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Proofs, Mathematical Practice and Argumentation

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Abstract In argumentation studies, almost all theoretical proposals are applied, in general, to the analysis and evaluation of argumentative products, but little attention has been paid to the creative process of arguing. Mathematics can be used as a clear example to illustrate some significant theoretical differences between mathematical practice and the products of it, to differentiate the distinct components of the arguments, and to emphasize the need to address the different types of argumentative discourse and argumentative situation in the practice. I consider some issues of recent papers associated with mathematical argumentation in an attempt to contribute to the discussion about the role of arguing in mathematical practice and in the evaluation of the products of this practice. I apply this discussion to learning environments to defend the thesis that argumentative practice should be encouraged when teaching technical subjects to convey a better understanding and to improve thought and creativity.

Keywords Argument · Argumentation · Mathematical practice · Learning · Proof

1 Introduction

Since antiquity, logic has been considered to be a normative theory of reasoning and as such has been applied to the analysis and evaluation of arguments. In the last

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century, Toulmin (1958) noticed a gap between logic as a theory of argument and real practice; his work gave rise to new ways of conceiving and theorizing about arguments. The study of fallacies was pivotal in the development of the field of argumentation and emphasis on avoiding fallacious arguments when arguing in natural settings was one of the driving forces behind the new theoretical proposals. In part because of these origins, almost all theoretical proposals in the field of argumentation apply to the analysis and evaluation of arguments, mostly in written texts.

Johnson (2000) states that a theory of argumentation, considered as a theory of the practice of argumentation, has to consider many aspects not included in the study of its products. He defines argumentation as "the socio-cultural activity of constructing, presenting, interpreting, criticizing and revising arguments" (p. 12). For Johnson, a theory of argument is a component of a theory of argumentation and he considers that proper work has to be done first towards a better theory of argument in order to have a balanced theory of argumentation.

Much work has been done to present actual examples of arguments as they appear in real practice but, nevertheless, it is still true that, as Hitchcock (2002, p. 288) remarks of Johnson's examples of argumentative interchanges, "[they] do not exhibit at first glance the features of one person interpreting and criticizing an argument and the argument's author revising it in response to this criticism", features he considers constitutive of the practice of argumentation.

The relationship between the theory and practice of argumentation has been reconsidered again in several recent papers, for example, those by Pinto (2001), Johnson (2005) and Kvernbekk (2012). The positions of the authors differ, but, when talking of practice, they usually consider specific arguments as they appear in argumentative exchanges in order to analyze the distance a normative theory of argument should maintain to be of any value to evaluate practice. In general, they want to assess the arguments as part of the activity, but not the activity as a whole.

The ideal pragma-dialectical model of a critical discussion is "based on analytical considerations regarding the most pertinent presentation of the constitutive parts of a problem-valid procedure for carrying out a particular kind of discursive task" (van Eemeren and Houtlosser 2005, p. 75). The emphasis is put in the activity, but here too, its application is devoted to the analysis and evaluation of the argumentative products. This is the case even in the latest attempts to look at the properties of what the above cited authors call activity types, defined as "conventionalized entities that can be distinguished by 'external' empirical observation of the communicative practices in the various domains […] of discourse" (p. 76).

In this paper, I consider some issues of recent papers associated with mathematical argumentation in an attempt to contribute to the discussion about the role of arguing in mathematical practice and in the evaluation of the products of this practice. I argue that, in mathematical practice, argumentation considered as a rational, social and communicative activity should be encouraged in order to improve collaborative, efficient and creative work, but this does not necessarily imply that direct application of the current theories of ordinary argumentation to evaluate its products should be undertaken. The particular constraints of mathematical activities, for example, the rigor required for mathematical definitions and proofs and their institutionalized forms, are sufficient for their evaluation in the different contexts in which they arise.

Application of problem-solving strategies has proved helpful for the successful accomplishment of mathematical tasks and the understanding of difficult mathematical concepts. In those cases, argumentation may be of help not only to raise and solve problems, formulate hypotheses, ask for justification of inferential steps, construct explanations and test one's understanding, but also to establish relationships between concepts and the application of methods in different situations. That is, argumentation may be of help if mathematics is considered as a critical and collaborative inquiry to look for a solution to a problem. Nevertheless, adaptation to the specific activity type and the actual context may have to be taken into account to design the tasks and argumentative practices that trigger collaborative and effective work.

2 A Look at Proofs, Arguments and Mathematical Practice

Discussion on the nature of mathematical proof has a long history that is beyond the scope of this paper. I refer only to some recent contributions that link the idea of proof, argument and the kind of processes that can be found in different contexts of mathematical practice in order to support my claim that argumentative practice is important to promote understanding, and to improve thought and creativity, including in mathematics.

Johnson (2000, p. 168) defines argument as "the distillate of the practice of argumentation". For him, in addition to the reasons to support a claim (the illative core of the argument), "an argument possesses a dialectical tier in which the arguer discharges his dialectical obligations". As he considers that mathematical proofs do not have this "dialectical tier", they are not (paradigmatic) arguments. As Tindale (2002) points out, "this concept of argument is hampered by an internal tension between the product an argument is and the process it captures" (p. 299) because mathematical proofs appear in many types of argumentative situation and the idea that mathematical proofs are more than chains of deductive inferences is nowadays supported in very different fields or disciplines.

From the field of argumentation, Aberdein (2009) presents an insightful recompilation of references in which authors appeal directly to studies on mathematical proof and mathematical practice. In this paper and in many others (see references), Aberdein considers that much of what mathematicians do, in particular proofs, may be understood as a "species of *argument*", considering it "an act of communication intended to lend support to a claim" (pp. 1–2). Several authors (for example, Alcolea Banegas 1998; Dove 2009; Aberdein 2010) consider that the way mathematicians analyze and assess mathematical reasoning is closer to the way informal logicians analyze and assess ordinary arguments than what the convention about mathematical proof asserts, namely, that the reconstruction of a mathematical proof should conform to a chain of valid deductive arguments. As a consequence, they think that elements of the new theories of argument(ation) may be of help to understand mathematical proofs as given in practice. Krabbe (2008) distinguishes different types of mathematical activity with various objectives and examines examples of strategic maneuvering in mathematical proofs.

From the field of the philosophy of mathematics, George Pólya and Imre Lakatos's pioneering work to present proofs based on their own heuristic experience did not resonate with mainstream work in philosophy of mathematics, except perhaps for the application of their proposals to mathematical education. The deductivist and logicist approaches to science in general, and to mathematics in particular, were the conventional approaches during the majority of the last century. The emphasis on the objects or results promoted losing sight of the processes by which they were obtained (Ferreirós 2010). Nowadays, there is a widespread eagerness to overcome the foundational view on mathematics that considers mathematical theorems a priori truths that are there, in the void, waiting to be discovered. Instead, there is a new emphasis on understanding how proofs are developed and built and many philosophers of mathematics are now concerned with the kinds of activities mathematicians perform, that is, how the practice of mathematics is actually carried out (Mancosu 2008). There is a turn to the practical (Gabbay and Woods 2005) and the dividing line between the pair product/process, the first conceived as an object of analysis subject to a normative evaluation, and the second seeking to accommodate the descriptive adequacy of real practices, has begun to blur. Aberdein (2011) provides us again with a good summary of references of works that try to integrate insights from many fields into a new philosophy of mathematical practice.

In traditional mathematics education, mathematics consists of readymade perfect products transmitted directly by the teacher in an authoritative but also authoritarian way. All is set up to be accepted. The process of discovery or of construction that led to these products is hardly considered. Communication is oriented mainly to the explanation of difficult steps in proofs, the resolution of repetitive problems as applications of the theory and the assessment of the solutions. Argumentation (in the ordinary sense of the term) among students or with the teacher is not usual. The focus is on knowing that something is the case, not on how it can be constructed. This fact can be explained in part by the difficulty that many students encounter in assimilating abstract concepts and by the need to address very long curricula. At elementary levels, a constructivist approach to mathematics is nowadays more common, but as soon as the contents of the curriculum accumulate, the traditional way of doing mathematics is still prevalent in many countries. As a consequence, many students give up on understanding mathematics and apply the results or methods in a mechanical and rote way. If we think of education as a way of pursuing a method to construct knowledge in the mind of the student, the classical approach to mathematics is clearly not the ideal.

If we look at the practice of mathematics in any particular setting, we soon realize that, to establish the right path of valid inferences that lead to the solution of a problem, we first have to perform many different activities. For example, we have to conceptualize the new ideas that can be of help to solve the problem and to do so, we have, maybe, to translate it into another more familiar domain. We also have to find a proof strategy able to solve the problem and, to arrive at that, we have to identify a promising direction to find the solution and/or to dismiss other directions.

We may have to explain to ourselves or to others the reasons to adopt or to reject this strategy, that is, to explain why we think this direction is appropriate or why it will not work. We have to confirm that the strategy works by being able to express the particular details that conform to it; to do so, we may have to express those technical or difficult details that lead to the solution to make it comprehensible to others and, at the same time, we may have to eliminate those details that at first seemed to be necessary, but that finally are not. In addition, we may have to go back to the first formulation of the problem to readdress its initial conditions by including additional preconditions to accommodate the solution found. We may also look for ways to adapt the problem and its solution to an actual problem in a specific field. We may want to refine the proof to make it clearer or more elegant. Finally, we may have to communicate the problem and its solution to different audiences.

The ideas involved in these tasks are in many cases tentative, incomplete or even incorrect and have to be developed or explained in order to be included in the final presentation of the solution or in the proof of the theorem. Not all this material is included in the final product, but all of this is part of the mathematical practice that leads to the solution. Those intermediate steps towards the solution are important to understand how mathematics works.

If we think of practice in mathematics as a set of complex activities and tasks to be performed in order to solve a mathematical problem, not only logical reasoning but also good arguing is a very valuable tool to improve understanding and creativity, as I try to show in the following sections.

3 Mathematics and Argumentation

The influence of Pólya and Lakatos or even of Toulmin's model[1] was crucial to the pioneers of mathematical argumentation. Recently the literature on this subject has increased and the publication of a special issue on "Mathematical Argumentation" in *Foundations of Science* (Aberdein and Dove 2009) and, afterwards, of the book *The Argument of Mathematics* (Aberdein and Dove 2013) are clear proof of the interest raised by this subject. In those papers, many different aspects of proofs have been reconsidered from the point of view of argumentation theory. For example, Dove (2009) presents many mathematical examples to try to show that the method by which mathematicians assess mathematical reasoning resembles the practice of informal logic or argumentation theory. Alcolea Banegas (1998), Aberdein (2005) and many others (see Aberdein 2009 for references) try to adapt Toulmin's layout to mathematics. Aberdein (2010, 2013) and Dove (2009) consider how some of the argumentation schemes in the work of Walton et al. (2008) may be of use in evaluating mathematics.

In many of these papers, the main concern is to show that there is argumentation in mathematics. To do so, the authors often discuss examples of problems, proofs or

[1] Toulmin's model was used to analyze a regular argument in Toulmin et al. (1979), p. 126. I would like to thank an anonymous referee for bringing my attention to this and other references that helped me improve this and others points. Many thanks are also due to the second referee for helping me improve this paper.

different kinds of mathematical error, either to show how mathematicians evaluate them in practice or to show that the way in which the neat final results were reached was to refine previous faulty results, using to that end ordinary communicative forms in a language that was not totally formalized. Either way, evaluation and refinement of results, the authors claim, can have a parallel in ordinary theories of argumentation.

To give an example, Dove (2009, pp. 140–141) comments on one of the faulty proofs in the work of Maxwell (1959) in which the proposed task is to prove that any given triangle is isosceles.[2]

First of all, maybe there could be someone, somewhere, who considers this a true problem and is trying to find a *proof* for it, but it is difficult for me to imagine such a situation, unless as a kind of mathematical joke. Dove acknowledges this fact and states that the proposed conclusion is obviously absurd but also makes the defense that there are cases in which believing that a mathematical conclusion is false is not enough to warrant rejection. Moreover, his aim is to use the example to show that informal logic techniques are employed to find errors in faulty proofs.

The proposed *proof* begins with a diagram of a triangle that clearly is not isosceles, which seems an odd illustration for the *proof* of the given proposition. After that, a very detailed notation is used; there is an appeal to two mathematical theorems (the angle bisection theorem and the sine rule) and a series of careful mathematical steps are given. That is, the method used is mathematical. Then, the supposed author of the *proof* makes the error of considering that, from the equality of sines, the equality of angles follows.

The error is trickier to find in the original example due to many of the factors already cited (use of graphics, notational details, the appeal to mathematical theorems, etc.). Dove thinks that being rational entails, somehow, wanting to discover the error to relieve the tension caused by the false claim in one's system of beliefs. As already mentioned, he uses this example to illustrate how people proceed when confronted with an evidently faulty result and states that "the process one uses to discover the mistake is analogous to the process one might use to criticize an unpalatable argument in non-mathematical settings" (p. 142).

Nevertheless, in a classroom setting, a teacher would not need much time to discover the error. It is a common error made time and again by students of elementary trigonometry and she would mark it as an error to be corrected without any more explanation or argumentative discussion.

A teacher could use a faulty proof like this one to introduce her students to the manipulation of mathematical notation or to the study of mathematical notions or well-known theorems. I think that if this kind of task is appropriately weighted to take into account the mathematical knowledge and the training of the students, informal argumentative dialogs can arise as part of the process of finding the error. However, in my opinion, the steps in the *proof* by Maxwell are too rigorous and the error is too simple to trigger productive argumentation in normal classroom settings.

[2] Examples from Maxwell can also be found in Aberdein (2010). A very similar example to this one from Wikipedia can be found in Krabbe (2008).

Dove is right to say that when trying to find an error in a proof or even when attempting to prove a proposition, mathematical practice does not need to formalize all the steps and that natural language mixed with mathematical notation is always used. Nevertheless, I think that, in order to learn more about the nature of mathematical practice and how its products are evaluated, we should be looking at real examples of this practice, including the contextual elements of the situations in which they are produced. For instance, an example of real practice is presented by Pease and Martin (2012) or in many of the works on mathematical education cited in Sect. 4 of this paper.

Other lines of defense on the argumentative nature of mathematics are the appeal to the axiom of choice (Alcolea Banegas 1998; Dove 2009) the surveyability of long proofs (Coleman 2009) and the use of mathematical diagrams (Larvor 2013). All of these try to underline the "challengeable" nature of at least some mathematical proofs and to defend the idea that some results can be accepted not only because of their validity, but also because they are useful for mathematics. This being true, it is also true that many proofs in mathematics are deductive.[3] However mathematical proofs are also communicative products. To be understood and, as a consequence, to be accepted, they have to be worded and presented in different ways to adapt to the audience to which they are directed. The presentation of the products is part of the rhetorical component of an argumentative process, but in many of those examples, the need to stress the dialectics in proofs does not reflect the rhetorical component of them. It is important to study the products to understand how mathematicians work, but to differentiate the different components in them we need to look at real products arising from and situated in ordinary mathematical practice. Kuhn (1992) states that "thinking as argument arises every time a significant decision must be made" (p. 157) and many difficult conscious decisions have to be made in conditions of uncertainty on the way to obtaining a proof. It is this point that I want to stress in the remainder of this section.

Pólya (1945, 1954) and Lakatos (1976) are cited in many works that try to emphasize the plausible and heuristic nature of mathematical practice but, then, examples tend to show the argument in the finished products instead of the argument in the practice. As noticed by many authors, mathematical practice is not always successful and in many cases it creates "knowledge" that is neither precise, rigorous nor certain (Chazan 1990). It is, I think, in this process that the dialectical part of the activity arises in a natural way. For example, when working toward a proof or looking for a solution to a problem uncertainty may be present, objections to an inferential step may be raised, explanations for a definition may be required or even an argumentative diagram may be of help to make clear where we stand or what we need to achieve a proof.

From a review of many of the papers cited above we can extract two main ideas. First, many authors try to justify the claim that mathematical products are argumentative to link aspects of proofs with theoretical notions in argumentation

[3] With this statement I am not claiming that mathematical proofs are or should be pure logical or formal proofs, but that their underlying inferential structure is deductive.

theory. Second, there is a manifest tension in these works between the examples of mathematical products considered as arguments and the process that leads to them.

At first glance, Krabbe (2008) seems to avoid this problem because he takes note of the "various contexts in which proofs occur and of the various objectives they may serve" (p. 453). He also proposes a list of contexts of proof and the supposed functions of reasoning in them by their association with different types of dialog:

1. Thinking up a proof to convince oneself of the truth of some theorem;
2. Thinking up a proof in dialogue with other people (inquiry dialogue; probative functions of reasoning);
3. Presenting a proof to one's fellow discussants in an inquiry dialogue (persuasion dialogue embedded in inquiry dialogue; persuasive and probative functions of reasoning);
4. Presenting a proof to other mathematicians, e.g. by publishing it in a journal (persuasion dialogue; persuasive and probative functions of reasoning);
5. Presenting a proof when teaching (information-seeking and persuasion dialogue; explanatory, persuasive, and probative functions of reasoning; Krabbe 2008, p. 457).

The first two types of activity are not considered argumentative owing to the characteristics associated by Walton and Krabbe (1995) with the type of dialog in which they occur. For Krabbe (2008), probative functions are intended to extend knowledge and are not argumentative. In order to have persuasive functions, the aim should be to convince another by overcoming her doubts (p. 457).

Nevertheless, in the situation of thinking up a proof or of looking for the solution to a problem there is uncertainty, and dialectic situations can appear (clearly when thinking up the proof with other people, but even when stating the problem to oneself). When trying to establish the inferential structure that glues together the initial conditions and the solution or the claim, there are many situations in which a choice has to be made in conditions of uncertainty. There may be many methods to try that could be of use. You may need to persuade the others (or yourself) that a particular path of inquiry is better than another or that some conjecture is adequate to solve a problem. There may also be situations in which, although the inference seems valid, you may ask yourself or other people to look for possible counterexamples to the claim. There may be cases in which the solution to a problem is already known, but, nevertheless, you may want to try it with specific examples before thinking about how to prove it, and so on.

Another difference between the first two situations and the other three in the list that might be invoked is that, when thinking up a proof, there is no need for language interaction. Language is an important tool for thinking in mathematics (Thurston 1994). The need to formulate careful definitions and the use of specific notation seem fundamental to advance the construction of a proof. Thinking up a proof with other people without linguistic interaction seems impossible and, if communication is needed, it is difficult to distinguish this case from that of a presentation of a proof to other people (case 4) except for the fact that, here, there is uncertainty involved and dialectical and rhetorical elements have, in my opinion, a role to play. I think that in the process of thinking up a proof, the dialog types are generally complex. It could be of use to separate the five types of activity for theoretical purposes, but at least the first two appear in many cases to be mixed up with one of the others.

A problem is usually proposed or considered in a specific contextual situation (be it a classroom context, an academic situation or even a proposal to solve a problem through the internet) and, although the solution to it may be unique, if we consider the definition of argumentation given by van Eemeren et al. (1996), which considers argumentation as a verbal and social activity of reason aimed at increasing (or decreasing) the acceptability of a standpoint, the activity of looking for a solution stands in accordance with this definition.

Krabbe considers the dialectical component of a proof to lie in the number of inferential steps it contains. I think that inferential steps correspond to the logical part of the proof, and dialectical and rhetorical components correspond to the communicative situations in which mathematical practice is undertaken. If by "the number of inferential steps" Krabbe is referring to the use or not of logical gaps, I agree that this depends on the audience towards which the communication of the proof is directed but, as I have already said I would consider the use or not of gaps as part of the rhetorical process of the communicative act.

Presentation of mathematical products is a communicative act and, as is usual in such acts, not all the communicational elements are made explicit. In mathematical proofs, gaps are intentionally left, and those gaps need not correspond to faulty inferential steps (Fallis 2003). In many cases, several steps of the inferential process are left out to facilitate communication and to adapt to the context. For example, a long proof with all the small inferential steps made explicit may be boring for working mathematicians. An outline of the proof or of the problem may be sufficient and more informative than a complete proof in a classroom or in a scientific meeting. We can express the difference by saying that, in this case, we are making someone see the proof versus letting someone know the proof (Vega Reñón 1999).

All the situations in the list are communicative events in which an audience is involved (in the first one, only oneself). When communicating mathematical work, abstract notation, uses of previous works and deductive inferences are always involved. Because of this, explanation of some inferential steps or redefinitions of some concepts may be needed to enhance understanding and to accept the proof. As Mancosu (2011) states, demands for explanation in proofs do not always come with a new proof, that is, the logical part of the proof may remain the same but new information to bridge an inferential gap may contribute to reinforcing the proof, making it clearer and more convincing for the addressee. I consider those exchanges part of the dialectical component of the argumentative exchange.

Besides, there could be some (easy) cases where persuasion (or conviction) could be reached only by understanding the inferential structure in the proof, even with gaps in it. In those cases, when presenting the proof to others, rhetorical and communicational elements would surely be present, but not necessarily dialectical moves. For example, in a presentation of a mathematical result, there can be (or not) demands for a better explanation of it, and there can be (or not) requests for a better, clearer or more detailed display of the steps in the proof or some of the concepts involved in it. The appeal to diagrams, images, analogies or rhetorical figures may be not only helpful but even essential in order to make the result understandable and, as a consequence, acceptable for the (mathematical) audience. As a result, doubts, objections and even a display of counterexamples or a rebuttal of the proof can occur; that is, an argumentative dialog may begin, but does not have to.

To cite an example, consider any proof involving the notion of infinity. For instance, we can ask a high school student or a philosophy student of formal logic to prove that the cardinal of the set of natural numbers is equal to the cardinal of the set of even numbers.[4] The proof is easy: all they have to do is to prove that the function f(x) = 2x from the set of natural numbers to that of even numbers is bijective. It follows directly from the definition of bijection and can be done in a mathematical way without much effort. Nevertheless, it is my experience that students do not accept easily the result either before or after the proof. They may acknowledge the validity of the proof but they may not be convinced of the truth of the claim. There are always questions, commentaries, or reactions against the claim because they are used to an intuitive notion of infinity that is more or less a synonym of "a lot".

Students usually insist that there are more elements in the set of natural numbers than in the other set, namely, all the odd numbers. That is, they try to put into question the claim by means of a supposed counterexample. An answer to this objection can be to draw a list to show, in a graphic way, how the elements of both sets can be paired. Similarly students may propose as a counterexample the existence of an injective function between the two sets and afterwards they may try to compare this case to that of finite sets. When they realize that it is not possible to have a non-bijective function between finite sets of the same cardinal, the teacher can define an infinite set as a set that can be put in bijective correspondence with a part of itself. Some may ask for examples of other infinite sets to see if the definition applies. It is usually necessary to answer all their objections and even use several rhetorical elements in the presentation of the proof (such as a large list showing how the elements can be paired effectively) to adapt to the audience and to persuade the students.

Something similar happens with the proof that there are infinities larger than others. Many students recognize Cantor's diagonal method as valid but they do not accept the result and, as the proof is more difficult than that of the former example, here they usually try to find a faulty step in the application of the method. Cantor's diagonal method is a proof obtained by contradiction. In the proof we suppose that all real numbers between 0 and 1 can be enumerated and we use this enumeration to construct a new number that is not in the list. Students with poor mathematical training usually think that there is a trick in the way the list of real numbers is created or, also, in the way of expressing the numbers in the list if abstract notation is used. They may also raise objections about the way of constructing the new number that is not in the list. In many cases it is necessary to provide an example of

[4] The problem need not be formulated exactly like this; it can be presented in a more informal way.

a more or less extensive list of real numbers to show in more detail how this new number can be constructed.

If prompted, students may also try to compare the case of the real numbers between 0 and 1 with the set of all the real numbers or with the set of all the points in a plane. Few of them would be able to achieve a mathematical proof but some of them may be able to give reasons for or against the claim that these sets have the same cardinal. If their mathematical training is not adequate, an informal explanation of the proof may be more convincing to them than the actual proof.

Other students may try to compare the cardinal of the set of rational numbers and that of the set of real numbers and, as a consequence, they may try to extend the above-stated result to the set of rational numbers to claim that the cardinal of this set is larger than the cardinal of the set of natural numbers. It may be necessary to demonstrate that they are wrong, for instance by providing a visual method to enumerate the rational numbers.

Surely the same theorems when presented to students of mathematics at the university level would not raise so many objections. Probably, the proofs, if correctly presented and understood, would be enough to convince them.

To finish, I think that the necessity of a final proof to assign persuasive functions to a mathematical situation prompted Krabbe's list. There is again the tension between the practice and the products of this practice. There is also the legacy of the deductivist approach to mathematics and, perhaps, the added difficulty of observing practice in different real situations.

As I have already stated, much more work is needed to observe and understand the relationship between actual mathematical practices and argumentation in different contexts in order to design protocols that can help in the development of a better understanding of, and to improve thought and creativity in, mathematics. Empirical research from the field of mathematical education could be of help to understand better how mathematics and argumentation are handled in the classroom. The analysis of Pease and Martin (2012) also represents a good step to explore another type of context, that of online collaborative work to solve a challenging mathematical problem. Finally, it is worth considering the first exploration of Van Bendegem and Van Kerkhove (2009) to situate mathematical arguments in context, by looking at their commentaries on the organization of a large research program to prove a difficult theorem, and on the mode of presentation of a paper by Pólya.

4 Arguing, Proving and Learning in Mathematics

Learning has traditionally been defined as the integration of new information with existing knowledge (Andriessen 2009). However, a learner's previous knowledge can inhibit the integration of new information because this knowledge may have proved efficient in different situations in the past (Balacheff 2010). In my opinion, a good way to overcome this problem is by argumentation. Mathematical ideas may not be a matter of opinion or belief, but arguing is important as a type of communication to fix a (shared) understanding of mathematical concepts, to improve inferential steps, and to question the solution to a problem; in other words to convince a student of the truth of a theorem or of the solution to a problem. Although some researchers in mathematical education consider that "the key role of proof is the promotion of mathematical understanding" (Hanna 2000, p. 5), a more careful look at what those authors mean by proof shows that, for them too, it means much more than mere syntax or a chain of valid deductive steps, and includes ordinary argumentative elements to promote understanding.

Schwarz (2009) presents an outline of the complex relationships between argumentation and learning. To begin with, there are different approaches to the definition of what constitutes learning, and each conception determines the role of the argumentation in the classroom.

For some psychologists, learning is a psychological change in the individual that can be observed indirectly between successive activities. For others, learning emerges through interactions. These two views may not be incompatible, but they have been considered as if they were so from a theoretical point of view and they can be representative of the way mathematical education has been undertaken throughout history.

While the traditional view maintains that the acquisition of mathematical skills is individually undertaken, a more accurate view considers that for an average student, interaction with peers and a teacher is essential. When adopting the second view, many researchers think that the role of argumentation is central, even in mathematics (Muller Mirza and Perret-Clermont 2009). Nevertheless, the relationship between argumentation and learning in mathematics is complex because, in learning, there are multiple processes involved and different forms of implementing them. In the following, and without aiming to be exhaustive, I will try to illustrate some of the processes that empirical findings show can be improved in mathematical learning by means of argumentative dialogs and a good design of the task. Most of what has been done in this respect comes from researchers in the field of mathematical education, and the references I will use are from researchers in this field.

One of the processes involved in mathematical learning is the conceptualization of mathematical ideas. Empirical research proves that argumentation may represent an important tool that can be used to intervene in the progressive construction of basic mathematical concepts and in the development of consciousness and systematic links when it is guided by careful mediation of the teacher and a well-designed task, which has to take into account the appropriateness of the activity with respect to the class (Douek and Scali 2000). As these authors have shown for elementary education, the relationship between uncertainty and communication through argumentation can serve as a basis to encourage questioning, expression and evolution of conceptualization in mathematics. They show how children can work from a particular task to measure the height of a particular plant with a ruler that does not have the zero mark at the edge, towards a general solution that involves the idea of the invariability of measurement through translation or the possibility of adding two different kinds of measurement (that of the height of the plant and that of the distance between the edge of the ruler and the zero mark). To reach those general conclusions, children have to give reasons to the teacher to justify a particular way of solving the problem and they have to back their findings with data. Also, when confronted with contradictory findings they have to dialog with peers to back up or rebut one of the contradictory proposals.

Outside the classroom, another example to show how conceptualization of mathematical ideas evolves rapidly through dialogical interaction of informal ideas is shown in the analysis of the third Mini-Polymath project (Tao 2011) in Pease and Martin (2012). They show that 23 % of the comments on the problem analyzed were made to propose definitions developed in a variety of ways: analogies to relate the problem to other areas better known by the participants, correction of misunderstandings by showing counter-examples to the others' proposals, use of conjectures that afterwards could be proven right or not, etc.

Another process in mathematical education is the acquisition of reasoning skills. Several researchers have stressed the psychological gap that separates arguing and proving in the classroom (Schwarz 2009). As we have already stated about the notion of infinity and Cantor's diagonal method, in many cases, the presentation of a proof is not persuasive enough to convince a student of its validity (more examples can be found in Duval 1991; Healy and Hoyles 2000). For instance, Healy and Hoyles (2000) present an empirical study of high-attaining 14- and 15-year-old students about proof in algebra. In this study they found that students differentiated between mathematical arguments they considered would receive the best marks and arguments they would adopt for themselves. Algebraic arguments were considered as pertaining to the first class but most students preferred empirical arguments and arguments that they could evaluate and that they found convincing and explanatory, and those preferences excluded algebra.

For average students, instead of an authoritarian presentation of a proof, argumentative dialogs between them and/or with the teacher may help to bridge the gap that separates arguing and proving because proofs are then conceived as constructions built up through an interactive process that looks for the understanding and the acknowledgment of the student, who has to explain all the steps of the inferential process. Nevertheless, careful guidance in the process may be needed to transform spontaneous and not fruitful or even authoritarian interchanges into argumentative situations that are of help in understanding the problem (Atzmon et al. 2006).

The educational system does not facilitate the development of good argumentative practices in non-elementary levels of education in mathematics. The pressure to cover all the material leaves insufficient time for arguing in mathematics classes, and thus students merely assume the value of proof in mathematics, even if they do not fully understand how it works. They assume that their lack of understanding is due to a lack of knowledge. Discussion to promote understanding is not common in mathematics classes in non-elementary levels of education. As soon as the curriculum becomes more advanced, this lack of understanding presents an obstacle to many students, and the gap between the application of the theory and the practical work they need to do to solve problems may widen. Repetition of techniques is a typical way to acquire mathematical knowledge. As a consequence, many students do not fully understand what they are doing and fail when a different kind of problem is proposed or integration of different concepts is needed.

One possible way to improve understanding in mathematics could be to substitute some of those repetitive tasks with works that involve argumentative dialogs with peers or with the teacher. Argumentative dialogs can be used to attain different goals, depending on the context in which they arise. For example, a mathematical problem in the classroom can be presented as a kind of collaborative task in which two or more parties work together to find a solution. Some good examples to illustrate this can be found in Schwarz et al. (2010). Those authors comment in an example by Prusak (2007) that to resolve a problem "includes tasks encouraging intuition through visualization, geometric constructions and measurements to progressively foster deduction" (p. 119). The example shows two students working in pairs who are asked to find a point located at an equal distance from the four vertices of a rectangle (depicting a park in which at each vertex there is an attraction, and the equidistant point to find is the location for the ticket booth of each attraction).

The students start to work separately, and their initial approaches are very different. One of them backs her hypothesis on a previously known theorem—diagonals in a rectangle are equal and intersect at their middle points—while the other, after drawing the diagonals of the rectangle, folds his worksheet twice at the middle points of the opposite sides to find out that the intersection point is the same point as the intersection of the diagonals. Afterwards he checks the distances to this point with a ruler. Later on they begin to work together and are allowed to use DG software to check their solution. The first student confirms the result of the second student (i.e. that the intersection point of heights coincides with the intersection point of diagonals in a rectangle) by providing an explanation based on a theorem (the lines in the folds are the heights of the isosceles triangles created by the diagonals), but the second student does not recognize this explanation and uses the software to measure and to verify his point.

In the second task, the park's shape becomes an equilateral triangle. The problem is solved adopting a common intuitive-visual approach. The students agree that for equilateral triangles all intersection points coincide, but they also conjecture that for other triangles the solution should be different because they use the DG software to modify the shape of the triangle. They have different possible solutions to this conjecture: for the first student the solution is the point of intersection of heights; for the second student it is the intersection of medians, and for both, using the software, another possible solution is the intersection of angle bisectors.

In the third task, they have a conflict when asked to analyze the case of a scalene triangle. The first student proposes her solution, the intersection of medians. The second student uses visual considerations to check and rule out the other two conjectures, that is, the intersection of heights and the intersection of angle bisectors as a solution to this task. As a consequence they are left with only a possibility. Student two admits that one cannot rely solely on visual considerations but that they may be used as counterexamples to refute conjectures. They check their remaining conjecture with the software and see that it is also wrong. As a consequence, student two thinks that there is not a solution to the problem. Student one does not accept this conclusion and continues looking for a solution. The drawings they have used to reject their previous conjectures remind her of a vague relationship between a triangle and its circumcircle. Student two tries to understand the new idea by drawing a circle circumscribing the triangle and marking approximately the center of the circle. They see that the center of the circle may be outside the triangle. Student one proposes considering this point as a possible solution to their task, that is, she proposes to start an abductive argumentation (if the solution is this point, then the distances from each vertex of the triangle to the point are equal, and the distances coincide because they are the radius of the circumscribed circle). They are not able to conclude whether or not they are wrong from their drawing but student one realizes that the three triangles that have the circumcenter as a common vertex, and two of the vertices of the scalene triangle as the two other vertices are isosceles. Taking into account this statement student two is able to establish the deductive steps they should take to construct the point and to prove that their answer is correct. He proposes considering each side of the scalene triangle as the base of one of the isosceles triangles and he shows student one how to build the circumcenter (first they have to find the middle points of each of the sides of the scalene triangle and then draw the perpendicular bisectors of the sides of the triangle). From this construction they can prove that the above mentioned triangles are isosceles and that they have the solution to the task. He also draws a sketch to convince student one but he realizes that "constructing the solution means an accurate step-by step planning based on logical necessity" (p. 122). They finally use the software to construct their solution.

The experiment shows that the students worked together, interchanging different kinds of argument to test their views and to solve the problem. Some arguments were based on theorems, others were counter-examples to refute conjectures and others were based on manipulation of geometrical forms by some specific software followed by visual considerations of the results of the manipulation. Some arguments were finally discarded but the consideration of all of them was necessary to complete the assigned tasks and to make it possible for both students to acknowledge that a proof should be constructed step-by-step following deductive reasoning. The students were not asked to give a formal inscription of their proof, which as Schwarz et al. state, is a gap difficult to bridge at this level of instruction, but they were able to progress from intuitive proposals to deductive thinking in order to convince each other and construct the solution. Their proposal could be used afterwards to generalize the results to any polygon that can be inscribed in a circle.

Another goal of an argumentative dialog can be that of developing competences related to critical reasoning. For instance, students could be asked to look at the different solutions proposed to solve a task in order to compare them, establish relations between them, and evaluate them. A simple first-order equation can be solved, for instance, by algebraic or by geometric means.[5] It can be solved by making a table with a table function of a graphic calculator or by drawing a graph and finding the solution as the intersection point of two lines (for example if the equation is expressed as $2x - 10 = x + 8$). Students can be questioned about the "legal" or mathematical validity of a step in their proof or about the possible

[5] This example is an adaptation of the example provided in Chazan and Sandow (2011).

generalization of this step or of the method chosen. They may be asked to justify the strategic choice of an assignation of values to make a table or a graph or questioned whether the assignation of these values would work if the solution was too big or not a natural number. They may be asked to justify the choice of the method used. The solution to a problem may be unique, but comparison of methods may help to improve the critical assessment of mathematical methods and could be important to understand notation or abstract concepts.

Moreover, in order to achieve good cognitive development in mathematics, the above cited experiments and many others show that it is important that the student learns to argue, but also that she argues to learn in different contexts with different goals (Andriessen et al. 2003).

Nevertheless, not all the dialogical attempts to use argumentation as a collaborative method to solve a problem and to improve understanding are successful (Andriessen 2009). Douek (2005) shows that mediation by an instructor may be needed to trigger productive argumentative practices, otherwise the accomplishment of a task could demand a large amount of time. For example, it could be necessary to question or reject statements that are not really helpful in understanding a problem, to integrate discussions or arguments provided by different students, and to generate and integrate new statements. For example, when proving that there are infinities larger than others, it is usually necessary to prompt questions to open or to extend the discussion to other sets of numbers that do not appear directly in the proof (the rational numbers, the real numbers, the plane). The intervention of the instructor is also usually necessary to give hints and to direct discussions that lead to the right conclusions about the cardinality of these sets.

As a consequence, it is important that teachers have pedagogical and theoretical skills to foster argumentation in the classroom. Schwarz et al. (2010) acknowledge that the design of activities to trigger productive argumentation is a difficult task that needs to take into consideration many things: the students' mathematical knowledge, the sociological composition of groups, the tools to be used in order to help the students to solve a problem, and also the different strategies to be taken by the teacher to trigger productive discussions or to redirect them in case of lack of new ideas. Only in this way can argumentation serve as a tool to promote understanding and reinforce reasoning skills, and as an efficient method to achieve good results in mathematics.

Researchers in the field of mathematical education mostly use only a general definition of argumentation stated by the pragma-dialectical school[6] and Toulmin's model as a method of diagramming an ongoing problem. The examples of mathematics used by many researchers in the field of argumentation are somehow artificial and, in my opinion, are not based on real practice. I think that both fields can benefit from the other's findings to advance into the design of tasks to trigger productive argumentation that helps in producing and refining methods of proof in mathematical problems.

The discovery part of a proof is possibly the most difficult phase of any mathematical work. As Kerber and Pollet (2007, p. 87) state, deduction systems

[6] This definition is cited, for example in (Schwarz et al. 2010) and in (Balacheff 2010).

may be suitable as proof checkers in many cases, but lack the capacity to act as proof assistants for the exploration and construction of new mathematical knowledge. Argumentation theory and direct observation of real mathematical practice may be of help to design protocols to facilitate mathematical work, but as Pease and Martin (2012) remark, we are still a long way from a system that could contribute in a human-like manner to a mathematical discussion that has the goal of solving a problem.

5 Conclusions

In this paper, I have presented several considerations from mathematics to try to show some of the differences between the product of a practice and the practice itself that are not reflected by careful analysis and evaluation of its products.

Almost all of what is done in mathematics is informal in the sense that it is not done in a pure formal system. In practice, in mathematical proofs, there are gaps and appeals to intuition (by the use of diagrams, for example), and proofs are not fully formalized. Controversies occur and are in practice dealt with without fully formalizing them. However, standards of rigor are specific and additional requirements of mathematical practice and proofs are always achieved and checked by the mathematical community according to those standards.

Proofs arise in dialogical contexts (even when thinking up a proof to convince oneself). Uncertainty is usually present in the period of discovery of a proof or while looking for the solution to a problem. As a consequence, in the process of proving, argumentation, as in ordinary contexts, is always present. As Pólya (1954) stated, "we secure our mathematical knowledge by demonstrative reasoning, but we support our conjectures by plausible reasoning" (p. vi).

Mathematical products are thought and presented in communicative situations that may demand specific forms of expressing them. In those situations rhetorical elements to convince the particular audience should be added and dialectical situations could arise, but not necessarily.

Mathematical practice is complex and, in many cases, collaborative work can be helpful to advance towards comprehension and solution of a problem. This is particularly clear in classroom settings, but it can also be seen in contexts involving more advanced mathematics. The Mini-Polymath projects are a good example of collaborative work over the internet to solve difficult conjectures and demanding problems in mathematics (Pease and Martin 2012). When collaboration is undertaken, argumentation is always present and may help to accomplish many mathematical tasks that go beyond those of analyzing and evaluating a proof. The use of argumentative diagrams may also be useful to organize the process towards the proof. As Pease and Martin state, careful consideration should be given to mathematical practice in order to design protocols that help in a human-like manner to improve mathematical thinking. To advance in this direction, more attention should be paid to the different contexts in which practice is undertaken in order to look for special requirements that apply in those contexts. Social dimensions of practice should be considered if we want to construct better ways of arguing and, as a consequence, of thinking, including in mathematics.

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