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# HARMONIC MAPPINGS OF THE HEXAGASKET TO THE CIRCLE

Donglei Tang

*(Nanjing Audit University, China)*

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**Abstract.** Harmonic mappings from the hexagasket to the circle are described in terms of boundary values and topological data. Explicit formulas are also given for the energy of the mapping. We have generalized the results in [10].

**Key words:** *hexagasket, harmonic mapping, self-similar Dirichlet form*

**AMS (2010) subject classification:** 28A80, 58E20

## 1 Introduction

Whenever there is a theory of harmonic functions on a space $X$, there should be a theory of harmonic mappings from $X$ to $Y$, where the target space $Y$ is any Riemannian manifold(see [1], [2] for more details). In this paper we take $Y$ to be the circle, and we want to show that Strichartz's method in [10] holds for the hexagasket. The hexagasket provides us one kind of possibility to show the method holds for all $n-$gasket if we observe the fact that for the general $n-$gasket we can find a boundary set is $V_0$ only consisting of 3 vertices(see [9] and Section 4.1 in [11]) if $n$ is not a multiple of 4.

The hexagasket[6],[9],[11] is generated by the i.f.s. consisting of 6 mappings in the plane, $F_i(x) = \frac{1}{3}(x-p_i)+p_i, i = 1,2,3,4,5,6$, where $p_1,\cdots,p_6$ are vertices of a regular hexagon. The usual boundary set $V_0 = \{p_1, p_2, p_3, p_4, p_5, p_6\}$. But in this paper we take a smaller boundary $V_0 = \{p_1, p_3, p_5\}$, and the hexagasket is also an affine nested fractal(see [7]).

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We approximate the hexagasket $K$ by a sequence of graphs $\Gamma_0,\Gamma_1,\cdots$ with vertices $V_0 \subseteq V_1 \subseteq V_2\cdots$, and $V_{k+1} = \cup_{j=1}^{6} F_j V_k$. The edge relation for $\Gamma_m$, denoted $x \sim_m y$, for $x,y \in V_m$ and $x \neq y$, is defined by the existence of a word $w = (w_1,\cdots,w_m)$ with length $|w| = m$ such that $x,y \in F_w K$, where $F_w = F_{w_1} \circ \cdots \circ F_{w_m}$. The simple energy form on $\Gamma_m$ is

$$E_m(u,v) = \sum_{x\sim_m y} (u(x)-u(y))(v(x)-v(y)), \tag{1.1}$$

and the renormalization energy $\varepsilon_m$ is given by

$$\varepsilon_m(u,v) = (\frac{7}{3})^m E_m(u,v), \tag{1.2}$$

where $u$ and $v$ denote continuous functions on $K$ and, by abuse of notation, their restriction to $V_m$.

We regard $V_0$ as the boundary of each graph $V_m$, and also of $K$. A function $h$ on $V_m$ (for $m \geq 1$) is called graph harmonic if it satisfies

$$h(x) = \frac{1}{n}\sum_{y\sim_m x} h(y), \quad \text{for} \quad \sharp\{y : y \sim_m x\} = n = 2 \text{ or } 4, \tag{1.3}$$

for all non-boundary point $x$. It is easy to see this is equivalent to the property that $h$ minimizes the energy $E_m(u,u)$ among all functions $u$ with the same boundary values.

The following proposition summaries the basic results(from [3], [4], [5], [6], [8], [11]) concerning the Dirichlet form and harmonic functions on $K$, and justifies the choice of renormalization factor $r$ in (1.2):

*Proposition* 1.1. (i) For any continuous function $u$ on $K$, the sequence $\varepsilon_m(u,u)$ is monotone increasing, so

$$\varepsilon(u,u) = \lim_{m\to\infty} \varepsilon_m(u,u) \tag{1.4}$$

is well-defined in $[0,\infty]$, and $\varepsilon(u,u) = 0$ if and only if $u$ is a constant.

Denote by $dom(\varepsilon)$ the set of continuous functions for which $\varepsilon(u,u) < \infty$. Then $dom(\varepsilon)$ modulo constants is a Hilbert space with the inner product

$$\varepsilon(u,v) = \lim_{m\to\infty} \varepsilon_m(u,v). \tag{1.5}$$

(ii) A function $h$ is called harmonic on $K$ if it minimizes the energy $\varepsilon(u,u)$ among functions with the same boundary values. Then $h$ is harmonic if and only if its restriction to the every $V_m$ is graph harmonic.

For a harmonic function $h$, $\varepsilon_m(h,h) = \varepsilon(h,h)$ for every $m$.

The space of harmonic functions is 3-dimensional, with each harmonic function determined uniquely from its boundary by means of the following harmonic algorithm: if the values of $h$ on $V_m$ are known, and the values $h(x)$ for $x \in V_{m+1}\setminus V_m$ is desired, find w with length $|w| = m$, such that $x \in F_w K$, and set

$$h(x) = D_w \rho(x). \tag{1.6}$$

here $D_w = D_{w_m} \circ \cdots \circ D_{w_1}$ for a word $w = w_1 \cdots w_m$. $D_{w_m}$ is analog stochastic matrix, $\rho(x)$ is the boundary value. The harmonic extension matrices for the hexagasket in the pointwise sense are

$$D_1 = \begin{pmatrix} 1 & 0 & 0 \\ \frac{4}{7} & \frac{2}{7} & \frac{1}{7} \\ \frac{4}{7} & \frac{1}{7} & \frac{2}{7} \end{pmatrix}, D_2 = \begin{pmatrix} \frac{3}{7} & \frac{3}{7} & \frac{1}{7} \\ \frac{2}{7} & \frac{4}{7} & \frac{1}{7} \\ \frac{4}{7} & \frac{2}{7} & \frac{1}{7} \end{pmatrix}, D_3 = \begin{pmatrix} \frac{2}{7} & \frac{4}{7} & \frac{1}{7} \\ 0 & 1 & 0 \\ \frac{1}{7} & \frac{4}{7} & \frac{2}{7} \end{pmatrix},$$

$$D_4 = \begin{pmatrix} \frac{1}{7} & \frac{4}{7} & \frac{2}{7} \\ \frac{1}{7} & \frac{3}{7} & \frac{3}{7} \\ \frac{1}{7} & \frac{2}{7} & \frac{4}{7} \end{pmatrix}, D_5 = \begin{pmatrix} \frac{2}{7} & \frac{1}{7} & \frac{4}{7} \\ \frac{1}{7} & \frac{2}{7} & \frac{4}{7} \\ 0 & 0 & 1 \end{pmatrix}, D_6 = \begin{pmatrix} \frac{4}{7} & \frac{1}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{1}{7} & \frac{4}{7} \\ \frac{3}{7} & \frac{1}{7} & \frac{3}{7} \end{pmatrix}.$$

Because $\varepsilon_m(u,u)$ is independent of $m$ when $u$ is harmonic, we have the simple expression for the hexagasket

$$\varepsilon_m(h,h) = (h(p_1) - h(p_3))^2 + (h(p_1) - h(p_5))^2 + (h(p_3) - h(p_5))^2.$$

for the energy of a harmonic function $h$. The main goal of this paper is to find the analog of (1.5), (1.6) for harmonic mappings of the hexagasket to a circle.

We will allow circles of arbitrary radius, so $S^1 = R/\tau Z$ for some $\tau > 0$. Every continuous function $u : K \longrightarrow S^1$ has local lifts $\tilde{u} : U \longrightarrow R$ for small enough neighborhoods $U$ in $K$, and if $u$ is topologically trivial, then we may take $U = K$. It is easy to see that $u$ is a harmonic mapping if and only if the lifts $\tilde{u}$ are harmonic functions.

Consider the edges of the triangle $T$ with vertices $(p_1, p_3, p_5)$. Since each edge is topologically trivial we can find lifts defined on the whole edge and define the increments $(\Delta)_j$ along the edge opposite $p_j$ by

$$(\Delta)_j = \tilde{u}(p_{2j+1}) - \tilde{u}(p_{2j-3}), \qquad j = 1,2,3, \tag{1.7}$$

for the appropriate lift. Note that we have $(\Delta)_1 + (\Delta)_2 + (\Delta)_3 = k\tau$ for $k = W(u,T)$, the winding number of the image $u(T)$ in $S^1$. More generally, let $k_w = W(u,T_w)$ where $T_w$ is the triangle $F_w T$. All but a finite number of $k_w$ are zero.

Consider a homotopy class of maps with $u(p_j)$ specified. This determines data $\{(\Delta)_j, k_w\}$ subject to the consistency conditions

$$(\Delta)_j \equiv \tilde{u}(p_{2j+1}) - \tilde{u}(p_{2j-3}) \bmod \ \tau, \qquad j = 1,2,3, \tag{1.8}$$

$$(\Delta)_1 + (\Delta)_2 + (\Delta)_3 = k\tau. \tag{1.9}$$

and all but a finite number of $k_w$ are zero. Conversely, every such data set determines a homotopy class. It is not hard to see that every homotopy class contains an energy minimizer, using the fact that points have positive capacity so that energy limits are automatically uniform limits and so stay within homotopy classes.

We have a simple expression for the normal derivatives at the boundary points. In general, the normal derivatives are defined by

$$\partial_n u(p_j) = \lim_{m\to\infty} (\frac{7}{3})^m (2u(p_j) - u(F_j^m p_{j+2}) - u(F_j^m p_{j-2})), \qquad j = 1,3,5, \tag{1.10}$$

whenever the limit exists. For harmonic functions the values on the right side are independent of $m$, so the limit exists trivially and

$$\partial_n u(p_j) = 2u(p_j) - u(p_{j+2}) - u(p_{j-2}), \qquad j = 1,3,5. \tag{1.11}$$

The paper is arranged as follows. In Section 2, we give a simple proof of Lemma 2.1, show details of the extension algorithm for harmonic mappings of the hexagasket, and give a necessary condition for the existence of a harmonic mapping in each homotopy class. In Section 3 we compute the Dirichlet form in terms of the data given in section 2.

## 2 The Extension Algorithm

In this section, we assume that $u(p_j)$ in $S^1$ are given, and the compatible data $\{(\Delta)_j, k_w\}$ are given to determine a homotopy class. Let $h$ denote a harmonic mapping in this class. Rather than give formulas for the values of $h$ at points, we will give formulas for the increments of $h$ along the edges of the triangles. For the hexagasket let

$$(\Delta)_j = \tilde{h}(p_{(2j+1)}) - \tilde{h}(p_{(2j-3)}), \qquad j = 1,2,3. \tag{2.1}$$

$$(\Delta_{wi})_j = \begin{cases} \tilde{h}(F_{wi}p_{(2j+1)}) - \tilde{h}(F_{wi}p_{(2j-3)}), & \text{for } i = 1,3,5, \\ \tilde{h}(F_{wi}p_{(2j+2)}) - \tilde{h}(F_{wi}p_{(2j-2)}), & \text{for } i = 2,4,6. \end{cases} \tag{2.2}$$

for any lift $\tilde{h}$ along this edge. Where $((2j-3)j(2j+1))$ is a permutation of $(135)$, and $((2j-2)j(2j))$ is a permutation of $(246)$. We want an inductive formula that enables us to compute these increments for the word of length $m+1$ in terms of increments for words of length $m$, since the data supplies us with the initial values for the empty word.

For the rest of this paper, we always assume that all the winding numbers $k_w$ are equal to zero.

**Lemma 2.1.** *For the hexagasket we have*

$$\begin{pmatrix} (\Delta_{wi})_1 \\ (\Delta_{wi})_2 \\ (\Delta_{wi})_3 \end{pmatrix} = A_i \begin{pmatrix} (\Delta_w)_1 \\ (\Delta_w)_2 \\ (\Delta_w)_3 \end{pmatrix}, i = 1,2,\cdots,6, \tag{2.3}$$

*where*

$$A_1=\begin{pmatrix}\frac{2}{7}&\frac{1}{7}&\frac{1}{7}\\ \frac{2}{7}&\frac{4}{7}&\frac{1}{7}\\ -\frac{4}{7}&-\frac{5}{7}&-\frac{2}{7}\end{pmatrix},A_2=\begin{pmatrix}\frac{1}{7}&\frac{1}{7}&-\frac{1}{7}\\ \frac{2}{7}&\frac{2}{7}&\frac{3}{7}\\ -\frac{3}{7}&-\frac{3}{7}&-\frac{2}{7}\end{pmatrix},A_3=\begin{pmatrix}-\frac{2}{7}&-\frac{4}{7}&-\frac{5}{7}\\ \frac{1}{7}&\frac{2}{7}&\frac{1}{7}\\ \frac{1}{7}&\frac{2}{7}&\frac{4}{7}\end{pmatrix},$$

$$A_4=\begin{pmatrix}\frac{3}{7}&\frac{2}{7}&\frac{2}{7}\\ -\frac{1}{7}&\frac{1}{7}&\frac{1}{7}\\ -\frac{2}{7}&-\frac{3}{7}&-\frac{3}{7}\end{pmatrix},A_5=\begin{pmatrix}\frac{4}{7}&\frac{1}{7}&\frac{2}{7}\\ -\frac{5}{7}&-\frac{2}{7}&-\frac{4}{7}\\ \frac{1}{7}&\frac{1}{7}&\frac{2}{7}\end{pmatrix},A_6=\begin{pmatrix}\frac{2}{7}&\frac{3}{7}&\frac{2}{7}\\ -\frac{3}{7}&-\frac{2}{7}&-\frac{3}{7}\\ \frac{1}{7}&-\frac{1}{7}&\frac{1}{7}\end{pmatrix}.$$

*Proof.* Without loss of generality, let $i=3$, then

$$\begin{aligned}\begin{pmatrix}(\Delta_{w3})_1\\(\Delta_{w3})_2\\(\Delta_{w3})_3\end{pmatrix}&=\begin{pmatrix}\tilde{h}(F_{w3}p_3)-\tilde{h}(F_{w3}p_5)\\ \tilde{h}(F_{w3}p_5)-\tilde{h}(F_{w3}p_1)\\ \tilde{h}(F_{w3}p_1)-\tilde{h}(F_{w3}p_3)\end{pmatrix}\\
&=\begin{pmatrix}-\frac{1}{7}&\frac{3}{7}&-\frac{2}{7}\\ -\frac{1}{7}&0&\frac{1}{7}\\ \frac{2}{7}&-\frac{3}{7}&\frac{1}{7}\end{pmatrix}\begin{pmatrix}\tilde{h}(F_wp_1)\\ \tilde{h}(F_wp_3)\\ \tilde{h}(F_wp_5)\end{pmatrix}=A_3\begin{pmatrix}(\Delta_w)_1\\(\Delta_w)_2\\(\Delta_w)_3\end{pmatrix}\\
&=A_3\begin{pmatrix}\tilde{h}(F_wp_3)-\tilde{h}(F_wp_5)\\ \tilde{h}(F_wp_5)-\tilde{h}(F_wp_1)\\ \tilde{h}(F_wp_1)-\tilde{h}(F_wp_3)\end{pmatrix}=\begin{pmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{pmatrix}\begin{pmatrix}\tilde{h}(F_wp_3)-\tilde{h}(F_wp_5)\\ \tilde{h}(F_wp_5)-\tilde{h}(F_wp_1)\\ \tilde{h}(F_wp_1)-\tilde{h}(F_wp_3)\end{pmatrix}.\end{aligned}$$

By the knowledge of linear algebra, we can see the matrix

$$A_3=\begin{pmatrix}-\frac{2}{7}&-\frac{4}{7}&-\frac{5}{7}\\ \frac{1}{7}&\frac{2}{7}&\frac{1}{7}\\ \frac{1}{7}&\frac{2}{7}&\frac{4}{7}\end{pmatrix}$$

gives one solution for the above equations.

As Strichartz([10]) has pointed out, the matrix given in (2.3) is not the only possible choice. These particular matrices are chosen because they have column sums equal to zero.

Our aim will be to add a correction term to (2.3). That is, we define $(\lambda_{wi})_j$ by

$$\begin{pmatrix}(\Delta_{wi})_1\\(\Delta_{wi})_2\\(\Delta_{wi})_3\end{pmatrix}=A_i\begin{pmatrix}(\Delta_w)_1\\(\Delta_w)_2\\(\Delta_w)_3\end{pmatrix}+\begin{pmatrix}(\lambda_{wi})_1\\(\lambda_{wi})_2\\(\lambda_{wi})_3\end{pmatrix},\qquad i=1,2,\cdots,6. \tag{2.4}$$

We will also require similar correction terms for normal derivatives. Write $(N_w)_j$ for (1.10). Note that for the hexagasket, the matching condition on adjacent cells at a junction point $x$

means

$$\begin{cases} (N_{w1})_2+(N_{w2})_3=0,\\ (N_{w1})_3+(N_{w6})_1=0,\\ (N_{w2})_2+(N_{w3})_1=0,\\ (N_{w3})_3+(N_{w4})_1=0,\\ (N_{w4})_3+(N_{w5})_2=0,\\ (N_{w5})_1+(N_{w6})_2=0, \end{cases} \tag{2.5}$$

The consistency condition $(I)$ for the hexagasket is

$$(N_{w(2j-1)})_j=(N_w)_j,\ j=1,2,3. \tag{2.6}$$

In the topologically trivial case we have simply

$$(N_w)_j=\lim_{m\to\infty}(\frac{7}{3})^{|w|}((\Delta_w)_{j-1}-(\Delta_w)_{j+1}), \qquad j=1,2,3. \tag{2.7}$$

by taking $m=0$ on the right side of $(1.11)$. In the general case we define the correction terms $(\delta_w)_j$ by

$$(N_w)_j=\lim_{m\to\infty}(\frac{7}{3})^{|w|}((\Delta_w)_{j-1}-(\Delta_w)_{j+1}+(\delta_w)_j), \qquad j=1,2,3. \tag{2.8}$$

Both correction terms $(\lambda_w)_j$ and $(\delta_w)_j$ will be zero if $|\,w\,|$ is large enough so that $h$ is topologically trivial on $F_wK$.

**Lemma 2.2.** *For any w,*

$$(N_w)_1+(N_w)_2+(N_w)_3=0. \tag{2.9}$$

*Proof.* Similar to the proof of Lemma2.2 in [10].

**Lemma 2.3.** *The following equations on $(\lambda_{wi})_j$ and $(\delta_w)_j$ are necessary for them to be associated with a harmonic mapping in the given hopotopy class:*

$$\begin{cases} (\lambda_{w3})_1-(\lambda_{w4})_2+(\lambda_{w5})_1=0,\\ (\lambda_{w5})_2-(\lambda_{w6})_3+(\lambda_{w1})_2=0,\\ (\lambda_{w1})_3-(\lambda_{w2})_1+(\lambda_{w3})_3=0,\\ (\lambda_{w4})_1-(\lambda_{w5})_3+(\lambda_{w6})_1=0,\\ (\lambda_{w6})_2-(\lambda_{w1})_1+(\lambda_{w2})_2=0,\\ (\lambda_{w2})_3-(\lambda_{w3})_2+(\lambda_{w4})_3=0, \end{cases} \tag{2.10}$$

$$(\lambda_{wi})_1+(\lambda_{wi})_2+(\lambda_{wi})_3=0, i=1,2,\cdots,6. \tag{2.11}$$

$$\begin{cases} (\lambda_{w1})_3-(\lambda_{w1})_1+(\lambda_{w2})_1-(\lambda_{w2})_2=(\delta_{w1})_2+(\delta_{w2})_3,\\ (\lambda_{w1})_1-(\lambda_{w1})_2+(\lambda_{w6})_2-(\lambda_{w6})_3=(\delta_{w1})_3+(\delta_{w6})_1,\\ (\lambda_{w2})_3-(\lambda_{w2})_1+(\lambda_{w3})_2-(\lambda_{w3})_3=(\delta_{w2})_2+(\delta_{w3})_1,\\ (\lambda_{w3})_1-(\lambda_{w3})_2+(\lambda_{w4})_2-(\lambda_{w4})_3=(\delta_{w3})_3+(\delta_{w4})_1,\\ (\lambda_{w4})_1-(\lambda_{w4})_2+(\lambda_{w5})_3-(\lambda_{w5})_1=(\delta_{w4})_3+(\delta_{w5})_2,\\ (\lambda_{w5})_2-(\lambda_{w5})_3+(\lambda_{w6})_3-(\lambda_{w6})_1=(\delta_{w5})_1+(\delta_{w6})_2, \end{cases} \tag{2.12}$$

$$\frac{7}{3}((\lambda_{wi})_{j-1}-(\lambda_{wi})_{j+1}+(\delta_{wi})_j)=(\delta_w)_j. \tag{2.13}$$

$$(\delta_w)_1+(\delta_w)_2+(\delta_w)_3=0. \tag{2.14}$$

*Proof.* Each edge in the triangle $F_wK$ splits into a union of 3 edges of triangles of the next level, yielding the consistency condition $(II)$, which is equivalent to (2.10). The definition of the winding number requires

$$(\Delta_{wi})_1+(\Delta_{wi})_2+(\Delta_{wi})_3=0. \tag{2.15}$$

which is equivalent to (2.11).

Condition (2.6) is easily seen to be equivalent to (2.13). Condition (2.5) after substituting (2.8) and (2.4) and simplifying, is equivalent to (2.12). Similarly, condition (2.9) is equivalent to (2.14).

**Lemma 2.4.** *There exists at least one solution of* (2.10), (2.11) *and* (2.12) *for* $(\lambda_{wi})_j$ *in terms of* $(\delta_{wi})_j$ *for the hexagasket,* $i=1,2,\cdots,6$, $j=1,2,3$.

*Proof.* If we assume that $(\lambda_{w6})_2$ is a free variables. The solution can be given as follows:

$$\begin{cases}
(\lambda_{w1})_1=\frac{1}{90}(-(\delta_{w4})_1+(\delta_{w4})_3+5(\delta_{w5})_1+(\delta_{w5})_2+19(\delta_{w6})_1+5(\delta_{w6})_2)\\
\qquad\qquad+\frac{1}{90}(-19(\delta_{w1})_2+19(\delta_{w1})_3)-5(\delta_{w2})_2-19(\delta_{w2})_3-5(\delta_{w3})_1-(\delta_{w3})_3),\\
(\lambda_{w1})_2=(\lambda_{w6})_2+\frac{1}{45}(-(\delta_{w4})_1-2(\delta_{w4})_3-7(\delta_{w5})_1-2(\delta_{w5})_2-26(\delta_{w6})_1-7(\delta_{w6})_2)\\
\qquad\qquad+\frac{1}{45}(-7(\delta_{w1})_2-26(\delta_{w1})_3)-2(\delta_{w2})_2-7(\delta_{w2})_3-2(\delta_{w3})_1-(\delta_{w3})_3),\\
(\lambda_{w1})_3=-(\lambda_{w6})_2+\frac{1}{30}((\delta_{w4})_1+(\delta_{w4})_3+3(\delta_{w5})_1+1(\delta_{w5})_2+11(\delta_{w6})_1+3(\delta_{w6})_2)\\
\qquad\qquad+\frac{1}{30}(11(\delta_{w1})_2+11(\delta_{w1})_3)+3(\delta_{w2})_2+11(\delta_{w2})_3+3(\delta_{w3})_1+(\delta_{w3})_3),\\
(\lambda_{w2})_1=\frac{1}{90}(-5(\delta_{w4})_1-(\delta_{w4})_3+(\delta_{w5})_1-(\delta_{w5})_2+5(\delta_{w6})_1+(\delta_{w6})_2)\\
\qquad\qquad+\frac{1}{90}(19(\delta_{w1})_2+5(\delta_{w1})_3)-19(\delta_{w2})_2+19(\delta_{w2})_3-19(\delta_{w3})_1-5(\delta_{w3})_3),\\
(\lambda_{w2})_2=-(\lambda_{w6})_2+\frac{1}{90}(-(\delta_{w4})_1+(\delta_{w4})_3+5(\delta_{w5})_1+(\delta_{w5})_2+19(\delta_{w6})_1+5(\delta_{w6})_2)\\
\qquad\qquad+\frac{1}{90}(-19(\delta_{w1})_2+19(\delta_{w1})_3)-5(\delta_{w2})_2-19(\delta_{w2})_3-5(\delta_{w3})_1-(\delta_{w3})_3),\\
(\lambda_{w2})_3=(\lambda_{w6})_2+\frac{1}{15}((\delta_{w4})_1-(\delta_{w5})_1-4(\delta_{w6})_1-(\delta_{w6})_2-4(\delta_{w1})_3)+4(\delta_{w2})_2+4(\delta_{w2})_3+(\delta_{w3})_3),\\
(\lambda_{w3})_1=-(\lambda_{w6})_2+\frac{1}{10}(3(\delta_{w4})_1+(\delta_{w4})_3+(\delta_{w5})_1+(\delta_{w5})_2+3(\delta_{w6})_1+(\delta_{w6})_2)\\
\qquad\qquad+\frac{1}{10}((\delta_{w1})_2+3(\delta_{w1})_3)+(\delta_{w2})_2+(\delta_{w2})_3+(\delta_{w3})_1+(\delta_{w3})_3),\\
(\lambda_{w3})_2=\frac{1}{90}(-19(\delta_{w4})_1-5(\delta_{w4})_3-(\delta_{w5})_1-5(\delta_{w5})_2+(\delta_{w6})_1-(\delta_{w6})_2)\\
\qquad\qquad+\frac{1}{90}(5(\delta_{w1})_2+(\delta_{w1})_3)+19(\delta_{w2})_2+5(\delta_{w2})_3+19(\delta_{w3})_1-19(\delta_{w3})_3),\\
(\lambda_{w3})_3=(\lambda_{w6})_2+\frac{1}{45}(-4(\delta_{w4})_1-2(\delta_{w4})_3-4(\delta_{w5})_1-2(\delta_{w5})_2-14(\delta_{w6})_1-4(\delta_{w6})_2)\\
\qquad\qquad+\frac{1}{45}(-7(\delta_{w1})_2-14(\delta_{w1})_3)-14(\delta_{w2})_2-7(\delta_{w2})_3-14(\delta_{w3})_1-4(\delta_{w3})_3),\\
(\lambda_{w4})_1=(\lambda_{w6})_2+\frac{1}{15}((\delta_{w4})_1+4(\delta_{w4})_3+4(\delta_{w5})_2-4(\delta_{w6})_1-(\delta_{w1})_2-4(\delta_{w1})_3-(\delta_{w2})_3+(\delta_{w3})_3),
\end{cases}$$

$$\begin{cases}
(\lambda_{w4})_2 = \frac{1}{90}(19(\delta_{w4})_1 - 19(\delta_{w4})_3 - 5(\delta_{w5})_1 - 19(\delta_{w5})_2 - (\delta_{w6})_1 - 5(\delta_{w6})_2) \\
\qquad + \frac{1}{90}((\delta_{w1})_2 - (\delta_{w1})_3) + 5(\delta_{w2})_2 + (\delta_{w2})_3 + 5(\delta_{w3})_1 + 19(\delta_{w3})_3), \\
(\lambda_{w4})_3 = -(\lambda_{w6})_2 + \frac{1}{18}(-5(\delta_{w4})_1 - (\delta_{w4})_3 + (\delta_{w5})_1 - (\delta_{w5})_2 + 5(\delta_{w6})_1 + (\delta_{w6})_2) \\
\qquad + \frac{1}{18}((\delta_{w1})_2 + 5(\delta_{w1})_3) - (\delta_{w2})_2 + (\delta_{w2})_3 - (\delta_{w3})_1 - 5(\delta_{w3})_3), \\
(\lambda_{w5})_1 = (\lambda_{w6})_2 + \frac{1}{45}(-4(\delta_{w4})_1 - 14(\delta_{w4})_3 - 7(\delta_{w5})_1 - 14(\delta_{w5})_2 - 14(\delta_{w6})_1 - 7(\delta_{w6})_2) \\
\qquad + \frac{1}{45}(-4(\delta_{w1})_2 - 14(\delta_{w1})_3) - 2(\delta_{w2})_2 - 4(\delta_{w2})_3 - 2(\delta_{w3})_1 - 4(\delta_{w3})_3), \\
(\lambda_{w5})_2 = -(\lambda_{w6})_2 + \frac{1}{30}((\delta_{w4})_1 + 3(\delta_{w4})_3 + 11(\delta_{w5})_1 + 3(\delta_{w5})_2 + 11(\delta_{w6})_1 + 11(\delta_{w6})_2) \\
\qquad + \frac{1}{30}(3(\delta_{w1})_2 + 11(\delta_{w1})_3) + (\delta_{w2})_2 + 3(\delta_{w2})_3 + (\delta_{w3})_1 + (\delta_{w3})_3), \\
(\lambda_{w5})_3 = \frac{1}{90}(5(\delta_{w4})_1 + 19(\delta_{w4})_3 - 19(\delta_{w5})_1 + 19(\delta_{w5})_2 - 5(\delta_{w6})_1 - 19(\delta_{w6})_2) \\
\qquad + \frac{1}{90}(-(\delta_{w1})_2 - 5(\delta_{w1})_3) + (\delta_{w2})_2 - (\delta_{w2})_3 + (\delta_{w3})_1 + 5(\delta_{w3})_3), \\
(\lambda_{w6})_1 = -(\lambda_{w6})_2 + \frac{1}{90}(-(\delta_{w4})_1 - 5(\delta_{w4})_3 - 19(\delta_{w5})_1 - 5(\delta_{w5})_2 + 19(\delta_{w6})_1 - 19(\delta_{w6})_2) \\
\qquad + \frac{1}{90}(5(\delta_{w1})_2 + 19(\delta_{w1})_3) + (\delta_{w2})_2 + 5(\delta_{w2})_3 + (\delta_{w3})_1 - (\delta_{w3})_3), \\
(\lambda_{w6})_3 = \frac{1}{90}((\delta_{w4})_1 + 5(\delta_{w4})_3 + 19(\delta_{w5})_1 + 5(\delta_{w5})_2 - 19(\delta_{w6})_1 + 19(\delta_{w6})_2) \\
\qquad + \frac{1}{90}(-5(\delta_{w1})_2 - 19(\delta_{w1})_3) - (\delta_{w2})_2 - 5(\delta_{w2})_3 - (\delta_{w3})_1 + (\delta_{w3})_3),
\end{cases} \tag{2.16}$$

Once we give one solution for $(\lambda_{wi})_j$ in terms of $(\delta_{wi})_j$ for the equations $(2.10)-(2.12)$, then following the method of Lemma 2.5 and 2.6 in [10] we can get a solution in terms of induction formulas for $(\delta_{wi})_j$.

Thus we have

**Theorem 2.1.** *There exists at least one harmonic mapping in each homopoty class for the hexagasket, and its values are determined by the increments along edges in $F_wK$. These increments for $w = \varnothing$ are given by the initial data, and then (2.4) determines them inductively.*

## 3 Energy Computation

Now we consider the expression of the Dirichlet form in terms of the data. In fact for each $m$, we have

$$\varepsilon = \sum_{|w|=m} \varepsilon_w, \tag{3.1}$$

where $\varepsilon_w$ denotes the contribution toward the energy form from the cell $F_wK$. When $m$ is large enough

$$\varepsilon_w = (\frac{7}{3})^m((\Delta_w)_1^2 + (\Delta_w)_2^2 + (\Delta_w)_3^2) = (\frac{7}{3})^m \parallel \Delta_w \parallel^2, \tag{3.2}$$

where we use the vector notation $\Delta_w = ((\Delta_w)_1, (\Delta_w)_2, (\Delta_w)_3)$ and $\parallel \cdot \parallel$ stands for the Euclidean norm. We seek an expression of the form

$$\varepsilon_w = (\frac{7}{3})^m(\parallel \Delta_w + \mu_w \parallel^2 + E_w), \tag{3.3}$$

for any $w$, where $\mu_w$ and $E_w$ are the vector and scalar correction terms involving only the topological data. Note that (3.3) does not determine $\mu_w$ and $E_w$ uniquely, since by (2.15) we may add an arbitrary multiple of the constant vector $(1,1,1)$ to $\mu_w$ and compensate by adjusting $E_w$.

**Theorem 3.1.** *The energy on the hexagasket $\varepsilon_w$ are given by (3.3) for*

$$\mu_w = \sum_{v\neq 0} B_{v_1}\cdots B_{v_m}\lambda_{wv}, \tag{3.4}$$

*where* $B_i = \frac{7}{3}A_i^T, i = 1,2,\cdots,6$, *and*

$$E_w = -\parallel \mu_w \parallel^2 + \sum_{v\neq 0}(\frac{7}{3})^{|v|}(\parallel \mu_{wv}+\lambda_{wv}\parallel^2 - \parallel \mu_{wv}\parallel^2). \tag{3.5}$$

*Proof.* Since

$$\varepsilon_w = \sum_{i=1}^{6}\varepsilon_{w_i} \tag{3.6}$$

Substitute (3.3) into (3.6) and attempt to obtain recursion relations for the correction terms. We obtain first

$$\parallel \Delta_w + \mu_w \parallel^2 + E_w = \frac{7}{3}(\sum_{i=1}^{6}\parallel \Delta_{wi}+\mu_{wi}\parallel^2 + \sum_{i=1}^{6}E_{wi}) \tag{3.7}$$

And then we substitute (2.4) into the right side of (3.7) to eliminate $\Delta_{wi}$. We claim that the quadratic terms in $\Delta_w$ are the same on both sides of (3.7). Indeed, on the left side we have $\parallel \Delta_w \parallel^2$, while on the right side we have

$$\begin{aligned}\frac{7}{3}\sum_{i=1}^{6}\parallel \Delta_{wi}\parallel^2 &= \frac{7}{3}\sum_{i=1}^{6}((\Delta_{wi})_1^2+(\Delta_{wi})_2^2+(\Delta_{wi})_3^2)\\ &= \parallel \Delta_w \parallel^2 + ((\Delta_w)_1+(\Delta_w)_2+(\Delta_w)_3)^2 + lower\ \ order\ \ terms\\ &= \parallel \Delta_w \parallel^2 + (\tau k_w)^2 + lower\ \ order\ \ terms.\end{aligned}$$

Next we equate the terms in (3.7) that are linear in $\Delta_w$ to obtain

$$\sum_{j=1}^{3}(\Delta_w)_j(\mu_w)_j = \frac{7}{3}\sum_{i=1}^{6}\sum_{j=1}^{3}((\mu_{wi})_j+(\lambda_{wi})_j)((\Delta_{wi})_j-(\lambda_{wi})_j)$$

Equating separately the factors of $(\Delta_w)_j$ yields the vector equation

$$\mu_w = \sum_{i=1}^{6}B_i(\mu_{wi}+\lambda_{wi}), \tag{3.8}$$

where

$$B_i = \frac{7}{3}A_i^T, \qquad i = 1,2,\cdots,6.$$

Equating everything that remains in (3.7) yields

$$E_w = - \| \mu_w \|^2 + (\tau k_w)^2 + \frac{7}{3}\sum_{i=1}^{6} E_{wi} + \frac{7}{3}\sum_{i=1}^{6} \| \mu_{wv} + \lambda_{wv} \|^2 . \tag{3.9}$$

Altogether we have shown that a solution of (3.8) and (3.9) gives a solution of (3.7), hence a valid formula of the form (3.3). But it is straightforward to see that (3.4) solves (3.8) and then (3.5) solves (3.9).

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School of Mathematics and Statistics
Nanjing Audit University
Nanjing, 210029
P. R. China

E-mail: tdonglei@nau.edu.cn