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SOME INTEGRAL INEQUALITIES FOR THE POLAR DERIVATIVE OF A POLYNOMIAL

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Abstract. If $P(z)$ is a polynomial of degree n which does not vanish in $|z| < 1$, then it is recently proved by Rather [*Jour. Ineq. Pure and Appl. Math.*, 9 (2008), Issue 4, Art. 103] that for every $\gamma > 0$ and every real or complex number α with $|\alpha| \geq 1$,

$$\left\{\int_0^{2\pi} |D_\alpha P(e^{i\theta})|^\gamma \mathrm{d}\theta\right\}^{1/\gamma} \leq n(|\alpha|+1)C_\gamma \left\{\int_0^{2\pi} |P(e^{i\theta})|^\gamma \mathrm{d}\theta\right\}^{1/\gamma},$$
$$C_\gamma = \left\{\frac{1}{2\pi}\int_0^{2\pi} |1+e^{i\beta}|^\gamma \mathrm{d}\beta\right\}^{-1/\gamma},$$

where $D_\alpha P(z)$ denotes the polar derivative of $P(z)$ with respect to α. In this paper we prove a result which not only provides a refinement of the above inequality but also gives a result of Aziz and Dawood [J. Approx. Theory, 54 (1988), 306-313] as a special case.

Key words: *polar derivative, polynomial, Zygmund inequality, zeros*

AMS (2010) subject classification: 30A10, 30C10, 30D15, 41A17

1 Introduction and Statement of Results

Let $P(z) = \sum_{v=0}^{n} a_v z^v$ be a polynomial of degree atmost n and $P'(z)$ its derivative, then

$$\max_{|z|=1} |P'(z)| \leq n \max_{|z|=1} |P(z)|, \tag{1.1}$$

and for every $\gamma \geq 1$,

$$\left\{\int_0^{2\pi} |P'(e^{i\theta})|^\gamma |rmd\theta\right\}^{1/\gamma} \leq n \left\{\int_0^{2\pi} |P(e^{i\theta})|^\gamma \mathrm{d}\theta\right\}^{1/\gamma}. \tag{1.2}$$

The inequality (1.1) is a classical result of Bernstein[11] (see also [14]), whereas the inequality (1.2) is due to Zygmund[15], who proved it for all trigonometric polynomials of degree n and not only for those of the form $P(e^{i\theta})$. Arestov[1] proved that (1.2) remains true for $0<\gamma<1$ as well. If we let $\gamma\to\infty$ in the inequality (1.2), we get (1.1).

The above two inequalities (1.1) and (1.2) can be sharpened if we restrict ourselves to the class of polynomials having no zeros in $|z|<1$. In fact, if $P(z)\neq 0$ in $|z|<1$, then (1.1) and (1.2) can be respectively replaced by

$$\max_{|z|=1}|P'(z)|\leq \frac{n}{2}\max_{|z|=1}|P(z)| \tag{1.3}$$

and

$$\left\{\int_0^{2\pi}|P'(e^{i\theta})|^\gamma \mathrm{d}\theta\right\}^{1/\gamma}\leq nB_\gamma\left\{\int_0^{2\pi}|P(e^{i\theta})|^\gamma \mathrm{d}\theta\right\}^{1/\gamma}, \tag{1.4}$$

where

$$B_\gamma=\left\{\frac{1}{2\pi}\int_0^{2\pi}|1+e^{i\alpha}|^\gamma \mathrm{d}\alpha\right\}^{-1/\gamma}.$$

The inequality (1.3) is conjectured by Erdös and later verified by Lax[9], whereas the inequality (1.4) is proved by De-Bruijn [7] for $\gamma\geq 1$. Further, Rahman and Schmeisser[12] have shown that (1.4) holds for $0<\gamma<1$ also. If we let $\gamma\to\infty$ in the inequality (1.4), we get (1.3).

The inequality (1.3) is further improved by Aziz and Dawood[4] by proving that if $P(z)\neq 0$ in $|z|<1$, then

$$\max_{|z|=1}|P'(z)|\leq \frac{n}{2}\left\{\max_{|z|=1}|P(z)|-\min_{|z|=1}|P(z)|\right\}. \tag{1.5}$$

Let $D_\alpha P(z)$ denote the polar derivative of the polynomial $P(z)$ with respect to a complex number α. Then

$$D_\alpha P(z)=nP(z)+(\alpha-z)P'(z).$$

The polynomial $D_\alpha P(z)$ is of degree at most $n-1$ and it generalizes the ordinary derivative $P'(z)$ in the sense that

$$\lim_{\alpha\to\infty}\frac{D_\alpha P(z)}{\alpha}=P'(z).$$

Aziz[3] extended the inequality (1.3) to the polar derivatives and proved that if $P(z)$ is a polynomial of degree n such that $P(z)\neq 0$ in $|z|<1$, then for every real or complex number α with $|\alpha|\geq 1$,

$$\max_{|z|=1}|D_\alpha P(z)|\leq \frac{n}{2}(|\alpha|+1)\max_{|z|=1}|P(z)|. \tag{1.6}$$

While seeking the desired extension of the inequality (1.6) to the L^γ norm, recently Govil et al. [8] have made an incomplete attempt by proving the following generalization of the inequalities (1.4) and (1.6).

Theorem A. *If $P(z)$ is a polynomial of degree n which does not vanish in $|z|<1$, then for $\gamma \geq 1$ and every real or complex number α with $|\alpha| \geq 1$,*

$$\left\{\int_0^{2\pi} |D_\alpha P(e^{i\theta})|^\gamma \mathrm{d}\theta\right\}^{1/\gamma} \leq n(|\alpha|+1)F_\gamma \left\{\int_0^{2\pi} |P(e^{i\theta})|^\gamma \mathrm{d}\theta\right\}^{1/\gamma}, \tag{1.7}$$

where

$$F_\gamma = \left\{\frac{1}{2\pi}\int_0^{2\pi} |1+e^{i\beta}|^\gamma \mathrm{d}\beta\right\}^{-1/\gamma}.$$

Unfortunately, the proof of Theorem A is not correct as is first pointed out by Aziz and Rather[5] who in the same paper have given a correct proof of the inequality (1.7) also. The inequality (1.7) is then independently proved by Rather[13] for $\gamma > 0$.

In this paper we prove the following more general result which in particular provides refinements and generalizations of the inequalities (1.6) and (1.7) and also extends the inequality (1.7) for $\gamma \in (0,1)$. Further, it also gives the inequality (1.5) as a special case. Actually, we prove

Theorem 1.1. *If $P(z)$ is a polynomial of degree n which does not vanish in $|z|<1$, then for $\gamma > 0$, every real or complex numbers $\alpha_1, \cdots, \alpha_k,\ k \leq n-1$ with $|\alpha_i| \geq 1,\ i = 1,2,\cdots,k$ and real or complex δ with $|\delta| \leq 1$,*

$$\left\{\int_0^{2\pi} \left|D_{\alpha_1}\cdots D_{\alpha_k} P(e^{i\theta}) + \frac{mn(n-1)\cdots(n-k+1)(|\alpha_1\cdots\alpha_k|-1)\delta}{2}\right|^\gamma \mathrm{d}\theta\right\}^{1/\gamma}$$
$$\leq n(n-1)\cdots(n-k+1)(|\alpha_1|+1)(|\alpha_2|+1)\cdots(|\alpha_k|+1)C_\gamma \left\{\int_0^{2\pi} |P(e^{i\theta})|^\gamma \mathrm{d}\theta\right\}^{1/\gamma}, \tag{1.8}$$

where

$$C_\gamma = \left\{\frac{1}{2\pi}\int_0^{2\pi} |1+e^{i\beta}|^\gamma \mathrm{d}\beta\right\}^{-1/\gamma}$$

and

$$m = \min_{|z|=1} |P(z)|.$$

In the limiting case, when $\gamma \to \infty$, the above inequality is sharp and the equality in (1.8) holds for $P(z) = (z+1)^n$, where $\alpha_i \geq 1,\ i = 1,2,\cdots,k$ are real.

If we let $\gamma \to \infty$ in (1.8) and choose the argument of δ with $|\delta| = 1$ suitably, we get the following refinement and generalization of (1.6).

Corollary 1.1. *If $P(z)$ is a polynomial of degree n such that $P(z) \neq 0$ in $|z| < 1$, then for every real or complex numbers $\alpha_1, \cdots, \alpha_k$, $k \leq n-1$ with $|\alpha_i| \geq 1$, $i = 1, 2, \cdots, k$*

$$\max_{|z|=1} |D_{\alpha_1} \cdots D_{\alpha_k} P(z)| \leq \frac{n(n-1)\cdots(n-k+1)}{2} \left\{ (|\alpha_1|+1)\cdots(|\alpha_k|+1) \max_{|z|=1} |P(z)| - (|\alpha_1 \cdots \alpha_k| - 1) \min_{|z|=1} |P(z)| \right\}. \tag{1.9}$$

The result is best possible and the equality holds in (1.9) *for $P(z) = (z+1)^n$ with real $\alpha_i \geq 1$, $i = 1, 2, \cdots, k$.*

If we put $k = 1$, in Theorem 1.1, we get the following result which is a refinement of (1.7) and is an extension for $\gamma \in (0, 1)$.

Corollary 1.2. *If $P(z)$ is a polynomial of degree n which does not vanish in $|z| < 1$, then for $\gamma > 0$, every real or complex number α with $|\alpha| \geq 1$ and real or complex δ with $|\delta| \leq 1$,*

$$\left\{ \int_0^{2\pi} \left| D_\alpha P(e^{i\theta}) + \frac{mn(|\alpha|-1)\delta}{2} \right|^\gamma \mathrm{d}\theta \right\}^{1/\gamma} \leq n(|\alpha|+1) C_\gamma \left\{ \int_0^{2\pi} |P(e^{i\theta})|^\gamma \mathrm{d}\theta \right\}^{1/\gamma}, \tag{1.10}$$

where C_γ, m are defined above. In the limiting case, when $\gamma \to \infty$, the above inequality is sharp and the equality in (1.10) holds for $P(z) = (z+1)^n$, where $\alpha \geq 1$ is real.

If we let $\gamma \to \infty$ in (1.10) and choose the argument of δ with $|\delta| = 1$ suitably, we get the following refinement of (1.6).

Corollary 1.3. *If $P(z)$ is a polynomial of degree n such that $P(z) \neq 0$ in $|z| < 1$, then for every real or complex number α with $|\alpha| \geq 1$,*

$$\max_{|z|=1} |D_\alpha P(z)| \leq \frac{n}{2} \left\{ (|\alpha|+1) \max_{|z|=1} |P(z)| - (|\alpha|-1) \min_{|z|=1} |P(z)| \right\}. \tag{1.11}$$

The result is best possible and the equality holds in (1.11) *for $P(z) = (z+1)^n$ with real $\alpha \geq 1$.*

Remark 1.1. If we divide both sides of (1.11) by $|\alpha|$ and let $|\alpha| \to \infty$, we get (1.5).

2 Lemmas

We need the following lemmas for the proof of Theorem 1.1.

Lemma 2.1. *If all the zeros of an nth degree polynomial $P(z)$ lie in a circular region C and if none of the points $\alpha_1, \alpha_2, \cdots, \alpha_k$ lie in the region C, then each of the polar derivatives*

$$D_{\alpha_1} \cdots D_{\alpha_k} P(z), \quad k = 1, 2, \cdots, n-1,$$

has all of its zeros in C.

This follows by repeated application of Laguarre's theorem (see [1] or [9, p.52]).

Lemma 2.2. *If $P(z)$ is a polynomial of degree n having no zeros in $|z|<1$ and $m=\min_{|z|=1}|P(z)|$, then for any real or complex numbers $\alpha_1,\cdots,\alpha_k,\ \ k\le n-1$ with $|\alpha_i|\ge 1,\ \ i=1,2,\cdots,k$*

$$|D_{\alpha_1}\cdots D_{\alpha_k}Q(z)|\ge mn(n-1)\cdots(n-k+1)|\alpha_1\alpha_2\cdots\alpha_k z^{n-k}|, \quad \text{for } |z|\ge 1, \tag{2.1}$$

where

$$Q(z)=z^n\overline{P(1/\bar{z})}.$$

Proof of Lemma 2.2. If $m=\min_{|z|=1}|P(z)|=0$, then the inequality (2.1) is obvious. Henceforth, we assume $m\neq 0$, so that all zeros of $P(z)$ lie in $|z|>1$. Now if λ is any real or complex number with $|\lambda|<1$, then

$$|\lambda m|<m\le|P(z)|, \quad \text{for } |z|=1. \tag{2.2}$$

Therefore, it follows by Rouche's theorem that the polynomial $F(z)=P(z)-\lambda m$ has all zeros in $|z|>1$ for every λ with $|\lambda|<1$.

If $G(z)=z^nF\overline{(1/\bar{z})}=Q(z)-\bar{\lambda}mz^n$, then all zeros of $G(z)$ lie in $|z|<1$. Hence, it follows by Lemma 2.1 that all zeros of

$$\begin{aligned} &D_{\alpha_1}\cdots D_{\alpha_k}(Q(z)-\bar{\lambda}mz^n)\\ &\quad=D_{\alpha_1}\cdots D_{\alpha_k}Q(z)-\bar{\lambda}mn(n-1)\cdots(n-k+1)\alpha_1\alpha_2\cdots\alpha_k z^{n-k} \end{aligned} \tag{2.3}$$

lie in $|z|<1$ for any $\alpha_1,\cdots,\alpha_k,\ \ k\le n-1$ with $|\alpha_i|\ge 1,\ \ i=1,2,\cdots,k$ and for every λ with $|\lambda|<1$. This implies

$$|D_{\alpha_1}\cdots D_{\alpha_k}Q(z)|\ge mn(n-1)\cdots(n-k+1)|\alpha_1\alpha_2\cdots\alpha_k z^{n-k}|, \quad \text{for } |z|\ge 1,$$

because if this is not true, then there is a point $z=z_0$ with $|z_0|\ge 1$, such that

$$|D_{\alpha_1}\cdots D_{\alpha_k}Q(z)|_{z=z_0}<mn(n-1)\cdots(n-k+1)|\alpha_1\alpha_2\cdots\alpha_k z_0^{n-k}|.$$

We take

$$\bar{\lambda}=\frac{\{D_{\alpha_1}\cdots D_{\alpha_k}Q(z)\}_{z=z_0}}{mn(n-1)\cdots(n-k+1)\alpha_1\alpha_2\cdots\alpha_k z_0^{n-k}},$$

so that $|\lambda|<1$ and from (2.3) with this choice of $\bar{\lambda}$, we get $[D_{\alpha_1}\cdots D_{\alpha_k}(Q(z)-\bar{\lambda}mz^n)]_{z=z_0}=0$, where $|z_0|\ge 1$, which contradicts the fact that all zeros of $D_{\alpha_1}\cdots D_{\alpha_k}(Q(z)-\bar{\lambda}mz^n)$ lie in $|z|<1$ and this completes the proof of lemma 2.2.

Lemma 2.3. *If $P(z)$ is a polynomial of degree n having no zeros in $|z|<1$ and $m=\min_{|z|=1}|P(z)|, Q(z)=z^n\overline{P(1/\bar{z})}$, then for any real or complex numbers $\alpha_1,\cdots,\alpha_k,\ \ k\le n-1$ with $|\alpha_i|\ge 1,\ \ i=1,2,\cdots,k$ we have*

$$
\begin{aligned}
&|D_{\alpha_1}\cdots D_{\alpha_k}P(z)|\\
&\quad\le |D_{\alpha_1}\cdots D_{\alpha_k}Q(z)|-mn(n-1)\cdots(n-k+1)(|\alpha_1\alpha_2\cdots\alpha_k|-1),\ \text{for}\ \ |z|=1.
\end{aligned}
$$

Proof of Lemma 2.3. Since $P(z)$ has all zeros in $|z|\ge 1$ and $m=\min_{|z|=1}|P(z)|$, then

$$m\le |P(z)|,\quad \text{for}\ \ |z|=1.$$

Therefore, for every real or complex number λ with $|\lambda|<1$, it follows by Rouche's theorem for $m>0$ that the polynomial $F(z)=P(z)-\lambda m$ has all zeros in $|z|>1$ and hence no zero in $|z|<1$.Thus the polynomial $T(z)=z^n\overline{F(1/\bar{z})}=Q(z)-\bar{\lambda}mz^n$ has all zeros in $|z|<1$ and

$$|F(z)|\le |T(z)|,\quad \text{for}\ \ |z|=1.$$

It follows again by Rouche's theorem that for every β, $|\beta|>1$, the polynomial $F(z)-\beta T(z)$ has all zeros in $|z|<1$ which implies by Lemma 2.1 that for every real or complex numbers $\alpha_1,\cdots,\alpha_k$ with $|\alpha_i|\ge 1,\ \ i=1,2,\cdots,k$ the polynomial $D_{\alpha_1}\cdots D_{\alpha_k}[F(z)-\beta T(z)]$ has all zeros in $|z|<1$. This implies

$$|D_{\alpha_1}\cdots D_{\alpha_k}F(z)|\le |D_{\alpha_1}\cdots D_{\alpha_k}T(z)|,\quad \text{for}\ \ |z|\ge 1, \tag{2.4}$$

The inequality (2.4) is clearly equivalent to

$$|D_{\alpha_1}\cdots D_{\alpha_k}(P(z)-\lambda m)|\le |D_{\alpha_1}\cdots D_{\alpha_k}(Q(z)-\bar{\lambda}mz^n)|,\quad \text{for}\ \ |z|\ge 1.$$

Equivalently,

$$
\begin{aligned}
&|D_{\alpha_1}\cdots D_{\alpha_k}P(z)-\lambda mn(n-1)\cdots(n-k+1)|\\
&\quad\le |D_{\alpha_1}\cdots D_{\alpha_k}Q(z)-\bar{\lambda}mn(n-1)\cdots(n-k+1)\alpha_1\alpha_2\cdots\alpha_k z^{n-k}|,
\end{aligned}
$$

which gives

$$
\begin{aligned}
&|D_{\alpha_1}\cdots D_{\alpha_k}P(z)|-mn(n-1)\cdots(n-k+1)|\lambda|\\
&\quad\le |D_{\alpha_1}\cdots D_{\alpha_k}Q(z)-\bar{\lambda}mn(n-1)\cdots(n-k+1)\alpha_1\alpha_2\cdots\alpha_k z^{n-k}|,
\end{aligned}
\tag{2.5}
$$

for $|z|\ge 1$ and for every λ with $|\lambda|<1$.
Now choosing the argument of λ suitably, so that on $|z|=1$,

$$
\begin{aligned}
&|D_{\alpha_1}\cdots D_{\alpha_k}Q(z)-\bar{\lambda}mn(n-1)\cdots(n-k+1)\alpha_1\alpha_2\cdots\alpha_k z^{n-k}|\\
&\quad= |D_{\alpha_1}\cdots D_{\alpha_k}Q(z)|-mn(n-1)\cdots(n-k+1)|\alpha_1\alpha_2\cdots\alpha_k|\ |\lambda|,
\end{aligned}
\tag{2.6}
$$

we get from (2.5) that on $|z|=1$,

$$\begin{aligned}&|D_{\alpha_1}\cdots D_{\alpha_k}Q(z)|\\&\quad\geq |D_{\alpha_1}\cdots D_{\alpha_k}P(z)|+|\lambda|mn(n-1)\cdots(n-k+1)(|\alpha_1\alpha_2\cdots\alpha_k|-1).\end{aligned} \tag{2.7}$$

The fact that the right hand side of (2.6) is non-negative follows from Lemma 2.2. Lemma 2.3 now follows by making $|\lambda|\to 1$ in (2.7).

Lemma 2.4. *If $P(z)$ is a polynomial of degree n then for every complex number α and $\gamma>0$,*

$$\begin{aligned}&\left\{\int_0^{2\pi}|D_\alpha P(e^{i\theta})|^\gamma\right\}^{1/\gamma}\mathrm{d}\theta\\&\quad\leq n(|\alpha|+1)\left\{\int_0^{2\pi}|P(e^{i\theta})|^\gamma\right\}^{1/\gamma}\mathrm{d}\theta.\end{aligned}$$

Lemma 2.5. *If $P(z)$ is a polynomial of degree n which does not vanish in $|z|<t, t\geq 1$ and $Q(z)=z^n\overline{P(1/\bar{z})}$, then for every real or complex number α, real β with $0\leq\beta<2\pi$ and $\gamma>0$,*

$$\begin{aligned}&\int_0^{2\pi}\int_0^{2\pi}|D_\alpha P(e^{i\theta})+e^{i\beta}t^2D_{\alpha/t^2}Q(e^{i\theta})|^\gamma\mathrm{d}\theta\mathrm{d}\beta\\&\quad\leq 2\pi n^\gamma(|\alpha|+t)^\gamma\int_0^{2\pi}|P(e^{i\theta})|^\gamma\mathrm{d}\theta.\end{aligned}$$

The above two lemmas are due to Rather[12].

Lemma 2.6. *If A, B and C are non-negative real numbers such that $B+C\leq A$, then for every real number α,*

$$|(A-C)e^{i\alpha}+(B+C)|\leq|Ae^{i\alpha}+B|.$$

This lemma is due to Aziz and Rather[5].

3 Proof of Theorems

Proof of Theorem 1.1. Since $P(z)$ is a polynomial of degree at most n and $Q(z)=z^n\overline{P(1/\bar{z})}$, therefore for each β, $0\leq\beta<2\pi$, $F(z)=P(z)+e^{i\beta}Q(z)$ is a polynomial of degree at most n so that $D_{\alpha_1}\cdots D_{\alpha_k}F(z)=D_{\alpha_1}\cdots D_{\alpha_k}P(z)+e^{i\beta}D_{\alpha_1}\cdots D_{\alpha_k}Q(z)$ is a polynomail of degree at most $n-k,\ k=1,2,\cdots,n-1$. By repeated application of Lemma 2.4, we have for each $\gamma>0$,

$$\int_0^{2\pi}\left|D_{\alpha_1}\cdots D_{\alpha_k}F(e^{i\theta})\right|^\gamma\mathrm{d}\theta\leq(n-k+1)^\gamma(|\alpha_k|+1)^\gamma\int_0^{2\pi}\left|D_{\alpha_1}\cdots D_{\alpha_{k-1}}F(e^{i\theta})\right|^\gamma\mathrm{d}\theta. \tag{3.1}$$

Equivalently,

$$\begin{aligned}&\int_0^{2\pi}\left|D_{\alpha_1}\cdots D_{\alpha_k}P(e^{i\theta})+e^{i\beta}D_{\alpha_1}\cdots D_{\alpha_k}Q(e^{i\theta})\right|^{\gamma}\mathrm{d}\theta\\
&\le (n-k+1)^{\gamma}(|\alpha_k|+1)^{\gamma}\int_0^{2\pi}\left|D_{\alpha_1}\cdots D_{\alpha_{k-1}}P(e^{i\theta})+e^{i\beta}D_{\alpha_1}\cdots D_{\alpha_{k-1}}Q(e^{i\theta})\right|^{\gamma}\mathrm{d}\theta\\
&\le (n-k+1)^{\gamma}(n-k+2)^{\gamma}(|\alpha_k|+1)^{\gamma}(|\alpha_{k-1}|+1)^{\gamma}\\
&\qquad\times\int_0^{2\pi}\left|D_{\alpha_1}\cdots D_{\alpha_{k-2}}P(e^{i\theta})+e^{i\beta}D_{\alpha_1}\cdots D_{\alpha_{k-2}}Q(e^{i\theta})\right|^{\gamma}\mathrm{d}\theta \qquad (3.2)\\
&\qquad\vdots\\
&\qquad\vdots\\
&\le (n-k+1)^{\gamma}\cdots(n-1)^{\gamma}(|\alpha_k|+1)^{\gamma}\cdots(|\alpha_2|+1)^{\gamma}\\
&\qquad\times\int_0^{2\pi}\left|D_{\alpha_1}P(e^{i\theta})+e^{i\beta}D_{\alpha_1}Q(e^{i\theta})\right|^{\gamma}\mathrm{d}\theta. \qquad (3.3)\end{aligned}$$

Integrating both sides of (3.1) with respect to β from 0 to 2π, we get with the help of Lemma 2.5 (for $t=1$) that for each $\gamma>0$,

$$\begin{aligned}&\int_0^{2\pi}\int_0^{2\pi}\left|D_{\alpha_1}\cdots D_{\alpha_k}P(e^{i\theta})+e^{i\beta}D_{\alpha_1}\cdots D_{\alpha_k}Q(e^{i\theta})\right|^{\gamma}\mathrm{d}\theta\mathrm{d}\beta\\
&\le (n-k+1)^{\gamma}\cdots(n-1)^{\gamma}(|\alpha_k|+1)^{\gamma}\cdots(|\alpha_2|+1)^{\gamma}\int_0^{2\pi}\int_0^{2\pi}\left|D_{\alpha_1}P(e^{i\theta})+e^{i\beta}D_{\alpha_1}Q(e^{i\theta})\right|^{\gamma}\mathrm{d}\theta\mathrm{d}\beta\\
&\le 2\pi n^{\gamma}(n-1)^{\gamma}\cdots(n-k+1)^{\gamma}(|\alpha_k|+1)^{\gamma}\cdots(|\alpha_1|+1)^{\gamma}\int_0^{2\pi}\left|P(e^{i\theta})\right|^{\gamma}\mathrm{d}\theta. \qquad (3.4)\end{aligned}$$

Now by Lemma 2.3, for each θ, $0\le\theta<2\pi$ and any complex numbers $\alpha_1,\cdots,\alpha_k,\ \ k\le n-1$ with $|\alpha_i|\ge 1,\ \ i=1,2,\cdots,k$ we have

$$\begin{aligned}&\left|D_{\alpha_1}\cdots D_{\alpha_k}P(e^{i\theta})\right|\\
&\qquad\le\left|D_{\alpha_1}\cdots D_{\alpha_k}Q(e^{i\theta})\right|-mn(n-1)\cdots(n-k+1)(|\alpha_1\cdots\alpha_k|-1),\quad\text{for } |z|=1.\end{aligned}$$

This implies

$$\begin{aligned}&\left\{\left|D_{\alpha_1}\cdots D_{\alpha_k}P(e^{i\theta})\right|+\frac{mn(n-1)\cdots(n-k+1)(|\alpha_1\cdots\alpha_k|-1)}{2}\right\}\\
&\qquad\le\left\{\left|D_{\alpha_1}\cdots D_{\alpha_k}Q(e^{i\theta})\right|-\frac{mn(n-1)\cdots(n-k+1)(|\alpha_1\cdots\alpha_k|-1)}{2}\right\}. \qquad (3.5)\end{aligned}$$

Take $A = |D_{\alpha_1}\cdots D_{\alpha_k}Q(e^{i\theta})|$, $B = |D_{\alpha_1}\cdots D_{\alpha_k}P(e^{i\theta})|$,
$C = \dfrac{mn(n-1)\cdots(n-k+1)(|\alpha_1\cdots\alpha_k|-1)}{2}$ in lemma 2.6, we get

$$B+C \le A-C \le A.$$

Hence for every real β, with the help of Lemma 2.6, we get

$$\begin{aligned}&\left|\left\{\left|D_{\alpha_1}\cdots D_{\alpha_k}Q(e^{i\theta})\right| - \frac{mn(n-1)\cdots(n-k+1)(|\alpha_1\cdots\alpha_k|-1)}{2}\right\}e^{i\beta}\right.\\ &\left.+\left\{\left|D_{\alpha_1}\cdots D_{\alpha_k}P(e^{i\theta})\right| + \frac{mn(n-1)\cdots(n-k+1)(|\alpha_1\cdots\alpha_k|-1)}{2}\right\}\right|\\ &\le \left|\left|D_{\alpha_1}\cdots D_{\alpha_k}Q(e^{i\theta})\right|e^{i\beta} + \left|D_{\alpha_1}\cdots D_{\alpha_k}P(e^{i\theta})\right|\right|.\end{aligned}$$

This implies for each $\gamma > 0$,

$$\int_0^{2\pi}\left|F(\theta)+e^{i\beta}G(\theta)\right|^{\gamma}\mathrm{d}\theta \le \int_0^{2\pi}\left|\left|D_{\alpha_1}\cdots D_{\alpha_k}P(e^{i\theta})\right| + e^{i\beta}\left|D_{\alpha_1}\cdots D_{\alpha_k}Q(e^{i\theta})\right|\right|^{\gamma}\mathrm{d}\theta, \tag{3.6}$$

where

$$F(\theta) = \left|D_{\alpha_1}\cdots D_{\alpha_k}P(e^{i\theta})\right| + \frac{mn(n-1)\cdots(n-k+1)(|\alpha_1\cdots\alpha_k|-1)}{2}$$

and

$$G(\theta) = \left|D_{\alpha_1}\cdots D_{\alpha_k}Q(e^{i\theta})\right| - \frac{mn(n-1)\cdots(n-k+1)(|\alpha_1\cdots\alpha_k|-1)}{2}.$$

Integrating both sides of (3.6) with respect to β from 0 to 2π, we get with the help of (3.4), that for each $\gamma > 0$,

$$\begin{aligned}&\int_0^{2\pi}\int_0^{2\pi}\left|F(\theta)+e^{i\beta}G(\theta)\right|^{\gamma}\mathrm{d}\theta\mathrm{d}\beta\\ &\le \int_0^{2\pi}\int_0^{2\pi}\left|\left|D_{\alpha_1}\cdots D_{\alpha_k}P(e^{i\theta})\right| + e^{i\beta}\left|D_{\alpha_1}\cdots D_{\alpha_k}Q(e^{i\theta})\right|\right|^{\gamma}\mathrm{d}\theta\mathrm{d}\beta\\ &\le 2\pi n^{\gamma}(n-1)^{\gamma}\cdots(n-k+1)^{\gamma}(|\alpha_k|+1)^{\gamma}\cdots(|\alpha_1|+1)^{\gamma}\int_0^{2\pi}|P(e^{i\theta})|^{\gamma}\mathrm{d}\theta.\end{aligned} \tag{3.7}$$

Now for every real β and $t \ge 1$, we have

$$|t+e^{i\beta}| \ge |1+e^{i\beta}|,$$

which implies for every $\gamma > 0$,

$$\int_0^{2\pi}|t+e^{i\beta}|^{\gamma}\mathrm{d}\beta \ge \int_0^{2\pi}|1+e^{i\beta}|^{\gamma}\mathrm{d}\beta.$$

If $F(\theta)\neq 0$, we take $t=\left|\dfrac{G(\theta)}{F(\theta)}\right|$ and since $t\geq 1$ by (3.5)

$$\begin{aligned}
&\int_0^{2\pi}|F(\theta)+e^{i\beta}G(\theta)|^\gamma \mathrm{d}\beta\\
&= |F(\theta)|^\gamma\int_0^{2\pi}\left|1+\frac{G(\theta)}{F(\theta)}e^{i\beta}\right|^\gamma \mathrm{d}\beta\\
&= |F(\theta)|^\gamma\int_0^{2\pi}\left|\frac{G(\theta)}{F(\theta)}+e^{i\beta}\right|^\gamma \mathrm{d}\beta\\
&= |F(\theta)|^\gamma\int_0^{2\pi}\left|\left|\frac{G(\theta)}{F(\theta)}\right|+e^{i\beta}\right|^\gamma \mathrm{d}\beta\\
&\geq |F(\theta)|^\gamma\int_0^{2\pi}\left|1+e^{i\beta}\right|^\gamma \mathrm{d}\beta\\
&= \left\{\left|D_{\alpha_1}\cdots D_{\alpha_k}P(e^{i\theta})\right|+\frac{mn(n-1)\cdots(n-k+1)(|\alpha_1\cdots\alpha_k|-1)}{2}\right\}^\gamma\int_0^{2\pi}|1+e^{i\beta}|^\gamma \mathrm{d}\beta.
\end{aligned}$$

For $F(\theta)=0$, this inequality is trivially true. Using this in (3.7), we conclude that for each $\gamma>0,\beta$ real and any real or complex numbers $\alpha_1,\cdots,\alpha_k$, $k\leq n-1$ with $|\alpha_i|\geq 1$, $i=1,2,\cdots,k$,

$$\begin{aligned}
&\int_0^{2\pi}|1+e^{i\beta}|^\gamma d\beta\int_0^{2\pi}\left\{\left|D_{\alpha_1}\cdots D_{\alpha_k}P(e^{i\theta})\right|+\frac{mn(n-1)\cdots(n-k+1)(|\alpha_1\cdots\alpha_k|-1)}{2}\right\}^\gamma \mathrm{d}\theta\\
&\leq 2\pi n^\gamma(n-1)^\gamma\cdots(n-k+1)^\gamma(|\alpha_k|+1)^\gamma\cdots(|\alpha_1|+1)^\gamma\int_0^{2\pi}|P(e^{i\theta})|^\gamma \mathrm{d}\theta.
\end{aligned}\tag{3.8}$$

Now using the fact that for every real or complex number δ with $|\delta|\leq 1$,

$$\begin{aligned}
&\left|D_{\alpha_1}\cdots D_{\alpha_k}P(e^{i\theta})+\frac{mn(n-1)\cdots(n-k+1)(|\alpha_1\cdots\alpha_k|-1)}{2}\delta\right|\\
&\leq\left|D_{\alpha_1}\cdots D_{\alpha_k}P(e^{i\theta})\right|+\frac{mn(n-1)\cdots(n-k+1)(|\alpha_1\cdots\alpha_k|-1)}{2},
\end{aligned}$$

the desired result follows from (3.8).

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