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A NOTE ON MULTILINEAR FRACTIONAL INTEGRALS

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Abstract. Shi and Tao[6] studied the boundedness of multilinear fractional integrals introduced by Kenig and Stein[3] on product of weighted L^p-spaces, and got some results. We give some remarks with respect to their results and correct some mistakes. We also consider another multilinear fractional integral introduced by Grafakos[2].

Key words: *multilinear fractional integral, $A_{p,q}$ weight, $A^{\alpha}_{p,q}$ weight*

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1 Introduction

Recently the study of multilinear singular integral operators has received increasing attention. We are interested in multilinear fractional integrals. There are two types: $I_{\alpha}^{(m)}$ is defined by Kenig-Stein[3], and $I^m_{\alpha,\theta}$ is defined by Grafakos[2]. Shi-Tao[6] studied the boundedness of $I_{\alpha}^{(m)}$ on product of weighted L^p spaces. They showed [6; Theorem1.1], but it must to be corrected. We will give a counterexample for [6; Theorem1.1], and give a simple proof of [6; Theorem1.2]. We also consider weighted estimates for $I^m_{\alpha,\theta}$.

Now we introduce some notations and definitions for our results. We say a locally integrable function w is a weight if $w \geq 0$, and we denote $\|f\|_{L^p(\mathbf{R}^n;w)} = \left(\int_{\mathbf{R}^n} |f(x)|^p w(x)\mathrm{d}x\right)^{1/p}$.

Next we define two multilinear fractional integrals.

Definition 1.1. Let m be an integer and $0 < \alpha < mn$. We define the m-linear fractional integral

$$
\begin{aligned}
&I_\alpha^{(m)}(f_1, f_2, \cdots, f_m)(x) \\
&\qquad := \int_{\mathbf{R}^n} \prod_{i=1}^{m} f_i(x - y_i) \, |(y_1, y_2, \cdots, y_m)|^{\alpha - mn} \, \mathrm{d}y_1 \mathrm{d}y_2 \cdots \mathrm{d}y_m.
\end{aligned}
$$

When $m = 1$, $I_\alpha^{(m)}(f_1, f_2, \ldots, f_m)$ is the usual fractional integrals $I_\alpha(f)$ (see [7]).

Kenig-Stein[3] investigated the boundedness of $I_\alpha^{(m)}$ from $L^{p_1} \times L^{p_2} \times \cdots \times L^{p_m}$ to L^p, where $1/p = \sum_{i=1}^{m} 1/p_i - \alpha/n$.

Definition 1.2. Let $0 < \alpha < n$, $\theta_i \neq 0$ $(1 \leq i \leq m)$, and θ_i all distinct. We define another multilinear fractional integral

$$
\begin{aligned}
&I_{\alpha,\theta}^{m}(f_1, f_2, \ldots, f_m)(x) \\
&\qquad := \int_{\mathbf{R}^n} \prod_{i=1}^{m} f_i(x - \theta_i y) \, |y|^{\alpha - n} \, \mathrm{d}y, \quad \text{where} \quad \theta = (\theta_1, \ldots, \theta_m).
\end{aligned}
$$

Grafakos[2] investigated the boundedness of $I_{\alpha,\theta}^{m}$ from $L^{p_1} \times L^{p_2} \times \cdots \times L^{p_m}$ to L^p, where $1/p = \sum_{i=1}^{m} 1/p_i - \alpha/n$.

To consider the weighted boundedness of these operators, the following Theorems A and B are important. In 1974, Muckenhoupt and Wheeden[5] proposed the next weighted norm inequalities:

Definition 1.3. We say a weight w is in the class $A_p(\mathbf{R}^n)(1 < p < \infty)$ if

$$
\sup_{Q \subset \mathbf{R}^n, Q:\text{cube}} \left(\frac{1}{|Q|} \int_Q w(x) \mathrm{d}x \right) \left(\frac{1}{|Q|} \int_Q w(x)^{-\frac{1}{p-1}} \mathrm{d}x \right)^{p-1} < \infty.
$$

Definition 1.4. We say a weight w is in the class $A_{p,q}(\mathbf{R}^n)(1 < p, q < \infty)$ if

$$
\sup_{Q \subset \mathbf{R}^n,\ Q:\text{cube}} \left(\frac{1}{|Q|} \int_Q w(x)^q \mathrm{d}x \right)^{\frac{1}{q}} \left(\frac{1}{|Q|} \int_Q w(x)^{-p'} \mathrm{d}x \right)^{\frac{1}{p'}} < \infty
$$

and $\dfrac{1}{p} + \dfrac{1}{p'} = 1$.

Theorem A[5]. *Let $1 < p, q < \infty$ such that $\dfrac{1}{q} = \frac{1}{p} - \dfrac{\alpha}{n} > 0$. If $w \in A_{p,q}(\mathbf{R}^n)$, then there exists a constant $C > 0$ such that*

$$
\|I_\alpha f\|_{L^q(\mathbf{R}^n : w^q)} \leq C \, \|f\|_{L^p(\mathbf{R}^n : w^p)}.
$$

Remark . We have $w\in A_{p,p}(\mathbf{R}^n)$ if and only if $w^p\in A_p(\mathbf{R}^n)$, and w belongs to $A_{p,q}(\mathbf{R}^n)$ if and only if $w^q\in A_{1+\frac{q}{p'}}(\mathbf{R}^n)$.

Also in 2001, Garcia-Cuerva and Martell[1] introduced the following:

Definition 1.5. For a couple of weights (u,v), we say $(u,v)\in A^{\alpha}_{p,q}(\mathbf{R}^n)$ if there exists $r>1$ and a constant $C>0$ such that for each cube $Q\subset\mathbf{R}^n$,

$$|Q|^{\frac{1}{q}+\frac{\alpha}{n}-\frac{1}{p}}\left(\frac{1}{|Q|}\int_Q u(x)^r\mathrm{d}x\right)^{\frac{1}{rq}}\left(\frac{1}{|Q|}\int_Q v(x)^{r(1-p')}\mathrm{d}x\right)^{\frac{1}{rp'}}\le C.$$

Theorem B[1]. *Let* $0<\alpha<n$ *and* $1<p<q<\infty$*. If* $(u,v)\in A^{\alpha}_{p,q}(\mathbf{R}^n)$*, then there exists a constant* $C>0$ *such that*

$$\left(\int_{\mathbf{R}^n}|I_\alpha(f)(x)|^q v(x)\mathrm{d}x\right)^{\frac{1}{q}}\le C\left(\int_{\mathbf{R}^n}|f(x)|^p u(x)\mathrm{d}x\right)^{\frac{1}{p}}.$$

Shi-Tao[6; Theorem 1.1] states the following Proposition:

Proposition. *Let* $0<\alpha<mn$*,* $f_i\in L^{p_i}(\mathbf{R}^n:w^{p_i})$ *with* $1<p_i<\dfrac{mn}{\alpha}$ *and* $w\in\cap_{i=1}^m A_{p_i,q_i}(\mathbf{R}^n)$*, where* $1/q_i=1/p_i-\alpha/mn$*. If* $1/p=1/p_1+1/p_2+\cdots+1/p_m-\alpha/n$*, then there exists a constant* $C>0$ *such that*

$$\left\|I_\alpha^{(m)}(f_1,f_2,\ldots,f_m)\right\|_{L^p(\mathbf{R}^n:w^p)}\le C\prod_{i=1}^m\|f_i\|_{L^{p_i}(\mathbf{R}^n:w^{p_i})}. \tag{1}$$

However, this result is not true in general. In fact, we will give a counterexample when $m=2$ in §3.

Also Shi-Tao[6] proved the following by using Welland's inequality. We will give a simple proof without using this inequality in §2.

Theorem C[6 Theorem 1.2]. *Let* $0<\alpha<mn$ *and* $1<p_i<mp$ *for every* $i=1,2,\cdots,m$*. If* $(u,v)\in\cap_{i=1}^m A^{\alpha/m}_{p_i,mp}$*, then there is a constant* C *such that*

$$\left\|I_\alpha^{(m)}(f_1,f_2,\ldots,f_m)\right\|_{L^p(\mathbf{R}^n:v)}\le C\prod_{i=1}^m\|f_i\|_{L^{p_i}(\mathbf{R}^n:u)}. \tag{2}$$

Recently, Moen[4] showed the weighted estimates for I_α^m under weaker condition on (u,v). We consider weighted estimates for $I^m_{\alpha,\theta}$ and obtain the following results.

Theorem 1. *Let* $0<\alpha<n$ *and* s *be the harmonic mean of* $p_1,p_2,\ldots,p_m>1$*. Assume that* $1<s<\dfrac{n}{\alpha}$ *and let* $\dfrac{1}{p}=\dfrac{1}{s}-\dfrac{\alpha}{n}$*. Then, for* $w(x):=\Pi_{i=1}^m w_i(x)$*,* $w_i^{p_i/s}\in A_{s,p}(\mathbf{R}^n)(1\le i\le m)$*, there exists a constant* $C>0$ *such that*

$$\left\|I^m_{\alpha,\theta}(f_1,f_2,\ldots,f_m)\right\|_{L^p(\mathbf{R}^n:w^p)}\le C\prod_{i=1}^m\|f_i\|_{L^{p_i}(\mathbf{R}^n:w_i^{p_i})}.$$

In particular, when $w_i \equiv 1$ for all i, it is the same as Grafakos [2; Theorem 1] for $1 < s < \dfrac{n}{\alpha}$.

Theorem 2. *Let $0 < \alpha < n$ and s be the harmonic mean of $p_1, p_2, \ldots, p_m > 1$ with $1 < s < p$. If $(u,v) \in A^{\alpha}_{s,p}(\mathbf{R}^n)$, then there exists a constant $C > 0$ such that*

$$\left\| I^m_{\alpha,\theta}(f_1, f_2, \ldots, f_m) \right\|_{L^p(\mathbf{R}^n:v)} \leq C \prod_{i=1}^{m} \|f_i\|_{L^{p_i}(\mathbf{R}^n:u)}.$$

We will prove Theorems 1 and 2 in the next section. Throughout this paper, we may use varying a constant C.

2 Proofs of the Results

To prove Theorem C, we need the next lemma.

Lemma 2.1.

$$\left| I^{(m)}_{\alpha}(f_1, f_2, \ldots, f_m)(x) \right| \leq \prod_{i=1}^{m} I_{\frac{\alpha}{m}}\left(|f_i|\right)(x).$$

Proof. Since

$$|(y_1, y_2, \ldots, y_m)| = (|y_1|^2 + |y_2|^2 + \cdots + |y_m|^2)^{\frac{1}{2}} \geq (|y_1| \cdot |y_2| \cdots |y_m|)^{\frac{1}{m}},$$

we have

$$\begin{aligned}\left| I^{(m)}_{\alpha}(f_1, f_2, \ldots, f_m)(x) \right| &\leq \int_{(\mathbf{R}^n)^m} \prod_{i=1}^{m} \frac{|f_i(x - y_i)|}{|y_i|^{n - \frac{\alpha}{m}}} \mathrm{d}y_1 \mathrm{d}y_2 \cdots \mathrm{d}y_m \\ &= \prod_{i=1}^{m} I_{\frac{\alpha}{m}}(|f_i|)(x),\end{aligned}$$

and

$$\left| I^{(m)}_{\alpha}(f_1, f_2, \ldots, f_m)(x) \right| \leq \prod_{i=1}^{m} I_{\frac{\alpha}{m}}(|f_i|)(x).$$

Another proof of Theorem C. By Lemma 2.1 and Hölder's inequality with the assumption $(u,v) \in \cap_{i=1}^{m} A^{\alpha/m}_{p_i,mp}$ and Theorem B, we have

$$\begin{aligned}&\left(\int_{\mathbf{R}^n} \left| I^{(m)}_{\alpha}(f_1, f_2, \ldots, f_m)(x)) \right|^p v(x) \mathrm{d}x \right)^{\frac{1}{p}} \\ \leq &\left(\int_{\mathbf{R}^n} \left| \prod_{i=1}^{m} \left(I_{\frac{\alpha}{m}}\left(|f_i|\right)(x) \right) \right|^p v(x) \mathrm{d}x \right)^{\frac{1}{p}} \\ \leq &\prod_{i=1}^{m} \left(\int_{\mathbf{R}^n} I_{\frac{\alpha}{m}}(|f_i|)(x)^{mp} v(x) \mathrm{d}x \right)^{\frac{1}{mp}} \\ \leq &C \prod_{i=1}^{m} \|f_i\|_{L^{p_i}(\mathbf{R}^n:u)}.\end{aligned}$$

Therefore, we get (2).

Next we prove Theorem 1. First we prove the following lemma.

Lemma 2.2. *Under the condition of Theorem 1, we have*

$$\left|I^m_{\alpha,\theta}(f_1,f_2,\ldots,f_m)(x)\right| \le \prod_{i=1}^m \left(I_\alpha\left(|f_i|^{\frac{p_i}{s}}\right)(x)\right)^{\frac{s}{p_i}}.$$

Proof. By $1/s = 1/p_1 + 1/p_2 + \cdots + 1/p_m$ and Hölder's inequality, we have

$$\begin{aligned}\left|I^m_{\alpha,\theta}(f_1,f_2,\ldots,f_m)(x)\right| &\le \int_{\mathbf{R}^n} \prod_{i=1}^m |f_i(x-\theta_i y)|\,|y|^{\alpha-n}\,\mathrm{d}y\\ &\le \prod_{i=1}^m \left(\int_{\mathbf{R}^n} |f_i(x-\theta_i y)|^{\frac{p_i}{s}}\,|y|^{\alpha-n}\,\mathrm{d}y\right)^{\frac{s}{p_i}}\\ &= \prod_{i=1}^m \left(I_\alpha\left(|f_i|^{\frac{p_i}{s}}\right)(x)\right)^{\frac{s}{p_i}},\end{aligned}$$

and we get the desired result.

Proof of Theorem 1. By Lemma 2.2, we have

$$\int_{\mathbf{R}^n} \left|I^m_{\alpha,\theta}(f_1,f_2,\ldots,f_m)(x)w(x)\right|^p \mathrm{d}x \le \int_{\mathbf{R}^n} \prod_{i=1}^m \left(\left(I_\alpha\left(|f_i|^{\frac{p_i}{s}}\right)(x)w_i^{p_i/s}\right)\right)^{\frac{s}{p_i}\cdot p} \mathrm{d}x.$$

By Hölder's inequality and Theorem A,

$$\begin{aligned}\left\|I^m_{\alpha,\theta}(f_1,f_2,\ldots,f_m)\right\|_{L^p(\mathbf{R}^n:w^p)} &\le \prod_{i=1}^m \left(\int_{\mathbf{R}^n} \left|I_\alpha\left(|f_i|^{\frac{p_i}{s}}\right)(x)w_i^{p_i/s}(x)\right|^p \mathrm{d}x\right)^{\frac{s}{p_i}\cdot\frac{1}{p}}\\ &\le C\prod_{i=1}^m \left(\int_{\mathbf{R}^n} \left(|f_i(x)|^{\frac{p_i}{s}} w_i^{p_i/s}(x)\right)^s \mathrm{d}x\right)^{\frac{1}{p_i}}\\ &= C\prod_{i=1}^m \|f_i\|_{L^{p_i}(\mathbf{R}^n:w_i^{p_i})}.\end{aligned}$$

Hence, we have

$$\left\|I^m_{\alpha,\theta}(f_1,f_2,\ldots,f_m)\right\|_{L^p(\mathbf{R}^n:w^p)} \le C\prod_{i=1}^m \|f_i\|_{L^{p_i}(\mathbf{R}^n:w_i^{p_i})}.$$

Theorem 2 can be proved in the same way as that of Theorem C by using Theorem B and Lemma 2.2, therefore we omit the proof.

3 A Counterexample

We give a counterexample for (1). We consider the case $m = 2$ and $p_1 = p_2$, that is, $1/p = 2/p_1 - \alpha/n$ and $1/q_1 = 1/q_2 = 1/p_1 - \alpha/2n$.

Let $0 < \beta < n(1-1/p_1)$ and $\gamma = (3/4)\beta + n/p_1$. We define the next functions and weight. Let $f_1(x) = f_2(x) = |x|^{-\gamma}\chi_{\{|x|\leq 1\}}(x)$ and $w(x) = |x|^{\beta}$.

By Remark in §1, we have $w \in A_{p_1,q_1}(\mathbf{R}^n)$. Since

$$p_i(-\gamma+\beta)+n = \beta p_i/4 > 0,$$

we have $f_i \in L^{p_i}(\mathbf{R}^n : w^{p_i})$. However, we get

$$\left\| I_\alpha^{(2)}(f_1, f_2) \right\|_{L^p(\mathbf{R}^n : w^p)} = \infty.$$

In fact, let

$$0 < |x| < \frac{1}{10}.$$

Then

$$\begin{aligned}\left| I_\alpha^{(2)}(f_1, f_2)(x) \right| &\geq C \int_{\frac{|x|}{2}\leq |y_1| \leq \frac{3}{2}|x|} \int_{\frac{|x|}{2}\leq |y_2| \leq \frac{3}{2}|x|} \frac{f_1(x-y_1)f_2(x-y_2)}{|(y_1,y_2)|^{2n-\alpha}} \mathrm{d}y_1 \mathrm{d}y_2 \\ &\geq C|x|^{-2\gamma+\alpha},\end{aligned}$$

and

$$\int_{\mathbf{R}^n} \left| I_\alpha^{(2)}(f_1, f_2)(x) \right|^p w(x)^p \mathrm{d}x \geq C \int_0^{\frac{1}{10}} t^{p(-2\gamma+\alpha+\beta)+n-1} \mathrm{d}t = \infty,$$

since

$$p(-2\gamma+\alpha+\beta+n/p) = -\beta p/2 < 0.$$

Therefore, we get the desired result.

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