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SOME RESULTS ABOUT BEST COAPPROXIMATION IN $L^P(S,X)$

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Abstract. As a counterpart to best approximation in normed linear spaces, the best coapproximation was introduced by Franchetti and Furi. In this paper, we shall consider the relation between coproximinality M in X and $L^p(S,M)$ in $L^p(S,X)$. Finally we give some results in cochebyshev subspaces and additional subspaces.

Key words: *coproximinal subspace, cochebyshev subspace, reflexive subspace*

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1 Introduction

Let X be a normed linear space and M be a nonempty subspace of X. Then a point $g_0 \in M$ is said to be a best coapproximation for $x \in X$ if for every $g \in M$,

$$\|g - g_0\| \leq \|x - g\|.$$

If each $x \in X$ has at least one best coapproximation in M, then M is called a coproximinal subspace of X. If M is a coproximinal subspace in X, then M is closed in X. If each $x \in X$ has a unique best coapproximation in M, then M is called a cochebyshev subspace of X.

Let M be a subspace of a normed linear space X, then for $x \in X$ we put

$$R_M(x) = \{g_0 \in M : \|g - g_0\| \leq \|x - g\| \ \ \forall g \in M\},$$

the set of all best coapproximations for x in M. It is clear that $R_M(x)$ is a closed, bounded and convex subset of X. The set-valued function R_M associated to each x in X is called a cometric projection opeator. Put

$$\check{M} = \{x \in X \ : \ \|g\| \leq \|g - x\| \ \ \forall g \in M\} = R_M^{-1}(0).$$

Let X be a Banach space and (S,M,μ) be a finite complete measure space. A function $\varphi : S \to X$ is said to be simple if its range contains only finitely many points $x_1,x_2,\ldots,x_n \in X$, and if $\varphi^{-1}(x_i)$ is measurable for all $i = 1,2,\ldots,n$. Such φ can be written as $\varphi = \sum_{i=1}^{n} x_i \chi_{E_i}$, where χ_{E_i} is the characteristic function of the set $E_i = \varphi^{-1}(x_i)$. A function $f : S \to X$ is said to be strongly measurable if there exists a sequence $\{\varphi_n\}$ of simple functions with $\lim_{n\to\infty} \|\varphi_n(t) - f(t)\| = 0$ almost everywhere $[d\mu]$.

The space of Bochner p-integrable functions is denoted by $L^p(S,X)$ which contains of all strongly measurable functions $f : S \to X$ such that

$$\int_S \|f(t)\|^p \mathrm{d}\mu(t) < \infty \quad , \quad 1 \le p < \infty \cdot$$

The norm in $L^p(S,X)$ is defined to be $\|f\|_p = \left(\int_S \|f(t)\|^p \mathrm{d}\mu(t)\right)^{\frac{1}{p}}$. It is known that $L^p(S,X)$ is a Banach space. It is clear that if M is a closed subspace of a Banach space X, then $L^p(S,M)$ is a closed subspace of $L^p(S,X)$, $1 \le p < \infty$.

Let π be a set valued mapping, taking each point of a measurable space S into a subset of a topological space Y. We say that π is weakly measurable if the set

$$\pi^{\omega}(F) = \{t \in S : \pi(t) \cap F \neq \emptyset\}$$

is measurable in S for every closed set F in Y.

We say that π has a measurable selection if there exists a measurable function $f : S \longrightarrow Y$ such that $f(t) \in \pi(t)$ for each $t \in S$.

We assume that X is a Banach space and M is a closed subspace of X. For each $f \in L^1(S,X)$, we defined the map $\pi_f : S \longrightarrow 2^M$ by

$$\pi_f(t) = \{g \in M : f(t) - g \in \check{M}\}, \quad t \in S.$$

The aim of this paper is to find more results about coapproximation in $L^p(S,X)$ that obtained in [4]. For this purpose we give a list of known facts needed in the proof of our main results.

Lemma 1.1[4]**.** *If $f : S \longrightarrow X$ is measurable in the classical sense and has essentially separable range, then f is strongly measurable.*

Lemma 1.2[4]**.** *Let $\varphi : S \longrightarrow 2^Y$ be a weakly measurable set valued function. If Y is a complete separable metric space, then φ has a measurable selection.*

2 Coproximinality in $L^p(S,X)$

In this section at first we give some lemmas proved in [5] which are required in the sequel.

Lemma 2.1[5]**.** *Suppose M is coproximinal and π is weakly measurable. If M is separable, then $L^1(S,M)$ is coproximinal in $L^1(S,X)$.*

Lemma 2.2[5]**.** *Let M be a closed subspace of X. Then $g\in L^1(S,M)$ is a best coapproximation for an element f of $L^1(S,X)$ if and only if for almost all $s\in S$, $g(s)$ is a best coapproximation for $f(s)$.*

Proposition 2.3. *Let M be a closed subspace of X and $L^1(S,M)$ be coproximinal in $L^1(S,X)$, then M is coproximinal in X.*

Proof. Let $a\in X\setminus M$. Define $f:S\longrightarrow X$ by $f(t)=a$ for any $t\in S$. Since $\mu(S)<\infty$, $f\in L^1(S,X)$ and so there exists $g\in R_{L^1(S,M)}(f)$. Therefore by Lemma 2.2 for any $t\in S$, $g(t)$ is a best coapproximation in M for $f(t)=a$.

Lemma 2.4. *Let M be a coproximinal subspace in X. Then every simple function in $L^1(S,X)$ has a best coapproximation in $L^1(S,M)$.*

Proof. Let $f=\sum_{i=1}^{n}x_i\chi_{E_i}$ and x_i' be a best coapproximation x_i in M. Define $f'=\sum_{i=1}^{n}x_i'\chi_{E_i}$. Clearly $f'\in L^1(S,M)$. For all $g\in L^1(S,M)$,

$$\begin{aligned}
||g-f'||_1 &= \int_S ||g(t)-f'(t)||\mathrm{d}\mu(t)\\
&= \sum_{i=1}^{n}\int_{E_i}||g(t)-x_i'||\mathrm{d}\mu(t)\\
&\le \sum_{i=1}^{n}\int_{E_i}||g(t)-x_i||\mathrm{d}\mu(t)\\
&= \int_S ||g(t)-f(t)||\mathrm{d}\mu(t)\\
&= ||g-f||_1
\end{aligned}$$

and so f' is a best coapproximation for f in $L^1(S,M)$.

Definition. Let $(S,\mathcal{A},\mu)$ be a measure space. A subset F of $L^1(S,X)$ is called uniformly integrable if for every $\varepsilon>0$, there exist $\delta>0$ such that for every $E\in\mathcal{A}$ with $\mu(E)<\delta$, then

$$\int_E ||f(t)||\mathrm{d}\mu(t)<\varepsilon \quad (f\in F).$$

Lemma 2.5[5]**.** *Let $(S,\mathcal{A},\mu)$ be a measure space, and $\{f_n\}$ convergence to f in $L^1(S,X)$. Then $F=\{f,f_1,f_2,\ldots\}$ is uniformly integrable.*

Theorem 2.6. *Let M be a reflexive subspace and coapproximinal of X. Then $L^1(S,M)$ is coapproximinal in $L^1(S,X)$.*

Proof. Suppose $f \in L^1(S,X)$. Then there exists $\{f_n\}$ of simple functions that

$$\lim_{n\to\infty}\int_S ||f_n(t)-f(t)||\mathrm{d}\mu(t)=0.$$

By Lemma 2.4 for any $\{f_n\}$ there exists a best coapproximation $\{f_n'\}$ in $L^1(S,M)$. Put

$$K=\{f,f_1,f_2,\ldots\}, \qquad K'=\{f_1',f_2',\ldots\}.$$

We show K' has a weakly convergent subsequence. Since M is a reflexive subspace, by Danford Theorem in [1] and Theorem 13.1 in chapter 5[2], it is sufficient to verify that K' is bounded and uniformly integrable. f' is a best coapproximation of f_n, consequence $||f_n'|| \le ||f_n||$ for every $n \ge 1$. $\{f_n\}$ is convergent in $L^1(S,X)$, and so is bounded. Let $\varepsilon > 0$, by Lemma 2.2 for every $t \in S$ and every $n \ge 1$ $f_n'(t)$ is a best coapproximation of $f_n(t) andso ||f_n'(t)|| \le ||f_n(t)||$.by Lemma 2.5 there exists $\delta > 0$ such that for every measurable subspace E of S with $\mu(E) < \delta$, we have $\int_E ||f_n(t)||\mathrm{d}\mu(t) < \varepsilon$. Hence for $\mu(E) < \delta$

$$\int_E ||f_n'(t)||d\mu(t) \le \int_E ||f_n(t)||\mathrm{d}\mu(t) < \varepsilon,$$

Therefore K' is uniformly integrable. Then there exists a subsequence of $\{f_n'\}$ such that it weakly converges to f'. Without lose of generality we can suppose $f_n' \rightharpoonup f'$.Now we show that f' is a best coapproximation in $L^1(S,M)$ for f. For every $P \in [L^1(S,M)]^*$ with $\|P\| \le 1$ and for every $g \in L^1(S,M)$

$$\begin{aligned}
|<f'-g,P>| &= \lim_{n\to\infty} |<f_n'-g,P>| \\
&\le \lim_{n\to\infty} ||f_n'-g||\ ||P|| \\
&\le \lim_{n\to\infty} ||f_n'-g|| \\
&\le \lim_{n\to\infty} ||f_n-g|| \\
&= ||f-g||.
\end{aligned}$$

Therefore

$$\sup\{|<f'-g,P>| \,:\, P \in [L^1(S,M)]^*,\ ||P|| \le 1\} \le ||f-g||.$$

Then

$$||f'-g|| \le ||f-g||.$$

Corollary 2.7. *Let M be a coapproximinal subspace of X. If* $\dim M < \infty$, *then $L^1(S,M)$ is coapproximinal in $L^1(S,X)$.*

Theorem 2.8. *Let $1 < p < \infty$. The following are equivalent.*

(i) *$L^1(S,M)$ is coproximinal in $L^1(S,X)$.*

(ii) *$L^p(S,M)$ is coproximinal in $L^p(S,X)$.*

Proof. Suppose $L^P(S,M)$ is coproximinal in $L^P(S,X)$. Define the maping

$$J : L^1(S,X) \longrightarrow L^p(S,X),$$
$$J(f)(t) = f(t)||f(t)||^{\frac{1}{p}-1}.$$

Hence $||J(f)(t)|| = ||f(t)||^{\frac{1}{p}}$. It is clear that J is bijective. Now suppose $f \in L^1(S,X)$. Without lose of generality we can suppose $f \neq 0$ a.e.(μ). Since J is surjective and $L^p(S,M)$ is coproximinal in $L^p(S,X)$, there exists $g \in L^1(S,M)$ such that for every $h \in L^1(S,M)$

$$||J(g) - J(h)||_p \leq ||J(f) - J(g)||_p.$$

By Lemma 2.2 for every $y \in M$ a.e.(μ)

$$||J(g)(t) - y|| \leq ||J(f)(t) - y||.$$

Therefore for every $y \in M$ a.e.(μ),

$$||g(t)||g(t)||^{\frac{1}{p}-1} - y|| \leq ||f(t)||f(t)||^{\frac{1}{p}-1} - y||.$$

Thus

$$||g(t)||g(t)||^{\frac{1}{p}-1}||f(t)||^{1-\frac{1}{p}} - y|| \leq ||f(t) - y||,$$

and so $w(t) = g(t)||g(t)||^{\frac{1}{p}-1}||f(t)||^{1-\frac{1}{p}}$ for every $t \in S$ is a best coapproximation for $f(t)$. By Lemma 2.2 w is a best coapproximation for f in $L^1(S,M)$.

Conversely, let $f \in L^1(S,X)$. Since $\mu(S) < \infty$, $L^p(S,X) \subseteq L^1(S,X)$ and so $f \in L^1(S,X)$. Then there exists $g \in L^1(S,M)$ that is a best coapproximation for f. By Lemma 2.2 for every $t \in S$ a.e.(μ), $g(t)$ is a best coapproximation for $f(t)$ and so $||g(t)|| \leq ||f(t)||$ for every $t \in S$ a.e.(μ) and therefore $g \in L^P(S,M)$. Now h is an arbitrary element of $L^p(S,M)$,

$$||h(t) - g(t)|| \leq ||h(t) - f(t)||.$$

Consequently,

$$||h - g||_p \leq ||h - f||_p.$$

Corollary 2.9. *If M be a closed subspace of a Hilbert space H, then for $1 \leq p < \infty$, $L^p(S,M)$ is coproximinal in $L^p(S,X)$.*

Corollary 2.10. *If M be a closed subspace of X, then for $1 \le p < \infty$, $g \in L^p(S,M)$ is a best coproximinal of $f \in L^p(S,X)$ if and only if for almost $t \in S$, $g(t)$ is a best coapproximation for $f(t)$.*

Theorem 2.11.

(i) *Let M be a cochebyshev subspace of X. Then by either condition of Theorem* 2.6 *or Lemma* 2.1 *$L^P(S,M)$ is cochebyshev in $L^p(S,X)$.*

(ii) *If $L^P(S,M)$ is cochebyshev in $L^P(S,X)$, then M is cochebyshev in X.*

Proof. (i) By either condition of Theorem 2.6 or Lemma 2.1, $L^p(S,M)$ is coproximinal in $L^p(S,X)$. Suppose $g,h \in L^p(S,M)$ are two best coapproximations of $f \in L^p(S,X)$, then by Corollary 2.10 for almost $t \in S, g(t)$ and $h(t)$ are two best coapproximations for $f(t)$ and so $g(t) = h(t)$ for almost $t \in S$. Thus $g = h$ a.e.(μ).

(ii) Suppose $L^p(S,M)$ is cochebyshev in $L^p(S,X)$, then by Proposition 2.3 and Theorem 2.8, M is coproximinal in X. Let $b,c \in M$ be two best coapproximations of $a \in X$. Define

$$f : S \longrightarrow X \ , \ h : S \longrightarrow M \ , \ g : S \longrightarrow M$$
$$f(t) = a, \qquad h(t) = c, \qquad g(t) = b.$$

Clearly $f \in L^p(S,X)$, $g,h \in L^p(S,M)$ and g,h are two best coapproximations in $L^p(S,M)$ for f. Since $L^P(S,M)$ is cochebyshev in $L^P(S,X)$, then $g = h$ and so $b = c$.

Theorem 2.12. *Let M,N are separable and coproximinal in X. If for any $f \in L^1(S,X)$ π_f is weakly measurable, then $L^1(S,M) + L^1(S,N)$ is coproximinal in $L^1(S,X)$.*

Proof. Let $f \in L^1(S,X)$ be arbitrary and so π_f is weakly measurable. Then from Lemma 1.2, π_f has a measurable selection. Hence there exists a measurable function $g : S \longrightarrow M$ such that $g(t) \in \pi_f(t)$ for all $t \in S$. Thus, $f(t) - g(t) \in \check{M}$ for all $t \in S$. Now, we define

$$\breve{g} : S \longrightarrow \check{M} \ \text{ by } \ \breve{g}(t) = f(t) - g(t) \ \text{ for all } \ t \in S. \tag{1}$$

Since M is separable, from Lemma 1.1, g is strongly measurable. But f is strongly measurable, therefore $\breve{g}$ is strongly measurable and

$$\|\breve{g}(t)\| \le \|\breve{g}(t) + g(t)\|. \tag{2}$$

Now, we show $\|g\|_1 < \infty$. For this, consider

$$\begin{aligned} \|g(t)\|_1 = \int_S \|g(t)\| \mathrm{d}\mu(t) &\le \int_S \|\breve{g}(t) + g(t)\| \mathrm{d}\mu(t) \\ &= \int_S \|f(t)\| \mathrm{d}\mu(t) \\ &= \|f\|_1 < \infty. \end{aligned}$$

Similarly for N there exists $h \in L^1(S,N)$ with the same conditions as g.

Now define $w = \frac{1}{2}(g+h)$. Clearly $g \in L^1(S,M) + L^1(S,N)$. Now suppose $k \in L^1(S,M) + L^1(S,N)$, then

$$\begin{aligned}
\|w-k\|_1 &= \int_S \|\frac{1}{2}(g(t)+h(t)) - k(t)\| \mathrm{d}\mu(t) \\
&\leq \int_S \|\frac{1}{2}\breve{g}(t) + \frac{1}{2}g(t) + \frac{1}{2}\breve{h}(t) + \frac{1}{2}h(t) - k(t)\| \mathrm{d}\mu(t) \\
&= \int_S \|\frac{1}{2}f(t) + \frac{1}{2}f(t) - k(t)\| \mathrm{d}\mu(t) \\
&= \|f-k\|_1.
\end{aligned}$$

Then $L^1(S,M) + L^1(S,N)$ is coproximinal in $L^1(S,X)$.

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