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ORIGINAL ARTICLE

On Einstein–Rosen Cosmic Strings in a Scalar Tensor Theory of Gravitation

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Abstract Explicit field equations of a scalar tensor theory of gravitation proposed by Saez and Ballester are obtained with the aid of Einstein–Rosen cylindrically symmetric metric in the presence of cosmic string source. The field equations being highly non–linear static and non–static cases have been considered separately. It is observed that in the static case the geometric strings do not exist while in the non–static case cosmological model does not exist in this theory.

Keywords Cosmic strings · Cylindrically symmetric · Gravitation

1. Introduction

It is well known that a gravitational scalar field, beside the metric of the space-time must exist in the frame work of the present unified theories. Hence there has been much interest in scalar-theories of gravitation. These theories have importance in the early universe where it is expected that the coupling to the matter of scalar field would be of the same order as that of metric, although the scalar coupling is negligible in the present time. There are two categories of gravitational theories involving a classical scalar field ϕ. In the first category the scalar field ϕ has the dimension of the inverse of the gravitational constant G among which the Brans–Dicke (1961) theory is of considerable importance. In the second category the theories involve a dimension less scalar field. Saez and Ballester (1985) developed a scalar-tensor theory in which the metric is coupled with a dimension less scalar field. This coupling gives a satisfactory description of weak fields. This theory suggests a possible way to solve the missing matter problem in non-flat FRW cosmologies, Saez (1985), Singh and Agrawal (1991), Sri Ram and Tiwan (1998) and Reddy and Venkateswara Rao (2001) are some of the authors who have been investigated several aspects of the Saez-Ballester (1985) scalar-tensor theory.

Most grand unified theories predict the formation of cosmic strings during phase transitions in the early universe (Linde, 1979). These cosmic strings are one dimensional objects of the false vacuum of the more symmetric grand unified phase in an otherwise homogenous space-time. Networks of strings intercommute, thus forming distribution of loops of all sizes. A cosmic string with cylindrical symmetry gives in general relativity an illustrative example in which a line source can be represented by an interior solution in the limit where its radius tends to zero. Linet (1985) and Tikekar et al (1994) discussed cylindrically symmetric string cosmological models in general relativity. It would be interesting to study cosmic strings which have received considerable attention in cosmology, in the frame work of scalar-tensor theories of gravitation. Barros et al (2001), Reddy (2003a) have discussed Bianchi type-I string cosmological model in Brans–Dicke theory while Reddy (2003b) has obtained a Bianchi type-I string cosmological model in the scalar-tensor theory proposed by Saez and Ballester.

In this paper, we consider the Einstein—Rosen cylindrically symmetric metric with a cosmic string source in the frame work of Saez–Ballester scalar-tensor theory of gravitation. It has been shown that when the energy density is equal to the tension density of the string (geometric string (Letelier, 1979) static cosmological model does not exist while non–static cosmological models with string source does not survive in this particular theory.

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2. Field equations and the model

We consider the cylindrically symmetric Einstein–Rosen metric.

$$ds^2 = e^{2\alpha-2\beta}(dt^2 - dr^2) - r^2 e^{-2\beta} d\phi^{-2} - e^{2\beta} dz^2 \quad (1)$$

Where α and β are functions of r and t only. We denote the coordinates ϕ, z, t as x^1, x^2, x^3, x^4 respectively.

The field equations given by Saez and Ballester (1985) for the combined scalar and tensor fields are

$$G_{ij} - \omega\phi^n\left(\phi_i, \phi_j - \frac{1}{2} g_{ij}\phi_k, \phi^k\right) = T_{ij} \quad (2)$$

and the scalar field, ϕ, satisfies the equations

$$2\phi^n \phi^i_i, + n\phi^{n-1}\phi_k, \phi^k = 0 \quad (3)$$

Where $G_{ij} = R_{ij} - \frac{1}{2} g_{ij} R$ is the Einstein tensor, ω and n are constants, T_{ij} is the energy tensor of the matter and comma and semicolon denotes partial and covariant differentiation respectively.

Also

$$T^{ij}_{i;} = 0 \quad (4)$$

is a consequence of the field Equations. (2) and (3).

The energy momentum tensor for a system of symmetric strings is

$$T_{ij} = \rho u_i u_j - \lambda x_i x_j \quad (5)$$

Here ρ is the rest energy density of the system of strings with massive particles attached to the strings and λ the tension density of the system of strings. As pointed out by Latelier (1983), λ may be positive or negative, u^i describes the system of four velocities and x^i represents a direction of anisotropy, i.e. the direction of strings.

We have

$$u^i u_i = -x^i x_i = 1 \text{ and } u^i x_i = 0 \quad (6)$$

We consider

$$\rho = \rho_p + \lambda$$

Where ρ_p is the rest energy density of particles and x^i to be along z - axis, so that

$$x^i = (0, 0, e^{-\beta}, 0) \quad (7)$$

Here the axial symmetry assumed implies that the scalar field ϕ shares the same symmetry as α and β as aconsequence of which we note that

$$\phi_2 = \phi_3 = 0 \quad (8)$$

Here after the lower suffixes 1, 2, 3, 4 after an unknown function denote partial differentiation with respect to r, ϕ. z and t respectively.

The field Equations (2), (3) and (4) for the metric (1) with the help of Equations (5)–(6) can, explicitly, be written.

$$\beta_1^2 - \frac{\alpha_1}{r} + \beta_4^2 + \frac{\omega}{2}\phi^n\left(\phi_1^2 - \phi_4^2\right) = 0 \quad (9)$$

$$\alpha_{11} + \beta_1^2 - \alpha_{44} - \beta_4^2 - \frac{\omega\phi^n}{2}\left(\phi_1^2 - \phi_4^2\right) = 0 \quad (10)$$

$$\begin{aligned}2\beta_{44} - 2\beta_{11} - \frac{2\beta_1}{r} + \alpha_{11} - \alpha_{44} - \beta_4^2 + \beta_1^2 \\ -\frac{\omega\phi^n}{2}\left(\phi_1^2 - \phi_4^2\right) = \lambda \exp(2\alpha - 2\beta)\end{aligned} \quad (11)$$

$$\beta_1^2 - \frac{\alpha_1}{r} + \beta_4^2 - \frac{\omega\phi^n}{2}\left(\phi_1^2 - \phi_4^2\right) = \rho \exp(2\alpha - 2\beta) \quad (12)$$

$$2\beta_1\beta_4 - \frac{\alpha_4}{r} - \omega\phi^n\phi_1\phi_4 = 0 \quad (13)$$

$$\left(\phi_{11} + \frac{\phi_1}{r} + \frac{n}{2}\frac{\phi_1^2}{\phi}\right) - \left(\phi_{44} + \frac{n}{2}\frac{\phi_4^2}{\phi}\right) = 0 \quad (14)$$

$$\rho_4 + (\rho - \lambda)(\alpha_4 - \beta_4) = 0 \quad (15)$$

$$\rho_1 + (\rho - \lambda)(\alpha_1 - \beta_1) + \frac{\rho}{r} = 0 \quad (16)$$

The field Eqs. (9)–(16) being highly non-linear we consider the following two cases:

2.1. (1) Static case

Here we consider α, β, ϕ and ρ as functions of r alone. In this case the field Eqs. (9)–(16) reduce to

$$\beta_1^2 - \frac{\alpha_1}{r} + \frac{\omega}{2}\phi^n\phi_1^2 = 0$$

$$\alpha_{11} + \beta_1^2 - \frac{\omega}{2}\phi^n\phi_1^2 = 0$$

$$2\beta_{11} + \frac{2\beta_1}{r} - \alpha_{11} - \beta_1^2 + \frac{\omega}{2}\phi^n\phi_1^2 = -\lambda \exp(2\alpha - 2\beta)$$

$$\beta_1^2 - \frac{\alpha_1}{r} + \frac{\omega}{2}\phi^n\phi_1^2 = \rho\exp(2\alpha - 2\beta)$$

$$\phi_{11} + \frac{\phi_1}{r} + \frac{n}{2}\frac{\phi_1^2}{\phi} = 0$$

$$\rho_1 + (\rho - \lambda)(\alpha_1 - \beta_1) + \frac{\rho}{r} = 0 \tag{17}$$

It can be easily, seen the above set of field equations, when the equation of state for the string model is $\rho = \lambda$ (geometric string), admits the exact solution given by

$$\alpha = \left(C_3^2 + \frac{\omega C_1^2}{2}\right)\log r + \omega C_1^2 C_3 (\log r)^2 + \frac{\omega}{12}C_1^4(\log r)^3$$

$$\beta = C_3\log r + \frac{\omega}{4}C_1^2(\log r)^2$$

$$\rho = \lambda = \frac{\omega C_1^2}{r^2}\exp(2\beta - 2\alpha)$$

$$\phi^{\frac{n}{2}+1} = \left(\frac{n+2}{2}\right)C_1\log r + \phi_0 \tag{18}$$

Where C_1, C_2, C_3 and ϕ_0 are constants of integration. It is seen that the solution (18) satisfies each of the field Equation (17) provided the constant $C_1 = 0$ which immediately gives us the Einstein–Rosen vacuum solution given by

$$ds^2 = r^{2A}(dt^2 - dr^2) - r^2e^{-2A}d\phi^2 - e^{2A}dz^2$$

with $\phi = \phi_0 =$ constant and A is a constant. This shows that static Einstein–Rosen geometric strings and Saez–Ballester scalar field does not coexist.

2.2. (II) Non-static case

Here we consider $\alpha, \beta, \phi, \rho$ and λ as homogeneous functions of cosmic time only. In this particular case, it is a simple matter to see that the field Equations (9)–(16) admit the solution.

$$\alpha = \text{constant}$$

$$\beta = \text{constant}$$

$$\lambda = \rho = 0 \tag{19}$$

with $\phi =$ constant.

This immediately implies that we are getting the empty flat space-time of Einstein's theory. Hence, it is interesting to note that homogeneous non-static cylindrically symmetric cosmic strings do not exist in the scalar-tensor theory formulated by Saez and Ballester.

3. Conclusions

Scalar fields play a central role in inflationary cosmology. Recently, string cosmology has become an attractive subject of interest. Here we have shown that in Saez-Ballester scalar-tensor theory of gravitation, the scalar fields and the geometric do not coexist in a static cylindrically space time. We have also observed that in the homogeneous non-static case cylindrical symmetric cosmic strings do not exist and in this particular case we simply, obtain the vacuum flat space-time of Einstein's theory.

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