

Universe with Time Dependent Deceleration Parameter and Λ Term in General Relativity

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Abstract A new class of exact solutions of Einstein's field equations with perfect fluid for an LRS Bianchi type-I spacetime is obtained by using a time dependent deceleration parameter. We have obtained a general solution of the field equations from which three models of the universe are derived: exponential, polynomial and sinusoidal form respectively. The behaviour of these models of the universe are also discussed in the frame of reference of recent supernovae Ia observations.

Keywords Cosmology · Deceleration parameter · Variable cosmological term

1. Introduction

The Bianchi cosmologies play an important role in theoretical cosmology and have been much studied since the 1960s. A Bianchi cosmology represents a spatially homogeneous universe, since by definition the spacetime admits a three-parameter group of isometries whose orbits are spacelike hyper-surfaces. These models can be used to analyze aspects of the physical Universe which pertain to or which may be affected by anisotropy in the rate of expansion, for example, the cosmic microwave background radiation, nucleosynthesis in the early universe, and the question of the isotropization of the universe itself (MacCallum, 1979). For simplification and

description of the large scale behaviour of the actual universe, locally rotationally symmetric [henceforth referred as LRS] Bianchi I spacetime have widely studied (Hajj-Boutros et al., 1987; Ram, 1989; Mazumder, 1994; Pradhan et al., 2001; Pradhan and Vishwakarma, 2000, 2002, 2004; Chakrabarty and Pradhan, 2001; Mohanty et al., 2003). When the Bianchi I spacetime expands equally in two spatial directions it is called locally rotationally symmetric. These kinds of models are interesting because Lidsey (1992) showed that they are equivalent to a flat (FRW) universe with a self-interacting scalar field and a free massless scalar field, but produced no explicit example. Some explicit solutions were pointed out in references (Aguirregabiria et al., 1993, 1966).

The Einstein's field equations are coupled system of high non-linear differential equations and we seek physical solutions to the field equations for their applications in cosmology and astrophysics. In order to solve the field equations we normally assume a form for the matter content or that spacetime admits killing vector symmetries (Kramer et al., 1980). Solutions to the field equations may also be generated by applying a law of variation for Hubble's parameter which was proposed by Berman (1983). In simple cases the Hubble law yields a constant value of deceleration parameter. It is worth observing that most of the well-known models of Einstein's theory and Brans-Deke theory with curvature parameter $k = 0$, including inflationary models, are models with constant deceleration parameter. In earlier literature cosmological models with a constant deceleration parameter have been studied by Berman (1983), Berman and Gomide (1988), Johri and Desikan (1994), Singh and Desikan (1997), Maharaj and Naidoo (1993), Pradhan et al. (2001), Pradhan and Aotemshi (2002) and others. But redshift magnitude test has had a chequered history. During the 1960s and the 1970s, it was used to draw very categorical conclusions. The deceleration parameter q_0 was then claimed to lie between 0 and

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1 and thus it was claimed that the universe is decelerating. Today’s situation, we feel, is hardly different. Observations (Knop et al., 2003; Riess et al., 2004) of Type Ia Supernovae (SNe) allow to probe the expansion history of the universe. The main conclusion of these observations is that the expansion of the universe is accelerating. So we can consider the cosmological models with variable cosmological term and deceleration parameter. The readers are advised to see the papers by Vishwakarma and Narlikar (2005) and Virey et al. (2005) and references therein for a review on the determination of the deceleration parameter from Supernovae data.

Motivated with the situation discussed above, in this paper we can focus upon the problem of establishing a formalism for studying the relativistic evolution for a time dependent deceleration parameter in an expanding universe. This paper is organized as follows. The metric and the field equations are presented in Section 2. In Section 3 we deal with a general solution. The Sections 4, 5, and 6 contain the three different cases for the solutions in exponential, polynomial and sinusoidal forms respectively. Finally in Section 7 concluding remarks will be given.

2. The metric and field equations

We consider the LRS Bianchi type-I metric in the form

$$ds^2 = dt^2 - A^2 dx^2 - B^2(dy^2 + dz^2), \tag{1}$$

where A and B are functions of x and t . The energy momentum-tensor in the presence of perfect fluid has the form

$$T_{ij} = (\rho + p)u_i u_j - p g_{ij}, \tag{2}$$

where ρ , p are the energy density, thermodynamical pressure respectively and u_i is the four velocity vector satisfying the relations

$$u_i u^i = 1, \tag{3}$$

The Einstein’s field equations (in gravitational units $c = 1$, $G = 1$) read as

$$R_{ij} - \frac{1}{2} R g_{ij} + \Lambda g_{ij} = -8\pi T_{ij}, \tag{4}$$

where R_{ij} is the Ricci tensor; $R = g^{ij} R_{ij}$ is the Ricci scalar. The Einstein’s field Equations (4) for the line element (1) has been set up as

$$\frac{2\ddot{B}}{B} + \frac{\dot{B}^2}{B^2} - \frac{B'^2}{A^2 B^2} = -8\pi p + \Lambda, \tag{5}$$

$$\dot{B}' - \frac{B'\dot{A}}{A} = 0, \tag{6}$$

$$\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\dot{A}\dot{B}}{AB} - \frac{B''}{A^2 B} + \frac{A'B'}{A^3 B} = -8\pi p + \Lambda, \tag{7}$$

$$\frac{2B''}{A^2 B} - \frac{2A'B'}{A^3 B} + \frac{B'^2}{A^2 B^2} - \frac{2\dot{A}\dot{B}}{AB} - \frac{\dot{B}^2}{B^2} = 8\pi\rho - \Lambda. \tag{8}$$

The energy conservation equation yields

$$\dot{\rho} + (p + \rho) \left(\frac{\dot{A}}{A} + \frac{2\dot{B}}{B} \right) + \dot{\Lambda} = 0, \tag{9}$$

where dots and primes indicate partial differentiation with respect to t and x respectively.

In order to completely determine the system, we choose a barotropic equation of state

$$p = \gamma\rho, \quad 0 \leq \gamma \leq 1. \tag{10}$$

3. Solution of the field equations

Equation (6), after integration, yields

$$A = \frac{B'}{\ell}, \tag{11}$$

where ℓ is an arbitrary function of x . Equations (5) and (7), with the use of Equation (11), reduces to

$$\frac{B}{B'} \frac{d}{dx} \left(\frac{\ddot{B}}{B} \right) + \frac{\dot{B}}{B'} \frac{d}{dt} \left(\frac{B'}{B} \right) + \frac{\ell^2}{B^2} \left(1 - \frac{B}{B'} \frac{\ell'}{\ell} \right) = 0. \tag{12}$$

If we assume $\frac{B'}{B}$ to be a function of x alone, then A and B are separable in x and t . Hence, after integrating Equation (12) gives

$$B = \ell S(t), \tag{13}$$

where S is a scale factor which is an arbitrary function of t . Thus from Equations (11) and (13), we have

$$A = \frac{\ell'}{\ell} S. \tag{14}$$

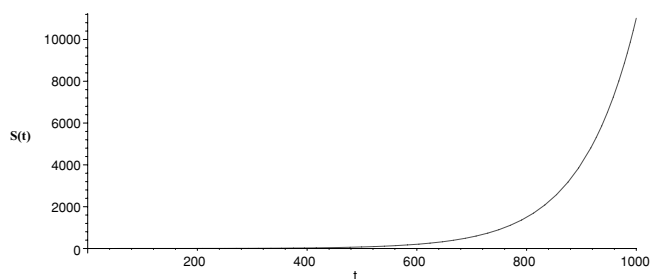
Now the metric (1) is reduced to the form

$$ds^2 = dt^2 - S^2 [dX^2 + e^{2X}(dy^2 + dz^2)], \tag{15}$$

where $X = \ln \ell$. The mass-density, pressure and Ricci scalar are obtained as

$$8\pi\rho = \frac{3}{S^2} [\dot{S}^2 - 1] + \Lambda, \tag{16}$$

Fig. 1 The plot of scale factor $S(t)$ vs time with parameters $k_1 = 0.01, k_2 = 0.5,$ and $\gamma = 0.5$



$$8\pi p = \frac{1}{S^2}[1 - \dot{S}^2 - 2\ddot{S}] - \Lambda, \tag{17}$$

$$R = \frac{6}{S^2}[\ddot{S} + \dot{S}^2 - 1]. \tag{18}$$

The function $S(t)$ remains undetermined. To obtain its explicit dependence on t , one may have to introduce additional assumption. To achieve this, we assume the deceleration parameter to be variable, i.e.

$$q = -\frac{S\ddot{S}}{\dot{S}^2} = -\left(\frac{\dot{H} + H^2}{H^2}\right) = b(\text{variable}), \tag{19}$$

where $H = \frac{\dot{S}}{S}$ is the Hubble parameter. The above equation may be rewritten as

$$\frac{\ddot{S}}{S} + b\frac{\dot{S}^2}{S^2} = 0. \tag{20}$$

The general solution of Equation (20) is given by

$$\int e^{\int \frac{b}{S} dS} dS = t + k, \tag{21}$$

where k is an integrating constant.

In order to solve the problem completely, we have to choose $\int \frac{b}{S} dS$ in such a manner so that Equation (21) be integrable.

Let us consider

$$\int \frac{b}{S} dS = \ln L(S), \tag{22}$$

which does not effect the nature of generality of solution. Hence from Equations (21) and (22), one can obtain

$$\int L(S)dS = t + k. \tag{23}$$

Of course the choice of $L(S)$ is quite arbitrary but, since we are looking for physically viable models of the universe consistent with observations, we consider the following three cases:

4. Solution in the exponential form

Let us consider $L(S) = \frac{1}{k_1 S}$, where k_1 is an arbitrary constant. In this case, on integrating, Equation (23) gives the exact solution

$$S(t) = k_2 e^{k_1 t}, \tag{24}$$

where k_2 is an arbitrary constant. Using Equations (10) and (24) in Equations (16)–(18), the mass-density, cosmological term and Ricci scalar are obtained as

$$\rho(t) = \left\{ \frac{(3\gamma + 1)}{\gamma + 1} - \frac{3}{8\pi} \right\} \frac{1}{k_2^2 e^{2k_1 t}} - \frac{2k_1^2}{(1 + \gamma)k_2} \frac{1}{e^{k_1 t}} - \left\{ \frac{(3\gamma + 1)}{\gamma + 1} - \frac{3}{8\pi} \right\} k_1^2, \tag{25}$$

$$\Lambda(t) = \left\{ \frac{1}{k_2^2 e^{2k_1 t}} - k_1^2 \right\} \left(\frac{3\gamma}{1 + \gamma} \right) - \frac{2k_1^2}{(1 + \gamma)k_2} \frac{1}{e^{k_1 t}}, \tag{26}$$

$$R = \frac{6(k_1^2 k_2 e^{k_1 t} - 1)}{k_2^2 e^{2k_1 t}} + 6k_1^2. \tag{27}$$

From Equation (24), since scale factor can not be negative, we find $S(t)$ is positive if $k_2 > 0$. From Fig.1, it can be deduced that at the early stages of the universe i.e. near $t = 0$, the scale factor of the universe had been approximately constant and had increased very slowly. At an specific time the universe has exploded suddenly and it has expanded to large scale. This fits nicely with Big Bang scenario.

From Equations (25) and (26), it is observed that $\rho(t) > 0$ and $\Lambda(t) > 0$ for $0 < t < \infty$ if $0 < k_2 < 1$. Figure 2 clearly shows this behaviour of $\rho(t)$.

From Equation (26), we observe that the cosmological term is a decreasing function of time and it approaches a small value as time progresses (i.e. the present epoch), which explains the small and positive value of Λ at present (Perlmutter et al., 1997, 1998, 1999; Riess et al., 1998, 2004; Garnavich et al., 1998a, b; Schmidt et al., 1998). Figure 3 clearly shows this behaviour of Λ as decreasing function of time. From Equation (27), we see that the Ricci scalar remain

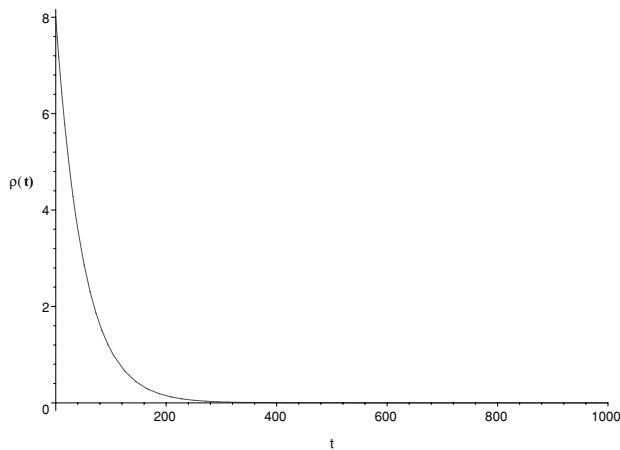


Fig. 2 The plot of energy density $\rho(t)$ vs time with parameters $k_1 = 0.01, k_2 = 0.5,$ and $\gamma = 0.5$

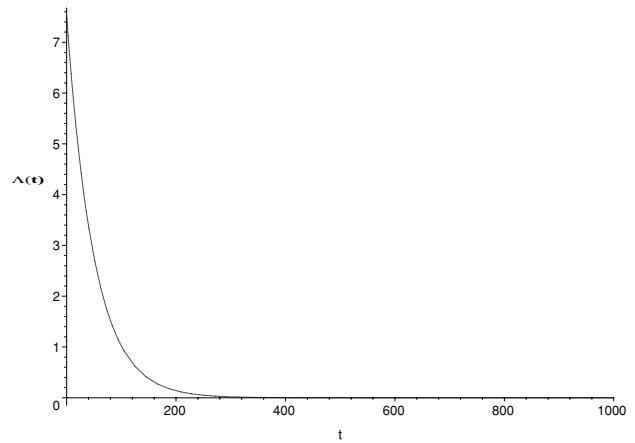


Fig. 3 The plot of cosmological term (Λ) vs time with parameters $k_1 = 0.01, k_2 = 0.5,$ and $\gamma = 0.5$

positive for

$$k_1 > \frac{1}{\sqrt{k_2(1+k_2)}}.$$

5. Solution in the polynomial form

Let $L(S) = \frac{1}{2k_3\sqrt{S+k_4}}$, where k_3 and k_4 are constants.

In this case, on integrating, Equation (23) gives the exact solution

$$S(t) = \alpha_1 t^2 + \alpha_2 t + \alpha_3, \tag{28}$$

where α_1, α_2 and α_3 are arbitrary constants. Using Equations (10) and (28) in Equations (16)–(18), the mass-density, cosmological term and Ricci scalar are obtained as

$$\rho(t) = \frac{[(1+3\gamma) - (1+3\gamma)(2\alpha_1 t + \alpha_2)^2 - 4\alpha_1]}{(1+\gamma)(\alpha_1 t^2 + \alpha_2 t + \alpha_3)^2}, \tag{29}$$

$$\Lambda(t) = \frac{[(2\alpha_1 t + \alpha_2)^2 - 2\alpha_1 - 1]}{4\pi(1+\gamma)(\alpha_1 t^2 + \alpha_2 t + \alpha_3)^2}, \tag{30}$$

$$R = \frac{6[(2\alpha_1 t + \alpha_2)^2 + 2\alpha_1 - 1]}{(\alpha_1 t^2 + \alpha_2 t + \alpha_3)^2} \tag{31}$$

From Equation (28), it is observed that $S(t) > 0$ for $0 \leq t < \infty$ if α_1, α_2 and α_3 are positive constants. From Figure 4, it is observed that the scale factor is a decreasing function of time which means that our universe is expanding.

In order to have $\rho > 0$ for $0 \leq t < \infty$, we must have $\alpha_2^2 > 2\alpha_1 + 1$. From Equation (31) we observe that Ricci scalar remains positive if $\alpha_2^2 > \frac{1-2\alpha_1}{6}$. This condition also implies that $\alpha_1 < \frac{1}{2}$. Figure 5 clearly shows the decreasing behaviour of $\rho(t)$ as time increases and is always positive the interesting

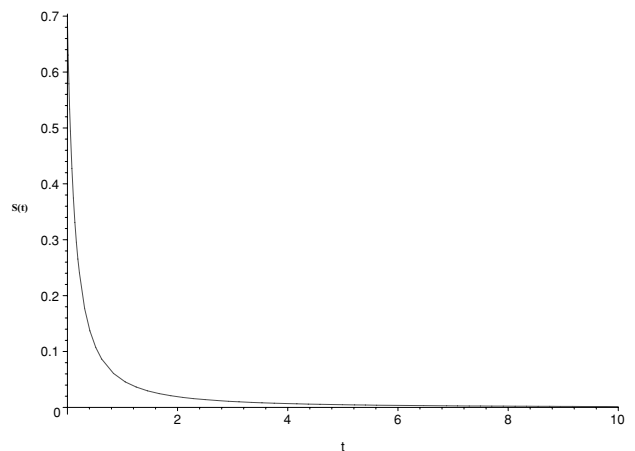


Fig. 4 The plot of scale factor $S(t)$ vs time with parameters $\alpha_1 = 1.00, \alpha_2 = 4.00, \alpha_3 = 1.00$ and $\gamma = 0.5$

point is that all physical parameters in our model are defined at $t = 0$ and we do not have any singularity.

It is observed from Equation (30) that $\Lambda(t)$ remains always negative but decreasing function of time. From the Fig. 6 it can be seen the behaviour of Λ as a decreasing function of time. By decreasing we mean its absolute magnitude approaches zero which is acceptable physically. A negative cosmological term adds to the attractive gravity of matter; therefore, universe with a negative cosmological term is invariably doomed to recollapse. A positive cosmological term resists the attractive gravity of matter due to its negative pressure. For most universe cosmological term eventually dominates over the attraction of matter and drives the universe to expands exponentially.

6. Solution in the sinusoidal form

If we set $L(S) = \frac{1}{\beta\sqrt{1-S^2}}$, where β is constant.

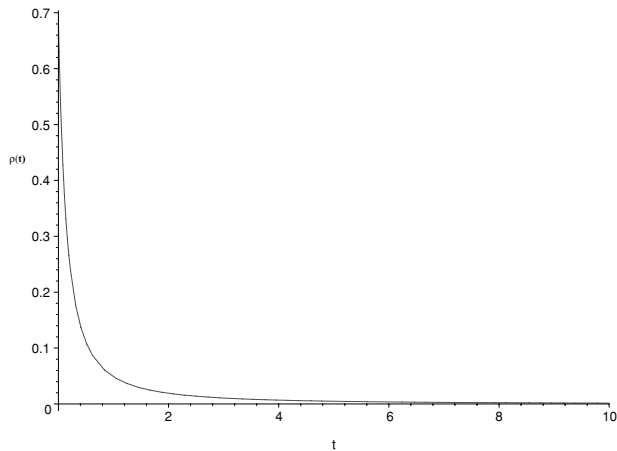


Fig. 5 The plot of energy density $\rho(t)$ vs time with parameters $\alpha_1 = 1.00, \alpha_2 = 4.00, \alpha_3 = 1.00$ and $\gamma = 0.5$

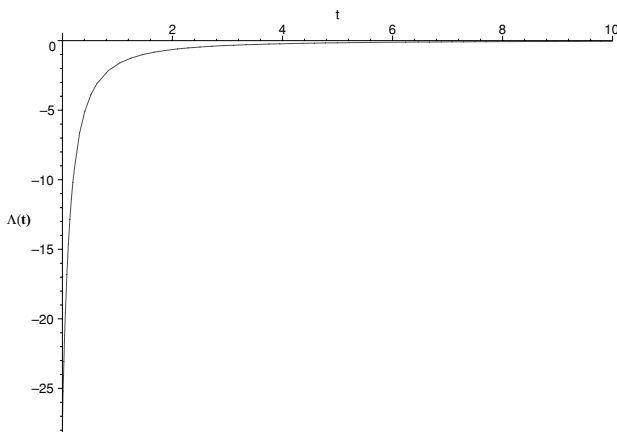


Fig. 6 The plot of cosmological term (Λ) vs time with parameters $\alpha_1 = 1.00, \alpha_2 = 4.00, \alpha_3 = 1.00$ and $\gamma = 0.5$

In this case, on integrating, Equation (23) gives the exact solution

$$S = M \sin(\beta t) + N \cos(\beta t) + \beta_1, \tag{32}$$

where M, N and β_1 are constants. Using Equations (10) and (32) in Equations (16)–(18), the mass-density, cosmological term and Ricci scalar are obtained as

$$4\pi(1 + \gamma)\rho = \frac{[\beta^2(M \cos(\beta t) - N \sin(\beta t))^2 + \beta^2(M \sin(\beta t) - N \cos(\beta t)) - 1]}{(M \sin(\beta t) + N \cos(\beta t) + \beta_1)^2} \tag{33}$$

$$(1 + \gamma)\Lambda = -\frac{[(1 + 3\gamma)\beta^2(M \cos(\beta t) - N \sin(\beta t))^2 - (1 + 3\gamma) - 2k_1^2(M \sin(\beta t) + N \cos(\beta t))]}{(M \sin(\beta t) + N \cos(\beta t) + \beta_1)^2} \tag{34}$$

$$R = \frac{6[(M\beta \cos(\beta t) - N\beta \sin(\beta t))^2 - \beta^2(M \sin(\beta t) + N \cos(\beta t)) - 1]}{(M \sin(\beta t) + N \cos(\beta t) + \beta_1)^2} \tag{35}$$

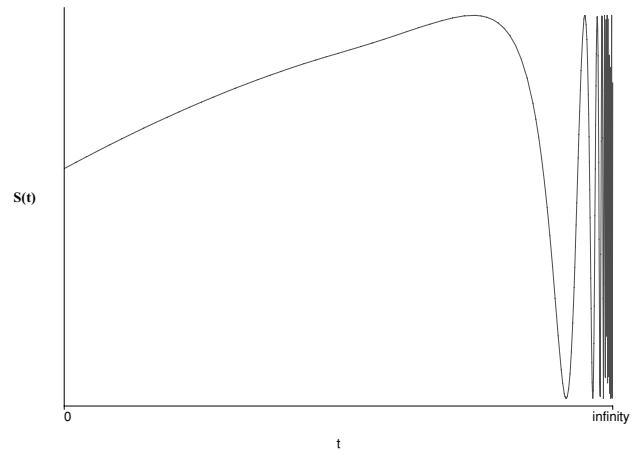


Fig. 7 The plot of scale factor $S(t)$ vs time with parameters $M = 2.00, N = 1.00, \beta = 10.00, \beta_1 = 0.2,$ and $\gamma = 0.5$

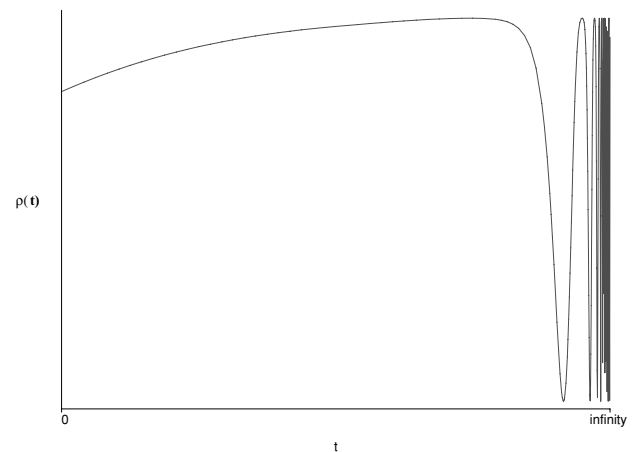


Fig. 8 The plot of energy density $\rho(t)$ vs time with parameters $M = 2.00, N = 1.00, \beta = 10.00, \beta_1 = 0.2,$ and $\gamma = 0.5$

Since, in this case, we have many alternatives for choosing values of M, N, β, β_1 , it is just enough to look for suitable values of these parameters, such that the physical initial and boundary conditions are satisfied. We are trying to find feasible interpretation and situations relevant to this case. Further study in this case is in progress.

From the Fig. 7 it is observed that at early stages of the universe, the scale of the universe increases gently and then decreases sharply, and afterwards it will oscillate for ever.

From Fig. 8 and Fig. 9 we conclude that at early stages of the universe the matter is created as a result of loss of vacuum energy and at a particular epoch it has started to oscillate for ever due to sinusoidal property. It worths mention here that in this case oscillation takes place in positive quadrant which has physical meaning.

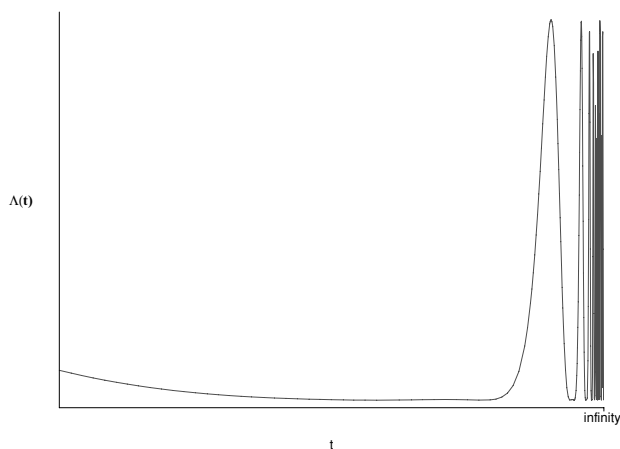


Fig. 9 The plot of cosmological term (Λ) vs time with parameters $M = 2.00$, $N = 1.00$, $\beta = 10.00$, $\beta_1 = 0.2$, and $\gamma = 0.5$

7. Conclusions

In this paper we have described a new class of LRS Bianchi type I cosmological models with a perfect fluid as the source of matter by applying a variable deceleration parameter. Generally, the models are expanding, non-shearing and isotropic in nature.

The cosmological term in the model in Section 4 is a decreasing function of time and this approaches a small value as time increases (i.e. present epoch). The value of the cosmological “term” for this model is found to be small and positive which is supported by the results from recent supernovae observations obtained by the High-Z Supernova Team and Supernova Cosmological Project (Perlmutter et al., 1997, 1998, 1999; Riess et al., 1998, 2004; Garnavich et al., 1998; Schmidt et al., 1998). The cosmological term in the model in Section 5 is also a decreasing function of time but it is always negative. A negative cosmological term adds to the attractive gravity of matter; therefore, universe with a negative cosmological term is invariably doomed to recollapse. The cosmological term in Section 6 also decreases while time increases to a specific instant. During this period as we can understand from Fig. 9, we will have enough matter creation to force the universe to oscillate for ever due to sinusoidal property of Λ . This means we always have annihilation and creation of matter permanently. At this point one more sentence may be added to our discussion and i.e. as the graphs for Λ and ρ in

this case the explosion of the universe at the early stages of its creation has been only a consequence of matter creation.

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References

- Aguirregabiria, J.M., Chimento, L.P.: *Class. Quant. Grav.* **13**, 3197 (1966)
- Aguirregabiria, J.M., Feinstein, A., Ibanez, J.: *Phys. Rev. D* **48**, 4662 (1993)
- Berman, M.S.: *Nuovo Cimento B* **74**, 182 (1983)
- Berman, M.S. Gomide, F.M.: *Gen. Rel. Grav.* **20**, 191 (1988)
- Chakrabarty, I., Pradhan, A.: *Grav. Cosm.* **7**, 55 (2001)
- Garnavich, P.M. et al.: *Astrophys. J.* **493**, L53 (astro-ph/9710123) (1998a); *Astrophys. J.* **509**, 74 (astro-ph/9806396) (1998b)
- Hajj-Boutros, J., Sfeila, J.: *Int. J. Theor. Phys.* **26**, 98 (1987)
- Johri, V.B., Desikan, K.: *Pramana – J. Phys.* **42**, 473 (1994)
- Knop, R.K. et al.: *Astrophys. J.* **598**, 102 (2003)
- Kramer, D., Stephani, H., MacCallum, M., Hertt, E.: *Exact Solutions of Einstein’s Field Equations*, Cambridge University Press, Cambridge (1980)
- Lidsey, J.E.: *Class. Quant. Grav.* **9**, 1239 (1992)
- MacCallum, M.A.H.: in *General Relativity: An Einstein Centenary Survey*, by Hawking, S.W., Israel, W. (ed.), Cambridge University Press, Cambridge (1979)
- Maharaj, S.D., Naidoo, R.: *Astrophys. Space Sci.* **208**, 261 (1993)
- Mazumder, A.: *Gen. Rel. Grav.* **26**, 307 (1994)
- Mohanty, G., Sahu, S.K., Sahoo, P.K.: *Astrophys. Space Sci.* **288**, 523 (2003)
- Perlmutter, S. et al.: *Astrophys. J.* **483**, 565 (astro-ph/9608192) (1997); 1998, *Nature* **391**, 51, (astro-ph/9712212) (1998); *Astrophys. J.* **517**, 565, (astro-ph/9608192). (1999)
- Pradhan, A., Tiwari, K.L., Beesham, A.: *Indian J. Pure Appl. Math.* **32**, 789 (2001)
- Pradhan, A., Vishwakarma, A.K.: *SUJST XII Sec. B*, 42 (2000)
- Pradhan, A., Vishwakarma, A.K.: *Int. J. Mod. Phys. D* **8**, 1195 (2002)
- Pradhan, A., Vishwakarma, A.K.: *J. Geom. Phys.* **49**, 332 (2004)
- Pradhan, A., Yadav, V.K., Chakrabarty, I.: *Int. J. Mod. Phys. D* **10**, 339 (2001)
- Pradhan, A., Aotemshi, I.: *Int. J. Mod. Phys. D* **9**, 1419 (2002)
- Ram, S.: *Int. J. Theor. Phys.* **28**, 917 (1989)
- Riess, A.G. et al.: *Astrophys. J.* **607**, 665 (2004)
- Riess, A.G. et al.: *Astron. J.* **116**, 1009 (astro-ph/9805201) (1998)
- Schmidt, B.P. et al.: *Astrophys. J.* **507**, 46 (astro-ph/9805200) (1998)
- Virey, J.-M., Taxil, P., Tilquin, A., Ealet, A., Tao, C., Fouchez, D.: On the determination of the deceleration parameter from Supernovae data, astro-ph/0502163 (2005)
- Vishwakarma, R.G., Narlikar, J.V.: *Int. J. Mod. Phys. D* **14**, 345 (2005)