = C_{1,1} + (C_{1,2} + C_{1,3}\gamma)\epsilon + (C_{1,4} + C_{1,5}\gamma)A_2 + (C_{1,6} + C_{1,7}\gamma)W_1, \quad (71)$$

where $A_{1,i}, B_{1,i}$ and $C_{1,i}$ $(i = 1, \ldots, 7)$ are as in Appendix 1.

5 Stability

The condition (i) of KAM theorem fails when $\omega_1 = 2\omega_2$ and $\omega_1 = 3\omega_2$.

5.1 Case (i)

When

$$\omega_1 = 2\omega_2. \quad (72)$$

Then from (72) and (28) we have

$$\mu^2\left(-\frac{27}{4} - \frac{3\epsilon}{2} - \frac{117A_2}{4} - \frac{221W_1}{15\sqrt{3}}\right) + \mu\left(\frac{27}{4} - \frac{107\epsilon}{100} + \frac{3021A_2}{100} + \frac{4291W_1}{120\sqrt{3}}\right) - \frac{4}{25} + \frac{407\epsilon}{200} - \frac{12A_2}{25} - \frac{23991W_1}{200\sqrt{3}} = 0. \quad (73)$$

Solving for μ we have

$$\mu_{c1} = 0.024294 - 0.312692\epsilon - 0.036851A_2 + 1.001052W_1. \quad (74)$$

5.2 Case (ii)

When

$$\omega_1 = 3\omega_2. \quad (75)$$

Proceeding as in case 5.1, we have

$$\mu^2\left(-\frac{27}{4} - \frac{3\epsilon}{2} - \frac{117A_2}{4} - \frac{99\sqrt{3}W_1}{20}\right) + \mu\left(\frac{27}{4} - \frac{93\epsilon}{100} + \frac{2979A_2}{100} + \frac{119\sqrt{3}W_1}{10}\right) - \frac{9}{100} + \frac{393\epsilon}{200} - \frac{27A_2}{100} - \frac{4777W_1}{400\sqrt{3}} = 0. \quad (76)$$

Solving for μ, we have

$$\mu_{c2} = 0.013516 - 0.29724\epsilon - 0.019383 A_2 + 1.007682 W_1. \quad (77)$$

Normalized Hamiltonian up to fourth order is

$$H = \omega_1 I_1 - \omega_2 I_2 + \frac{1}{2}(A I_1^2 + 2B I_1 I_2 + C I_2^2) + \cdots. \quad (78)$$

Calculating the determinant D occurring in condition (ii) of KAM theorem, we have

$$D = -(A\omega_2^2 + 2B\omega_1\omega_2 + C\omega_1^2).$$

Putting the values of A, B and C and if $u = \omega_1\omega_2$, we have

$$D = \frac{644u^4 - 541u^2 + 36}{8(4u^2 - 1)(25u^2 - 4)} + (D_2 + D_3\gamma)\epsilon + (D_4 + D_5\gamma)A_2 + (D_6 + D_7\gamma)W_1. \quad (79)$$

The second condition of KAM theorem is satisfied if, in the interval $0 < \mu < \mu_{c0}$ (where μ_{c0} as in (26)) the mass parameter does not take the value μ_{c3}, which makes $D = 0$. To find μ_{c3}, we note that when $\epsilon = A_2 = W_1 = 0$, then from (79), D becomes zero if and only if

$$644u^4 - 541u^2 + 36 = 0.$$

This implies that

$$u^2 = \frac{541 - \sqrt{199945}}{1288} = 0.072863 = u_0 \text{ (say)}. \quad (80)$$

Writing $u^2 = \frac{27(1-\gamma^2)}{16}$, $\gamma = 1 - 2\mu$ and then solving above, we get

$$\gamma = \gamma_0 = 0.978173\ldots, \qquad \mu = \mu_0 = 0.010914\ldots. \quad (81)$$

When ϵ, A_2, W_1 are not zero, we assume that D is zero if

$$\mu = \mu_0 + \alpha_1\epsilon + \alpha_2 A_2 + \alpha_3 W_1, \quad (82)$$

$$\gamma = \gamma_0 - 2(\alpha_1\epsilon + \alpha_2 A_2 + \alpha_3 W_1), \quad (83)$$

where $\gamma_0 = 1 - 2\mu_0$,

$$u^2 = u_0 + (u_1 + \alpha_1 u_2)\epsilon + (u_3 + \alpha_2 u_4)A_2 + (u_5 + \alpha_3 u_6)W_1, \quad (84)$$

with

$$u_1 = \frac{27}{16}\gamma_0^2 + \frac{9}{8}\gamma_0 + \frac{9}{8}, \qquad u_3 = \frac{117(1-\gamma_0^2)}{16},$$

$$u_2 = u_4 = \frac{27\gamma_0}{4}, \qquad u_6 = \frac{27\gamma_0}{4\sqrt{3}},$$

$$u_5 = \frac{27\gamma_0^2 + 165\gamma_0 + 35}{16\sqrt{3}}$$

and α_i $(i = 1, 2, 3)$ are to be determined. From (79), D is zero when

$$D = \frac{644u^4 - 541u^2 + 36}{8(4u^2 - 1)(25u^2 - 4)} + (D_2 + D_3\gamma)\epsilon + (D_4 + D_5\gamma)A_2 + (D_6 + D_7\gamma)W_1 = 0. \quad (85)$$

Making use of (84) in (85) and equating to zero the coefficients of ϵ, A_2 and W_1, we get

$$\alpha_1 = -\frac{1}{u_2(1288u_0 - 541)}\{(1288u_0 - 541)u_1 + 8(D_2^0 + D_3^0\gamma_0)(4u_0 - 1)(25u_0 - 4)\}, \quad (86)$$

$$\alpha_2 = -\frac{1}{u_4(1288u_0 - 541)}\{(1288u_0 - 541)u_3 + 8(D_4^0 + D_5^0\gamma_0)(4u_0 - 1)(25u_0 - 4)\}, \quad (87)$$

$$\alpha_3 = -\frac{1}{u_6(1288u_0 - 541)}\{(1288u_0 - 541)u_5 + 8(D_6^0 + D_7^0\gamma_0)(4u_0 - 1)(25u_0 - 4)\}, \quad (88)$$

where D_n^0 $(n = 2, 3, 4, \ldots, 7)$ are D_n given as in Appendix 2, as evaluated for the unperturbed problem. Numerical computation yields,

$$\alpha_1 = -0.120489\ldots, \qquad \alpha_2 = -0.373118\ldots,$$

$$\alpha_3 = 2.904291\ldots.$$

Then we have

$$\mu_{c3} = \mu_0 + \alpha_1\epsilon + \alpha_2 A_2 + \alpha_3 W_1 = 0.010914 - 0.120489\epsilon - 0.373118 A_2 + 2.904291 W_1. \quad (89)$$

Hence in the interval $0 < \mu < \mu_{c0}$, both the conditions of KAM theorem are satisfied and therefore the triangular point is stable except for three mass ratios μ_{ci} $(i = 1, 2, 3)$.

6 Analytical study

6.1 Observation I

Consider $A_2 = 0$, $q_1 = 1$ $(W_1 = 0)$ then problem reduced to the classical restricted three body problem. From (4, 5) we get

$$x = \frac{1}{2} - \mu, \qquad y = \pm\frac{\sqrt{3}}{2}$$

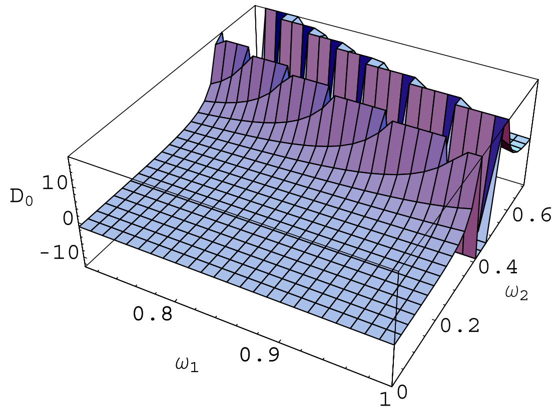

Fig. 1 $A_2=0,\ q_1=1$

from (26) stability is assured when $\mu<\mu_{c0}$ where $\mu_{c0}=0.038521$. The relation between ω_1,ω_2 in (27, 28) are given by

$$\omega_1^2+\omega_2^2=1,\qquad \omega_1^2\omega_2^2=\frac{27}{16}1-\gamma^2$$
$$\left(0<\omega_2<\frac{1}{\sqrt{2}}<\omega_1<1\right). \tag{90}$$

From (74, 77, 89) we have found that the triangular points are stable in the range of linear stability except the three mass ratios

$$\mu_{c1}=0.024294, \tag{91}$$

$$\mu_{c2}=0.013516, \tag{92}$$

$$\mu_{c3}=0.010914, \tag{93}$$

and the D occurring in the second condition of KAM theorem we have found from (79)

$$D=\frac{644u^4-541u^2+36}{8(4u^2-1)(25u^2-4)}, \tag{94}$$

where $u=\omega_1\omega_2$.

All the above results, are exactly similar with the results as in Deprit and Deprit-Bartholome (1967).

Now we have $\epsilon=1-q_1, W_1=\frac{(1-\mu)\epsilon}{c_d}$, suppose $D=D_0$. We draw the Fig. 1 which describes the instability range in classical case and Fig. 2 views the points $\omega_1=0.924270, \omega_2=0.381742, D_0=0$, when $A_2=0$, $q_1=1$, $\frac{1}{\sqrt{2}}<\omega_1<1$, $0<\omega_2<\frac{1}{\sqrt{2}}$ and the value of $\gamma=0.978173$.

6.2 Observation II

Consider the case when $A_2=0$, $q_1\neq 1$ ($W_1\neq 0$) i.e. photogravitational restricted three body problem with P-R drag when bigger primary is supposed to be radiating body and

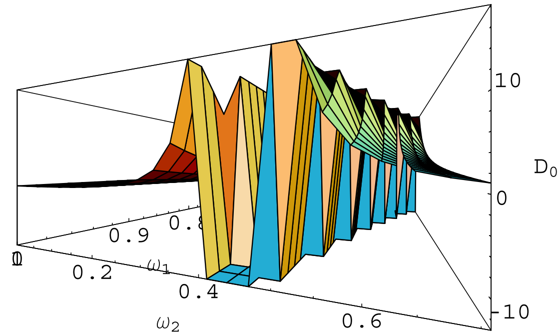

Fig. 2 $A_2=0,\ q_1=1,\ \omega_1=0.924270,\ \omega_2=0.381742,\ D_0=0$

small primary is being spherical symmetric. The coordinates of triangular equilibrium points are given by

$$x=x_0\left\{1-\frac{W_1[(1-\mu)+\mu\frac{\delta^2}{2}]}{3\mu(1-\mu)x_0y_0}\right\}, \tag{95}$$

$$y=y_0\left\{1-\frac{W_1\delta^2[2\mu-1-\mu\frac{\delta^2}{2}]}{6\mu(1-\mu)y_0^3}\right\}, \tag{96}$$

this result coincides with Schuerman (1980), where $x=x_0=\frac{\delta^2}{2}-\mu$, $y=y_0=\pm\delta(1-\frac{\delta^4}{4})^{1/2}$, $q_1=1=\delta$, $x=\frac{1}{2}-\mu$, $y=\pm\frac{\sqrt{3}}{2}$. Substituting $\epsilon=1-q_1$, $W_1=\frac{(1-\mu)\epsilon}{c_d}$, $\mu=\mu_{ci}$ $(i=0,1,2,3)$, $A_2=0$ in (74, 77, 89), we have found that the triangular equilibrium points are stable in the range of stability except three mass ratios

$$\mu_{c1}=0.024294-0.312692(1-q_1)+\frac{0.976732(1-q_1)}{c_d}, \tag{97}$$

$$\mu_{c2}=0.013516-0.29724(1-q_1)+\frac{0.994062(1-q_1)}{c_d}, \tag{98}$$

$$\mu_{c3}=0.010914-0.120489(1-q_1)+\frac{2.87259(1-q_1)}{c_d}. \tag{99}$$

We have observed from Table 1 and Fig. 3, the mass ratio increases, accordingly as the radiation pressure increases, these results are similar but not identical to those of Papadakis (1999).

6.3 Observation III

When $A_2\neq 0$, $q_1=1$ ($W_1=0$) i.e. in this observation we have considered the smaller primary as an oblate spheroid, the radiation pressure (P-R drag) is not considered. The triangular equilibrium points are given by

$$x=\frac{1-2\mu-A_2}{2}, \tag{100}$$

Table 1 $A_2 = 0, q_1 \neq 1$ $(W_1 \neq 0)$

q_1	μ_{c1}	μ_{c2}	μ_{c3}
0.95	0.00866	−0.001346	0.00488921
0.96	0.011786	0.0016263	0.006094
0.97	0.014913	0.0045987	0.007299
0.98	0.018040	0.0075712	0.008504
0.99	0.02117	0.010544	0.00970878
1.00	0.024294	0.013516	0.0109137

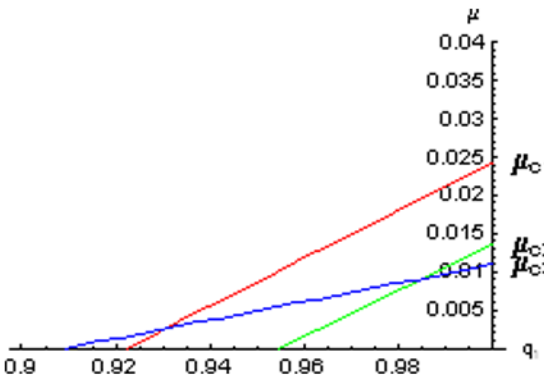

Fig. 3 Stability region μ_{ci} $(i = 1, 2, 3)$–q_1, when $A_2 = 0$, $W_1 \neq 0$

$$y = \pm\frac{\sqrt{3}}{2}\left\{1 - \frac{A_2}{3}\right\} \tag{101}$$

which are similar but not identical to results as in (Bhatnagar and Hallan 1983) and Chandra and Kumar (2004). In this case triangular equilibrium points are stable in the nonlinear sense except three mass ratios at which Moser's condition fails. Which are given by

$$\mu_{c1} = 0.024294 - 0.036851A_2, \tag{102}$$

$$\mu_{c2} = 0.013516 - 0.019383A_2, \tag{103}$$

$$\mu_{c3} = 0.010914 - 0.373118A_2. \tag{104}$$

The stability region are shown in the diagram A_2–μ_{ci} $(i = 1, 2, 3)$, Fig. 4, the outer line is corresponding to μ_{c1}, second line due to μ_{c2} and innermost line is due to μ_{c3}. It is clear from Table 2 that μ decreases as A_2 increases. These results agree with Markellos et al. (1996); Bhatnagar and Hallan (1983).

6.4 Observation IV

When $A_2 \neq 0$, $q_1 \neq 1$ $(W_1 \neq 0)$ this is the most generalized case which is being considered.The triangular equilibrium points are given by (4, 5) clearly they are the functions of oblateness coefficient A_2 and P-R drag term W_1.

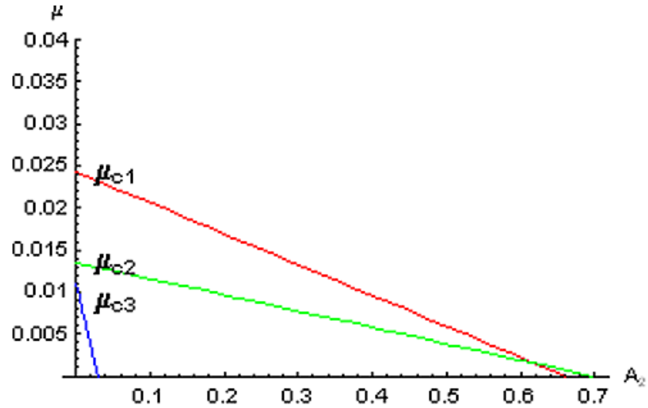

Fig. 4 Stability region μ_{ci} $(i = 1, 2, 3)$–A_2, when $q_1 = 1$, $W_1 = 0$

Table 2 $A_2 \neq 0, q_1 = 1$ $(W_1 = 0)$

A_2	μ_{c1}	μ_{c2}	μ_{c3}
0.0	0.024294	0.01352	0.010914
0.1	0.020609	0.01158	−0.026398
0.2	0.016924	0.009639	−0.06371
0.3	0.013239	0.007701	−0.101022
0.4	0.009554	0.005763	−0.138334
0.5	0.005869	0.003825	−0.175645
0.6	0.002184	0.001886	−0.212957
0.7	−0.001501	−0.000052	−0.250269

Substituting $\epsilon = 1 - q_1, W_1 = \frac{(1-\mu)\epsilon}{c_d}$, $\mu = \mu_{ci}$ $(i = 0, 1, 2, 3)$ in (74, 77, 89), we get the new formulae

$$\mu_{c1} = 0.024294 - 0.312692(1 - q_1) - 0.036851A_2 + \frac{0.976732(1 - q_1)}{c_d}, \tag{105}$$

$$\mu_{c2} = 0.013516 - 0.29724(1 - q_1) - 0.019383A_2 + \frac{0.994062(1 - q_1)}{c_d}, \tag{106}$$

$$\mu_{c3} = 0.010914 - 0.120489(1 - q_1) - 0.373118A_2 + \frac{2.87259(1 - q_1)}{c_d}. \tag{107}$$

Using (105–107) we have drawn μ–A_2–q_1, 3D diagrams, Fig 5. You can see in the first diagram, the uppermost plane is due to μ_{c1}, middle plane is due to μ_{c2} and innermost plane is due to μ_{c3}, second view value of $\mu_{c0} = 0.035829$. From these diagrams, we reached at the conclusion that the stability region is reduced due to P-R drag and oblateness effect of smaller primary. But still the triangular equilibrium points are stable in the range of linear stability except three mass ratios at which KAM theorem fails, while they are unstable in linear case (see Murray 1994; Ishwar and Kushvah 2006).

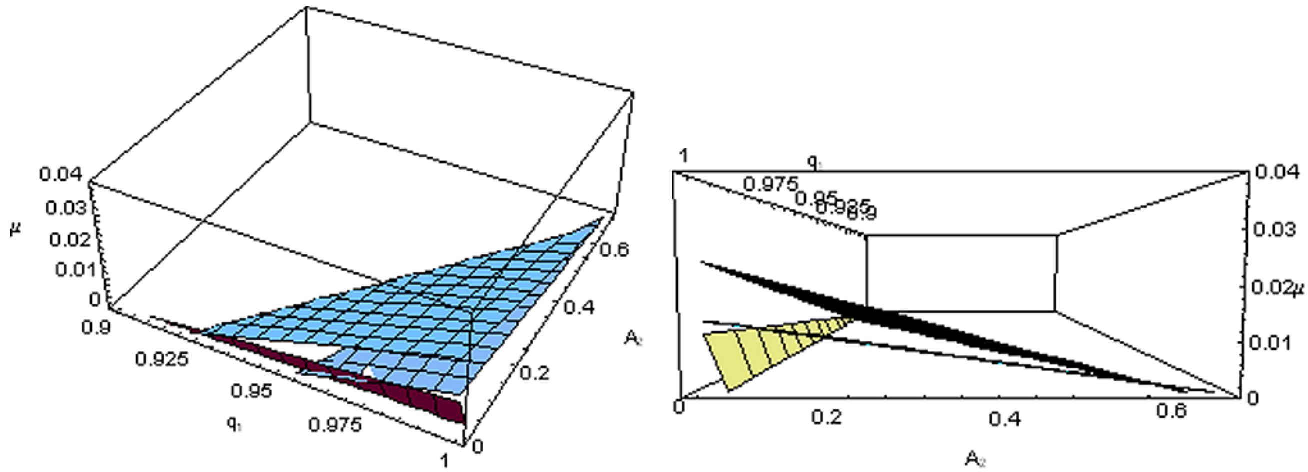

Fig. 5 Both the graphs show the stability region μ_{ci} $(i = 1, 2, 3)$–q_1–A_2, second graph view $\mu_{c0} = 0.035829$

7 Conclusion

Using Whittaker (1965) method we have seen that the second order part H_2 of the Hamiltonian is transformed into the normal form $H_2 = \omega_1 I_1 - \omega_2 I_2$ and the third order part H_3 of the Hamiltonian in $I_1^{1/2}$, $I_2^{1/2}$ zero. We conclude that the stability region is reduced due to P-R drag and oblateness effect of smaller primary. But still the triangular equilibrium points are stable in the nonlinear sense in the range of linear stability except for three mass ratios μ_{ci} $(i = 1, 2, 3)$ at which KAM theorem fails, while they are unstable in linear case (see Murray 1994; Ishwar and Kushvah 2006). These results agree with those found by Deprit and Deprit-Bartholome (1967) and others.

Acknowledgements We express our gratitude to D.S.T. Government of India, New Delhi for sanctioning a project DST/MS/140/2K dated 02/01/2004 on this topic. We are also grateful to IUCAA Pune for providing financial assistance for visiting library and computer facility.

Appendix 1

Coefficients $A_{1,i}$, $B_{1,i}$ and $C_{1,i}$ $(i = 1, \dots, 7)$ are given by:

$$A_{1,1} = -\frac{9}{8(-1+2\omega_1^2)^2(-1+5\omega_1^2)} + \frac{259\omega_1^2}{24(-1+2\omega_1^2)^2(-1+5\omega_1^2)} - \frac{205\omega_1^4}{18(-1+2\omega_1^2)^2(-1+5\omega_1^2)} + \frac{31\omega_1^6}{18(-1+2\omega_1^2)^2(-1+5\omega_1^2)},$$

$$A_{1,2} = \frac{1}{36(1-2\omega_1^2)^2(-1+5\omega_1^2)} - \frac{13\omega_1^2}{18(1-2\omega_1^2)^2(-1+5\omega_1^2)} + \frac{13\omega_1^4}{27(1-2\omega_1^2)^2(-1+5\omega_1^2)} + \frac{167\omega_1^6}{72(1-2\omega_1^2)^2(-1+5\omega_1^2)} + \frac{107\omega_1^8}{108(1-2\omega_1^2)^2(-1+5\omega_1^2)},$$

$$A_{1,3} = \frac{1}{2(1-2\omega_1^2)^3(-1+5\omega_1^2)^2} - \frac{421\omega_1^2}{32(1-2\omega_1^2)^3(-1+5\omega_1^2)^2} - \frac{19\omega_1^4}{2(1-2\omega_1^2)^3(-1+5\omega_1^2)^2} - \frac{8141559\omega_1^6}{32(1-2\omega_1^2)^3(-1+5\omega_1^2)^2} + \frac{29\omega_1^8}{(1-2\omega_1^2)^3(-1+5\omega_1^2)^2},$$

$$A_{1,4} = \frac{1319}{436(1-2\omega_1^2)^2(-1+5\omega_1^2)} - \frac{12639\omega_1^2}{436(1-2\omega_1^2)^2(-1+5\omega_1^2)} + \frac{14275\omega_1^4}{436(1-2\omega_1^2)^2(-1+5\omega_1^2)} - \frac{799\omega_1^6}{218(1-2\omega_1^2)^2(-1+5\omega_1^2)},$$

$$A_{1,5} = \frac{57}{52(1-2\omega_1^2)^3(-1+5\omega_1^2)^2} - \frac{525\omega_1^2}{52(1-2\omega_1^2)^3(-1+5\omega_1^2)^2} - \frac{475\omega_1^4}{26(1-2\omega_1^2)^3(-1+5\omega_1^2)^2} + \frac{1559\omega_1^6}{26(1-2\omega_1^2)^3(-1+5\omega_1^2)^2} + \frac{283\omega_1^8}{13(1-2\omega_1^2)^3(-1+5\omega_1^2)^2},$$

$$A_{1,6} = -\frac{2747\omega_1^2}{10368\sqrt{3}(-1+2\omega_1^2)} + \frac{41(9+4\omega_1^2)}{9216\sqrt{3}(-1+2\omega_1^2)^2} - \frac{93899(9+4\omega_1^2)}{331776\sqrt{3}(-1+2\omega_1^2)} + \frac{12875\omega_1^2(9+4\omega_1^2)}{82944\sqrt{3}(-1+2\omega_1^2)^2},$$

$$A_{1,7} = -\frac{1337}{6144\sqrt{3}(-1+2\omega_1^2)} + \frac{779\omega_1(9+4\omega_1^2)}{10368\sqrt{3}(-1+2\omega_1^2)^2} + \frac{41(9+4\omega_1^2)}{18432\sqrt{3}(-1+2\omega_1^2)^2} - \frac{227347\omega_1^2(9+4\omega_1^2)}{331776\sqrt{3}(-1+2\omega_1^2)} - \frac{37259(9+4\omega_1^2)}{82944\sqrt{3}(-1+2\omega_1^2)} + \frac{6517\omega_1^2(9+4\omega_1^2)}{3072\sqrt{3}(-1+2\omega_1^2)^2(4\omega_1^2-\omega_2^2)},$$

$$B_{1,1} = \frac{43\omega_1\omega_2}{6(1-5\omega_1^2)(-1+2\omega_1^2)(1-5\omega_2^2)(1-2\omega_2^2)} + \frac{32\omega_1^3\omega_2^3}{3(1-5\omega_1^2)(-1+2\omega_1^2)(1-5\omega_2^2)(1-2\omega_2^2)},$$

$$B_{1,2} = \frac{309\omega_1\omega_2}{8(1-5\omega_1^2)(1-2\omega_1^2)(1-5\omega_2^2)(1-2\omega_2^2)} + \frac{5904\omega_1\omega_2}{(-1+2\omega_1^2)(9+4\omega_2^2)^2} - \frac{407\omega_1^3\omega_2^3}{6(1-5\omega_1^2)(1-2\omega_1^2)(1-5\omega_2^2)(1-2\omega_2^2)},$$

$$B_{1,3} = \frac{1800\omega_1\omega_2}{(-1+2\omega_1^2)(9+4\omega_2^2)^2} + \frac{10083-614070\omega_1^2\omega_2^2+400800\omega_1^4\omega_2^4-3035216\omega_1^6\omega_2^6-260802\omega_1^8\omega_2^8}{8\omega_1\omega_2(9-59\omega_1^2+62\omega_1^4+40\omega_1^6)(9-59\omega_2^2+62\omega_2^4+40\omega_2^6)},$$

$$B_{1,4} = \frac{247\omega_1\omega_2}{4(1-5\omega_1^2)(1-2\omega_1^2)(1-5\omega_2^2)(1-2\omega_2^2)} + \frac{6817\omega_1^3\omega_2^3}{36(1-5\omega_1^2)(1-2\omega_1^2)(1-5\omega_2^2)(1-2\omega_2^2)},$$

$$B_{1,5} = \frac{1800\omega_1\omega_2}{(-1+2\omega_1^2)(9+4\omega_2^2)^2} + \frac{-89211+2042998\omega_1^2\omega_2^2+1028577\omega_1^4\omega_2^4\omega_1^2+16052098\omega_1^6\omega_2^6+1215804\omega_1^8\omega_2^8}{32\omega_1\omega_2(-1+5\omega_1^2)^2(-1+5\omega_2^2)^2(-9+14\omega_1^2+8\omega_1^4)(-9-14\omega_2^2+8\omega_2^4)},$$

$$B_{1,6} = \frac{1599\sqrt{3}(9+192\omega_1\omega_2+\omega_2^2)}{512\omega_1\omega_2(-1+2\omega_1^2)(9+4\omega_2^2)^2},$$

$$B_{1,7} = -\frac{3\sqrt{3}(2398599-9031680\omega_2^2-369\omega_1\omega_2^3+574\omega_1\omega_2^5+15744\omega_1^2\omega_2^6+328\omega_2^7}{512\omega_1^2\omega_2^4(-1+2\omega_1^2)(-9+14\omega_2^2+8\omega_2^4)} - \frac{192(-41601+41\omega_1^2)\omega_2^4}{512\omega_1^2\omega_2^4(-1+2\omega_1^2)(-9+14\omega_2^2+8\omega_2^4)},$$

$$C_{1,1} = \frac{9}{8(-1+2\omega_2^2)^2(-1+5\omega_2^2)} + \frac{205\omega_2^2}{24(-1+2\omega_2^2)^2(-1+5\omega_2^2)} - \frac{205\omega_2^4}{18(-1+2\omega_2^2)^2(-1+5\omega_2^2)} + \frac{31\omega_2^6}{18(-1+2\omega_1^2)^2(-1+5\omega_1^2)},$$

$$C_{1,2} = \frac{1}{36(1-2\omega_2^2)^2(-1+5\omega_2^2)} - \frac{13\omega_2^2}{18(1-2\omega_2^2)^2(-1+5\omega_2^2)} + \frac{13\omega_2^4}{27(1-2\omega_2^2)^2(-1+5\omega_2^2)} - \frac{167\omega_2^6}{72(1-2\omega_2^2)^2(-1+5\omega_2^2)} + \frac{107\omega_2^8}{108(1-2\omega_2^2)^2(-1+5\omega_2^2)},$$

$$C_{1,3}=\frac{1}{2(1-2\omega_2^2)^3(-1+5\omega_2^2)^2}-\frac{421\omega_2^2}{32(1-2\omega_2^2)^3(-1+5\omega_2^2)^2}-\frac{19\omega_2^4}{2(1-2\omega_2^2)^3(-1+5\omega_2^2)^2}$$
$$-\frac{407\omega_2^6}{16(1-2\omega_2^2)^3(-1+5\omega_2^2)^2}+\frac{29\omega_2^8}{(1-2\omega_1^2)^3(-1+5\omega_1^2)^2},$$

$$C_{1,4}=\frac{1319}{436(1-2\omega_2^2)^2(-1+5\omega_2^2)}-\frac{12639\omega_2^2}{436(1-2\omega_2^2)^2(-1+5\omega_2^2)}+\frac{14275\omega_2^4}{436(1-2\omega_2^2)^2(-1+5\omega_2^2)}$$
$$-\frac{799\omega_2^6}{218(1-2\omega_2^2)^2(-1+5\omega_2^2)},$$

$$C_{1,5}=\frac{57}{52(1-2\omega_2^2)^3(-1+5\omega_2^2)^2}+\frac{525\omega_2^2}{52(1-2\omega_2^2)^3(-1+5\omega_2^2)^2}-\frac{475\omega_2^4}{26(1-2\omega_2^2)^3(-1+5\omega_2^2)^2}$$
$$+\frac{1559\omega_2^6}{26(1-2\omega_2^2)^3(-1+5\omega_2^2)^2}+\frac{283\omega_2^8}{13(1-2\omega_2^2)^3(-1+5\omega_2^2)^2},$$

$$C_{1,6}=-\frac{287\sqrt{3}(-3+32\omega_2^2+48\omega_2^4)}{1024\omega_2^2(-9+14\omega_2^2+8\omega_2^4)},$$

$$C_{1,7}=-\frac{\sqrt{3}82\omega_1^2(3-38\omega_2^2+16\omega_2^4+96\omega_2^6)}{512(9+4\omega_2^2)(-\omega_1^2+4\omega_2^2)(\omega_2^2-2\omega_2^3)^2}+\frac{3\sqrt{3}\omega_2^2(-142911+195110\omega_2^2+74728\omega_2^4+66784\omega_2^6)}{512(9+4\omega_2^2)(-\omega_1^2+4\omega_2^2)(\omega_2^2-2\omega_2^3)^2},$$

Appendix 2

$$D_2=\frac{567(-151+16\omega_1^2)}{16384(-1+2\omega_1^2)^2(9+4\omega_1^2)^2}+\frac{\omega_1^2\omega_2^2}{884736}\left\{\frac{1620864}{(-1+2\omega_1^2)^2}+\frac{2507364}{(1-2\omega_1^2)}+\frac{706482}{(-1+2\omega_1^2)^2(4\omega_1^2-\omega_2^2)}\right.$$
$$+\frac{71663616000}{(-1+2\omega_1^2)(9+4\omega_2^2)}+\frac{8062156800}{(-1+2\omega_2^2)(9+4\omega_2^2)^2}+\frac{1074954240}{(-1+2\omega_1^2)^2(9+4\omega_2^2)}$$
$$\left.+\frac{112969617408}{(1-5\omega_1^2)^2(-1+2\omega_1^2)^2(9+4\omega_1^2)(1-5\omega_2^2)^2(-1+2\omega_2^2)^2(9+4\omega_2^2)}+\frac{17146183680}{(9-14\omega_2^2-8\omega_2^4)}\right\}$$
$$+\frac{1028577\omega_1^4\omega_2^4}{16(1-5\omega_1^2)^2(-1+2\omega_1^2)^2(9+4\omega_1^2)(1-5\omega_2^2)^2(-1+2\omega_2^2)^2(9+4\omega_2^2)}$$
$$+\frac{8026049\omega_1^6\omega_2^6}{8(1-5\omega_1^2)^2(-1+2\omega_1^2)^2(9+4\omega_1^2)(1-5\omega_2^2)^2(-1+2\omega_2^2)^2(9+4\omega_2^2)}$$
$$+\frac{303951\omega_1^8\omega_2^8}{4(1-5\omega_1^2)^2(-1+2\omega_1^2)^2(9+4\omega_1^2)(1-5\omega_2^2)^2(-1+2\omega_2^2)^2(9+4\omega_2^2)},$$

$$D_3=\frac{3}{8192(-1+2\omega_1^2)}\left\{819+\frac{8064}{(2\omega_1+\omega_2)(\omega_1+2\omega_2)(9+4\omega_2)^2}\right.$$
$$\left.-\frac{6883328}{(1-5\omega_1^2)(-1+2\omega_1^2)(9+4\omega_1^2)(1-5\omega_2^2)(-1+2\omega_2^2)(9+4\omega_2^2)}\right\}$$
$$+\frac{\omega_1^2\omega_2^2}{147456}\left\{\frac{706240}{(-1+2\omega_1^2)}+\frac{289737}{(1-2\omega_1^2)}(4\omega_1^2-\omega_2^2)-\frac{530841600}{(-1+2\omega_1^2)(9+4\omega_2^2)^2}\right.$$
$$+\frac{59719680}{(-1+2\omega_2^2)(9+4\omega_2^2)^2}+\frac{59719680\omega_2^2}{(-1+2\omega_1^2)(9+4\omega_2^2)^2}+\frac{3317760}{(-1+2\omega_2^2)^2(9+4\omega_2^2)}$$

$$+\frac{71516160}{9-14\omega_2^2-8\omega_2^4}+\frac{24772608}{(\omega_1^2-4\omega_2^2)(-1+2\omega_2^2)(9+4\omega_2^2)}$$

$$\left.+\frac{22637076480}{(1-5\omega_1^2)(-1+2\omega_1^2)(9+4\omega_1^2)(1-5\omega_2^2)(-1+2\omega_2^2)(9+4\omega_2^2)}\right\}$$

$$-\frac{100200\omega_1^4\omega_2^4}{(1-5\omega_1^2)(-1+2\omega_1^2)(9+4\omega_1^2)(1-5\omega_2^2)(-1+2\omega_2^2)(9+4\omega_2^2)}$$

$$+\frac{758804\omega_1^6\omega_2^6}{(1-5\omega_1^2)(-1+2\omega_1^2)(9+4\omega_1^2)(1-5\omega_2^2)(-1+2\omega_2^2)(9+4\omega_2^2)}$$

$$+\frac{130401\omega_1^8\omega_2^8}{2(1-5\omega_1^2)(-1+2\omega_1^2)(9+4\omega_1^2)(1-5\omega_2^2)(-1+2\omega_2^2)(9+4\omega_2^2)},$$

$$D_4=\frac{1}{294912}\left\{243\left\{\frac{58477}{(1-2\omega_1^2)^2}+\frac{89216}{(9-14\omega_1^2-8\omega_1^4)}+\frac{7872}{(-1+2\omega_2^2)^2}+\frac{33456}{(9-14\omega_2^2-8\omega_2^4)}\right\}\right.$$

$$+2\omega_1^2\omega_2^2\left\{\frac{5864788}{(1-2\omega_1^2)}-\frac{186165}{(-1+2\omega_1^2)^2(4\omega_1^2-\omega_2^2)}+\frac{1885814784}{(\omega_1^2-4\omega_2^2)(9-14\omega_2^2-8\omega_2^4)}\right.$$

$$\left.\left.+\frac{18210816}{(1-7\omega_1^2+10\omega_1^4)(1-7\omega_2^2+10\omega_2^4)}\right\}-\frac{111689728\omega_1^4\omega_2^4}{(1-7\omega_1^2+10\omega_1^4)(1-7\omega_2^2+10\omega_2^4)}\right\},$$

$$D_5=\frac{1}{49152}\left\{9\left\{-\frac{2457}{(1-2\omega_1^2)^2}+\frac{6426}{(-9+14\omega_1^2+8\omega_1^4)}-\frac{30450688}{(-1+5\omega_1^2)^2(9-14\omega_1^2-8\omega_1^4)(-1+5\omega_2^2)^2(9-14\omega_2^2-8\omega_2^4)}\right\}\right.$$

$$+\omega_1^2\omega_2^2\left\{\frac{90048}{(1-2\omega_1^2)^2}+\frac{139298}{(1-2\omega_1^2)^2}+\frac{39249}{(1-2\omega_1^2)^2(4\omega_1^2-\omega_2^2)}+\frac{447897600}{(-1+2\omega_1^2(9+4\omega_2^2)}+\frac{952565760}{(9-14\omega_2^2-8\omega_2^4)}\right.$$

$$\left.+\frac{6276089856}{(1-5\omega_1^2)^2(-9+14\omega_1^2+8\omega_1^4)(1-5\omega_2^2)^2(-9+14\omega_2^2+8\omega_2^4)}-\frac{594542592}{(\omega_1^2-4\omega_2^2)(-9+14\omega_2^2+8\omega_2^4)}\right\}$$

$$+\frac{3159788544\omega_1^4\omega_2^4}{(1-5\omega_1^2)^2(-9+14\omega_1^2+8\omega_1^4)(1-5\omega_2^2)^2(-9+14\omega_2^2+8\omega_2^4)}$$

$$+\frac{49312045056\omega_1^6\omega_2^6}{(1-5\omega_1^2)^2(-9+14\omega_1^2+8\omega_1^4)(1-5\omega_2^2)^2(-9+14\omega_2^2+8\omega_2^4)}$$

$$\left.+\frac{3734949888\omega_1^8\omega_2^8}{(1-5\omega_1^2)^2(-9+14\omega_1^2+8\omega_1^4)(1-5\omega_2^2)^2(-9+14\omega_2^2+8\omega_2^4)}\right\},$$

$$D_6=\frac{1}{82944\sqrt{3}}\left\{29889\left\{\frac{52}{(-1+2\omega_1^2)^2}+\frac{7}{(9-14\omega_2^2-8\omega_2^4)}\right\}\right\}$$

$$+2\omega_1^2\omega_2^2\left\{-\frac{738}{(1-2\omega_1^2)^2}+\frac{93899}{(-1+2\omega_1^2)^2}+\frac{91445760}{(\omega_1^2-4\omega_2^2)(9-14\omega_2^2-8\omega_2^4)}\right\},$$

$$D_7=\frac{1}{110592\sqrt{3}}\left\{27\left\{\frac{5904}{(-1+2\omega_1^2)^2}+\frac{122157}{(\omega_1^2-4\omega_2^2)(-1+2\omega_1^2)^2}-\frac{758086}{(\omega_1^2-4\omega_2^2)(-1+2\omega_1^2)}+\frac{5904}{(9-14\omega_2^2-8\omega_2^4)}\right\}\right.$$

$$+2\omega_1^2\omega_2^2\left\{-\frac{492}{(1-2\omega_1^2)^2}+\frac{370964}{(-1+2\omega_1^2)}-\frac{58653}{(4\omega_1^2-\omega_2^2)(-1+2\omega_1^2)^2}\right.$$

$$+\frac{13893120}{(9+4\omega_2^2)^2(1-2\omega_2^2)}-\frac{116702208}{(\omega_1^2-4\omega_2^2)(-1+2\omega_2^2)^2(9+4\omega_2^2)^2}$$

$$\left.\left.+\frac{103680}{(9+4\omega_2^2)(1-2\omega_2^2)^2}+\frac{870912}{(\omega_1^2-4\omega_2^2)(-1+2\omega_2^2)^2(9+4\omega_2^2)^2}+\frac{62519040}{(-9+14\omega_2^2+8\omega_2^4)}-\frac{246177792}{(\omega_1^2-4\omega_2^2)(9+4\omega_2^2)^2}\right\}\right\}.$$

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