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ORIGINAL ARTICLE

Cosmological Model with a Viscous Fluid in a Kaluza-Klein Metric

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Abstract A Cosmological model with a viscous fluid in Kaluza-Klein metric is obtained assuming a time-dependent equation of state. The solution is in fact a generalization of an earlier work by Hajj and Boutros for a perfect fluid. It is also found that dimensional reduction of the extra space takes place such that the five-dimensional universe naturally evolves into an effective four-dimensional one. The dynamical behavior of the model is examined and it is also found that with a decrease in extra space the observable 3D space entropy increases thus accounting for the large value of entropy observable at present.

Keywords Cosmological model · Viscous fluid · Kaluza-Klein metric · Time-dependent equation of state

1. Introduction

The study of higher-dimensional cosmological models is motivated mainly by the possibility of geometrically unifying the fundamental interactions of the universe. In the context of the Kaluza-Klein and super string theories higher dimensions have recently acquired much significance. It has also been suggested that experimental detection of the time variation of the fundamental constants could provide strong evidence for the existence of extra dimensions (Alverez and Cavela, 1983; Marciano, 1984; Randjbar-Daemi et al., 1984). The earlier suggestion of Kaluza-Klein regarding the topology of the extra dimension has now been replaced by what is called 'Spontaneous Compactification' where 4D space-time expands while the extra dimensions contract to unobserved plankian length scale or remain constant (Chodos and Delweiler, 1980).

Several workers have recently obtained exact solutions using higher-dimensional space-time for both cosmological and non-cosmological cases with or without matter (Chatterjee et al., 1990; Banerjee et al., 1990; Mayers and Perry, 1986). But as far as our information goes there has not been much work in literature where viscous fluid has been considered with time-dependent equation of state in higher dimensions. The presence of viscosity in the fluid content introduces many homogeneous cosmological model (Collins, 1971; Belinski and Khalatnikov, 1976). The dissipative mechanism not only modifies the nature of singularity usually occurring for perfect fluid but can also successfully account for the large entropy per baryon in the present Universe. In view of the above we consider in this paper exact solution for a imperfect fluid with time-dependent equation of state $p = \lambda(t)\rho$. The solution obtained in the paper is realistic in the sense that physical properties such as mass density, viscosity coefficient are positive throughout. It is further observed that the entropy of the Universe increases both due to the decreasing metric component of the fifth dimension with time and also due to the presence of the viscous terms. As $g_{55} \to 0$, and $\xi \to 0$, our solution reduces to that for 4D space-time given by Hajj-Boutros (1991). Hence the solution is in fact, a generalization of an earlier work by Hajj et al. for a perfect fluid.

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2. Einstein's field equations

We take the 5D line element in the form

$$ds^2 = dt^2 - \frac{R^2}{(1 + kr^2/4)}(dx^2 + dy^2 + dz^2) - A^2\, dm^2 \quad (1)$$

where k characterizes the spatial curvature. Unlike Wesson, the fifth co-ordinate is taken to be space-like and the metric coefficients are assumed to be functions of time only, i.e., A and R are functions of time only.

The energy–momentum tensor of the viscous fluid (Landu and Lifshitz, 1959) is given by

$$T_{\mu\nu} = (\rho + p_1)V_\mu V_\nu + p_1\, g_{\mu\nu} \quad (2)$$

with

$$p_1 = p - \xi\theta$$

where $\theta = v^a_{;a}$ and $v^a\, v_a = 1$ and ρ, p and ξ stand for mass density, pressure and bulk viscosity, respectively.

By use of a comoving co-ordinate system $v^\mu = \delta^\mu_0$ in the field equations

$$R_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu} R = T_{\mu\nu} \quad (3)$$

From Eqs. (1)–(3) we get

$$[(R^\bullet)^2 + k]/R^2 + (R^\bullet A^\bullet)/RA = \rho/3 \quad (4)$$

$$2R^{\bullet\bullet}/R + [(R^\bullet)^2 + k]/R^2 + 2(R^\bullet A^\bullet)/RA + A^{\bullet\bullet}/A$$
$$= -p + \xi(3R^\bullet/R + A^\bullet/A) \quad (5)$$

$$R^{\bullet\bullet}/R + [(R^\bullet)^2 + k]/R^2 = 0 \quad (6)$$

['•' is differentiation with respect to 't']

According to Wesson's theory the fifth dimension is parameterization of the rest mass, hence we must have $p_5 = 0$.

Hence from Eq. (6) we get

$$R^2 = K_1 + K_2 t - kt^2 \quad (7)$$

where K_1 and K_2 are arbitrary integration constants.

3. Solutions of field equations

In this section, we shall consider some restrictions on the behavior of the bulk viscosity coefficient.

Following Belinski and Khalatnikov (1977), we assume bulk viscosity co-efficient to be power function of matter density. Thus,

$$\xi = \xi_0\, \rho^n, \;\; p = \lambda(t)\rho, \quad \xi_0 \text{ and } n \text{ are constants.} \quad (8)$$

3.1. Case 1 ($n = 0$ and $k < 0$)

In this case one can adjust K_1 and K_2 such that Eq. (7) becomes a perfect square, we get

$$R = t \quad (9)$$

As mentioned earlier, we assume an equation of state $p = \lambda(t)\rho$ and using Eq. (9), the field equations (4)–(6) give a differential equation of the form

$$(A^{\bullet\bullet}/A) + (A^\bullet/A)[\{(2 + 3\lambda)/t\} - \xi_0]$$
$$+ [\{(3\lambda - 1)(1 + k)\}/t^2] - (3\xi_0/t) = 0 \quad (10)$$

From the condition of exactness of the above linear equation (10), a straight forward calculation (putting $k = -1$) gives

$$\lambda(t) = c\, t^{-1} - (\xi_0 t/6) \quad (11)$$

where c is an arbitrary integration constant.

Putting the value of $\lambda(t)$ from Eq. (11) in Eq. (10), we get

$$A^{\bullet\bullet} + A^\bullet\{(2/t) + (3c/t^2) - (3\xi_0/2)\} - A(3\xi_0/t) = 0 \quad (12)$$

Substituting $(A^\bullet/A) = u$, Eq. (12) transforms into

$$u^\bullet = -u^2 - u\{(2/t) + (3c/t^2) - (3\xi_0/2)\} + (3\xi_0/t)$$
$$= h(t)u^2 + g(t)u + f(t) \quad (13)$$

This is a Riccati type of equation which does not give any simple solution. As a test case one can choose a very simple form. As a particular solution of the Riccati equation, let $u = u_1 = (a/t) + b$. Using this expression, we get from Eq. (13), $a = -1$ or $a = -2$ for $c = 0$.

For $a = -2$, we do not get A in a closed form. Hence we do not consider this case.

For $a = -1$, the Eq. (13) is satisfied. By applying standard procedure (Murphy $p - 16$), we obtain

$$[u + (1/t)]\left[C_1 + \int e^{3t\xi_0/2} dt\right] = e^{3t\xi_0/2} \quad (14)$$

where C_1 is an arbitrary integration constant. We get from Eq. (14)

$$A = \frac{A_0}{t[C_1 + (2/3)\xi_0 e^{3t\xi_0/2}]} \tag{15}$$

where A_0 is another arbitrary integration constant. This is a new solution for a viscous fluid. As $t \to \infty$, $A \to 0$, we get dimensional reduction phenomena. Now we put $C_1 = -C_1$ and from the field equation (4)–(6), it also follows that

$$\rho = \frac{1}{t[(2\xi_0/3) - C_1 e^{-3t\xi_0/2}]} \tag{16}$$

$$P = \lambda(t)\rho = \frac{\xi_0}{6[C_1 e^{-3t\xi_0/2} - (2\xi_0/3)]} \tag{17}$$

The mass density vanishes at $t \to \infty$ with an initial singularity at $t = 0$, and also always remains positive.

We also study the dynamical behavior of the model. The four volume $V = R^3 A$ starts from zero at $t = 0$ and vanishes at

$$t_0 = (2/3\xi_0)\ln[3(C_1 + 1)/2\xi_0] \tag{18}$$

It also follows that the extra scale factor, starting from an infinite extension at the Big-Bang reduces to the Plankian length at the same point of the time t_0.

3.2. Case 2

When the viscous factor is absent, i.e., $\xi_0 = 0$ for c not equal to zero, we get $\lambda(t) = ct^{-1}$ and by solving the differential equation (10), we get back the solutions originally derived by Bhui (Bhui et al., 2005) for a perfect fluid.

3.3. Case 3

λ (t) should reproduce the observation values for different epochs of the evolution of the Universe. When $\lambda = 0$, 1, 1/3, we get back the different solutions originally derived by Banerjee (Banerjee et al., 1990) for viscous fluid.

4. Conclusions

We have presented here an interesting solution of the five-dimensional world in Kaluza-Klein theory, with an energy momentum tensor containing viscous fluid only. From the solution of the metric coefficient A, it follows that with time A goes to zero giving rise to the phenomenon of the dimensional reduction for an expanding model. When the viscosity coefficient ξ_0 vanishes, and $g_{55} \to 0$ our solution goes over to the solution of Hajj and Boutrous (1991) for a perfect fluid. From the expressions for mass density and viscosity coefficient it is shown that they are, in general positive and vanish at infinity with an initial singularity at $t = 0$.

We also consider the case of entropy production. From the field equations it follows that

$$\rho^\bullet + 3(\rho + p)(R^\bullet/R) = -\rho(A^\bullet/A) + (3R^\bullet/R)\xi\theta \tag{19}$$

If S be the entropy per baryon and n be the no density one can define in our case in a comoving frame (considering $p_5 = 0$).

$$nT(dS/dt) = \rho^\bullet + 3(\rho + p)(R^\bullet/R) \tag{20}$$

which being combined with (19) gives

$$nTS^\bullet = (-\rho A^\bullet/A) + (3R^\bullet/R)^2\xi \tag{21}$$

The relation (21) shows that the decreasing value of A, the first term causes the rise in entropy and second – a contribution from the bulk viscosity too enhances the increase rate of the entropy as used.

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