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UNIFORM CONVERGENCE AND SEQUENCE OF MAPS ON A COMPACT METRIC SPACE WITH SOME CHAOTIC PROPERTIES

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Abstract. Recently, C. Tain and G. Chen introduced a new concept of sequence of time invariant function. In this paper we try to investigate the chaotic behavior of the uniform limit function $f: X \to X$ of a sequence of continuous topologically transitive (in strongly successive way) functions $f_n: X \to X$, where X is a compact interval. Surprisingly, we find that the uniform limit function is chaotic in the sense of Devaney. Lastly, we give an example to show that the denseness property of Devaney's definition is lost on the limit function.

Key words: *uniform convergence, chaos in the sense of Devaney, topological transitivity in strongly successive way*

AMS (2010) subject classification: 37D45, 54H20

1 Introduction

In [2], chaos of a time invariant continuous function on a metric space has been extensively studied. The authors also introduced several concepts, such as chaos for a sequence of time invariant functions. It is also well- known that if a sequence of continuous functions converges uniformly, then the uniform limit function is continuous. Also H. R. Flores [1] gives sufficient conditions for the topological transitivity of uniform limit function $f: X \to X$ of a sequence of continuous functions $f_n: X \to X$, where (X, d) is a compact metric space. Also chaos of a continuous function has been discussed extensively in [2, 3, 5 and 7]. But Devaney's definition[3] has certainly become one of the most popular. It is also the most purely topological and thus in some sense the simplest definition. Devaney's definition utilises the concept familiar to any student of basic point set topology. This motivates us to give further investigation whether the uniform limit function of a sequence of continuous functions on a compact and perfect metric space is chaotic in the sense of Devaney. To do this we require the main results of the two papers [4] and [6].

In this paper we take the definition of sequence of topologically transitive functions in successive way and we also modify this definition in a stronger sense. We then show that if $f_n : X \to X$ is a continuous sequence of continuous topologically transitive functions in strongly successive way, (X,d) is a compact metric space, then the uniform limit function $f : X \to X$ is also topologically transitive in Theorem-I. Then by the theorem of Vellekoop [4] we can say that the uniform limit function is chaotic in the sense of Devaney. Lastly we give an example to show that, although the uniform limit function is chaotic, the denseness property of Devaney's definition is lost on the limit function.

Throughout this paper we use some mathematical notations. We give them one by one.

i) If A is any set then we denote the boundary of A by $B(A)$.

ii) The radius of any set A is denoted by $Rad(A)$.

iii) If $\varepsilon > 0$ is arbitrary, we denote the ε-neighborhood of any point x by $S_\varepsilon(x)$.

2 Definitions

We now give some elementary definitions which is very important for our paper.

1. Uniform Convergence: Let (X,d) be a compact metric space and $f_n : X \to X$ be a sequence of continuous functions defined on X, $n = 1,2,\cdots$. Let $f : X \to X$ be a continuous function such that, $d(f_n(x), f(x)) < \varepsilon$, for all $n \geq n_0$ and for all x, where n_0 is a positive integer (depending on ε only), then we say that $\{f_n\}_{n=1}^{\infty}$ is uniformly convergent to f. If $\{f_n\}_{n=1}^{\infty}$ is uniformly convergent to f we then write $f_n \longrightarrow f$ uniformly.

2. Sensitive Dependence on Initial Conditions: A continuous function $f : X \to X$, where (X,d) is a compact metric space, has sensitive dependence on initial conditions if there exists $\delta > 0$ such that for any $x \in S$ and any neighborhood $N(x)$ of x, there exists $y \in N(x)$ and $n \geq 0$ such that $d(f^n(x), f^n(y)) > \delta$.

3. Topological Transitivity: The function $f : X \to X$ is said to be topologically transitive if for any pair of non-empty open sets $K, L \subset X$ there exists $n > 0$ such that $f^n(K) \cap L \neq \phi$, where (X,d) is a compact metric space.

4. Dense Set of Periodic Points: Let $f : X \to X$ be a continuous function on a compact metric space (X,d). If the set of all periodic points of f is dense in X, then f is said to have a dense set of periodic points.

5. Chaos in the Sense of Devaney: Let $f : X \to X$ be a continuous function on a compact metric space (X,d). Then f is said to be chaotic in the sense of Devaney if

i) f is topologically transitive on X;

ii) f has sensitive dependence on initial conditions;

iii) the set of periodic points is dense in X.

6. Topological Transitivity in Successive Way[2]: Let (X,d) be a compact metric space and $f_n : X \to X$ a sequence of continuous functions, $n = 1,2,\cdots$. If for any two non-empty open subsets U and V of X, there exists a positive integer k such that $f_k(U) \cap V \neq \phi$, then the sequence of functions $\{f_n\}_{n=1}^{\infty}$ is said to be topologically transitive on X in successive way.

We now modify this definition slightly.

7. Topological Transitivity in Strongly Successive Way : Let (X,d) be a compact metric space and $f_n : X \to X$ a sequence of continuous functions, $n = 1,2,\cdots$. If

i) for any two non-empty open subsets U and V of X, there exists a positive integer k such that $f_k(U) \cap V \neq \phi$ and

ii) for any two pair of distinct non-empty open subsets U_1, V_1 and U_2, V_2 of X, there exist positive integers $k_1 \neq k_2$ such that $f_{k_1}(U_1) \cap V_1 \neq \phi$ and $f_{k_2}(U_2) \cap V_2 \neq \phi$, then the sequence of function $\{f_n\}_{n=1}^{\infty}$ is said to be topologically transitive on X in strongly successive way.

8. Sensitive Dependence on Initial Conditions in Successive Way[2]: Let (X,d) be a compact metric space and $f_n : X \to X$ be a sequence of continuous functions, $n = 1,2,\cdots$. If there exists a constant $\delta > 0$ such that for any point $x \in X$ and any neighborhood $N(x)$ of x, there exist a point $y \in N(x)$ and a positive integer k such that $d(f_k(x), f_k(y)) > \delta$, then the sequence of functions $\{f_n\}_{n=1}^{\infty}$ is said to be sensitive dependence on initial conditions in successive way.

3 Main Theorems

Theorem-I. *Let (X,d) be a compact metric space and suppose that $\{f_n\}_{n=1}^{\infty}$ is a continuous sequence of functions from X into X such that $f_n \longrightarrow f$ uniformly. If the sequence $\{f_n\}_{n=1}^{\infty}$ is topologically transitive on X in strongly successive way, then f is topologically transitive.*

Proof. Let U_1 and V_1 be any two non-empty open subsets of X. Since the sequence $\{f_n\}_{n=1}^{\infty}$ is topologically transitive on X in the strongly successive way, there exists a positive integer k_1 such that $f_{k_1}(U_1) \cap V_1 \neq \phi$. Let $\varepsilon', \varepsilon'' > 0$. Consider an open ball $U_2 \subset U_1$ such that $Rad(U_2) = \dfrac{\varepsilon'}{2}$ and the minimum distance of $B(U_2)$ from $B(U_1)$ is $\dfrac{\varepsilon'}{100}$. Similarly we take an open ball $V_2 \subset V_1$

such that $Rad(V_2) = \dfrac{\varepsilon''}{2}$ and the minimum distance of $B(V_2)$ from $B(V_1)$ is $\dfrac{\varepsilon''}{100}$. Again by transitivity (in strongly successive way), there exists a positive integer k_2 such that $f_{k_2}(U_2) \cap V_2 \neq \phi$. We now take an open ball $U_3 \subset U_2$ such that $Rad(U_3) = \dfrac{\varepsilon'}{3}$ and the center of U_3 is same as that of U_2. Similarly we take an open ball $V_3 \subset V_2$ such that $Rad(V_3) = \frac{\varepsilon''}{3}$ and the center of V_3 is same as that of V_2. Then by transitivity (in strongly successive way) again, there exists a positive integer k_3 such that $f_{k_3}(U_3) \cap V_3 \neq \phi$. See Figure-1 and Figure-2. We now continue this process repeatedly. Then by our modified definition $\{k_n\}_{n=1}^{\infty}$ is an infinite subset of N. So, we rearrange this set as a sequence by taking the least element first then the next lowest element and so on. We now denote this rearrangement by $\{k_{n'}\}_{n=1}^{\infty}$. That is, $k_{1'}$ the least element of $\{k_n\}_{n=1}^{\infty}$.

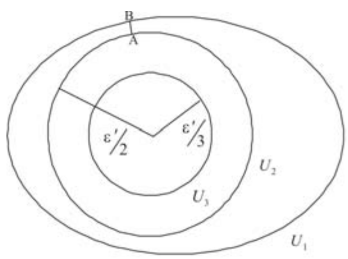

Figure 1.

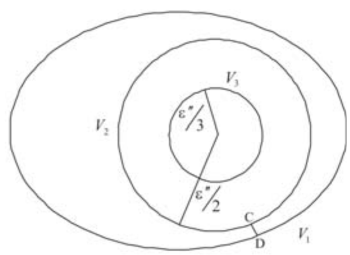

Figure 2.

By our construction AB$= \varepsilon'/100$ and CD$= \varepsilon''/100$. Also, $f_{k_1}(U_1) \cap V_1 \neq \phi$, $f_{k_2}(U_2) \cap V_2 \neq \phi$, $f_{k_3}(U_3) \cap V_3 \neq \phi$ and so on.

Then the following facts are noticeable:

a) U_i and V_i are open sets such that $U_{i+1} \subset U_i$ and $V_{i+1} \subset V_i$, for all $i = 1, 2, \cdots$.

b) There exists a sequence of positive integers $\{k_{n'}\}_{n=1}^{\infty}$ such that $f_{k_{n'}}(U_{n'}) \cap V_{n'} \neq \phi$, for all n'.

c) U_i's (and V_i's) are all open sets such that center of U_i's (and V_i's) are same for $i = 2, 3, \cdots$.

d) By **a)** and **b)** it can be easily proved that $f_{k_{n'}}(U_1) \cap V_1 \neq \phi$, for all $k_{n'}$.

With those facts as above we now proof Theorem-I. Obviously $f_{k_{n'}} \longrightarrow f$ uniformly for $n = 2, 3, \cdots$, since $\{f_{k_{n'}}\}_{n=2}^{\infty}$ is a subsequence of $\{f_{k_{n'}}\}_{n=1}^{\infty}$.

Then for $\varepsilon = \frac{\varepsilon''}{1000}$, $d(f_{k_{n'}}(x), f(x)) < \varepsilon$, for all $n' \geq m'$ and for all $x \in X$, with some

$$m' \in N - \{1\}. \tag{1}$$

We now show that $f^l(U_1) \cap V_1 \neq \phi$, for an $l > 0$.

Let

$$y \in f_{k_{m'}}(U_{m'}) \cap V_{m'}. \tag{2}$$

Then $y \in V_{m'}$ and $y \in f_{k_{m'}}(U_{m'})$. So, there exists $x \in U_{m'}$ such that $f_{k_{m'}}(x) = y$. Again from (1) we get $d(f_{k_{m'}}(x), f(x)) < \varepsilon$.

So, $f(x) \in S_\varepsilon(f_{k_{m'}}(x))$, that is

$$f(x) \in S_\varepsilon(y). \tag{3}$$

From (2) we get $y \in f_{k_{m'}}(U_1)$ and $y \in V_1$. Now $x \in U_{m'} \Rightarrow x \in U_1 \Rightarrow f(x) \in f(U_1)$. Since $m' \neq 1$, then by the definition of ε, (3) and also our construction above we see $f(x) \in V_1$. Hence, $f(U_1) \cap V_1 \neq \phi$.

So, we conclude that f is topologically transitive.

Hence, by an application of the theorem of Vellekoop in [4] and Theorem-I above we get our desired result.

Theorem-II. *Let X be a compact interval of real numbers and $f_n : X \to X$ a sequence of continuous function that converges uniformly to f, then f is chaotic on X whenever f_n are topologically transitive in strongly successive way.*

4 Conclusions

In [1], H. R. Flores shows that if $f_n : X \to X$ is a topologically transitive continuous sequence of functions on a metric space (X,d), then the uniform limit function f is not necessarily topologically transitive and then he gives the sufficient condition for transitivity of f. But in this paper we show that if $f_n : X \to X$ is a sequence of topologically transitive continuous functions in strongly successive way on a compact metric space (X,d), then the uniform limit function f is also topologically transitive (Theorem-I) in X. Hence by the theorem of Vellekoop [4], we can say that if $f_n : X \to X$ is a sequence of topologically transitive continuous functions in strongly successive way on a compact interval, then the uniform limit function is chaotic. So Theorem-II is a very important one, because we do not apply any condition for proving f to be chaotic.

Lastly we give an example to show that the denseness of periodic points will be lost on the limit function.

Example-I. Consider the sequence of translation maps $T_{\frac{(n+1)^{2n}}{(n^2+1)^n}} : S^1 \to S^1$ by

$$T_{\frac{(n+1)^{2n}}{(n^2+1)^n}}(x) = x + 2\pi \frac{(n+1)^{2n}}{(n^2+1)^n},$$

where S^1 is the unit circle on the plane. Now obviously $\dfrac{(n+1)^{2n}}{(n^2+1)^n}$ is a rational number for all $n = 1, 2, \cdots$. Hence, $\dfrac{(n+1)^{2n}}{(n^2+1)^n} = \dfrac{p}{q}$, where p, q are integers and $q \neq 0$. Then we see that

$T^q_{\frac{p}{q}}(x) = x + 2\pi q.\frac{p}{q} = x + 2\pi p = x$. So all points of S^1 are periodic with periods q. Hence, the set of all periodic points of S^1 are obviously dense in S^1. But note that

$$\lim_{n\to\infty} \frac{(n+1)^{2n}}{(n^2+1)^n} = e^2, \text{where } e^2 \text{ is an irrational number.}$$

Then by Jacobi's Theorem we get that $T_{e^2}(x)$ has dense orbit for each $x \in S^1$. Thus there is no periodic point. This proves that the denseness of periodic points will be lost on the limit function.

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References

[1] Flores, H. R., Uniform Convefgence and Transitivity, Chao. Sol. Frac., 38(2008), 148-153.

[2] Tian, C. and Chen, G., Chaos of a Sequence of Maps in a Metric Space, Chao. Sol. Frac., 28(2006), 1067-1075.

[3] Devaney, R. L., An Introduction to Chaotic Dynamical Systems, 2nd Edition, Addison-Wesley, Redwood City, CA, 1989.

[4] Vellekoop, M. and Berglund, R., On Intervals, Transitivity = Chaos, Amer. Math. Mon., 101:4(1994), 353-355.

[5] Robinsion, C., Dynamical System: Stability, Symbolic Dynamics and Chaos, 2nd Edition, CRC Press, Boca Raton, FL., 1999.

[6] Banks, J., Brooks, J., Cairns, G., Davis, G. and Stacey, P., On Devaney's Definition of Chaos, Amer. Math. Mon., 99:4(1992), 332-334.

[7] Touhey, P., Yet Another Definition of Chaos, Amer. Math. Mon., 104:4(1997), 411-414.

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