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FUNCTIONS APPROXIMATED BY ANY SEQUENCE OF INTERPOLATION POLYNOMIALS

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Abstract. In this note, we seek for functions f which are approximated by the sequence of interpolation polynomials of f obtained by any prescribed system of nodes.

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1 Introduction

It is very well known that interpolation by polynomials is a basic and useful method to approximate functions. But in order to obtain better approximation by interpolation polynomials to an approximated function, it is of much importance to select good nodes for the approximated function. Before showing examples, we give a theorem.

Theorem A. (see Theorem 2 in p.337 in Kincaid and Cheney[2]) *Let f be an n times continuously differentiable function on $[-1,1]$ and let p be the polynomial of degree at most n that interpolates the function f at $n+1$ distinct nodes $x_0,\dots,x_n$ in $[-1,1]$,* i.e., $p(x_i)=f(x_i), i=0,\dots,n$. *To each $x\in[-1,1]$ there exists a point $\xi_x\in(-1,1)$ such that*

$$f(x)-p(x)=\frac{1}{(n+1)!}f^{(n+1)}(\xi_x)\prod_{i=0}^{n}(x-x_i).$$

Here we give two examples.

Example 1. Let $f(x)=\sin\pi x,\ x\in[-1,1]$. Let

$$X_n:(-1\leqq)x_0^{(n)}<x_1^{(n)}<\cdots<x_{k_n}^{(n)}(\leqq 1),\qquad n\in\mathbf{N}$$

be any prescribed system of nodes, where $\lim_{n\to\infty} k_n = \infty$ and let $p_{f,X_n}(x), n \in \mathbf{N}$ be the polynomials of degree at most k_n that interpolates the function f at nodes of X_n. By Theorem A, we have

$$|f(x) - p_{f,X_n}(x)| \leqq \frac{1}{(k_n+1)!}\pi^{k_n+1}\left|\prod_{i=0}^{k_n}(x - x_i^{(n)})\right| \leqq \frac{2^{k_n+1}\pi^{k_n+1}}{(k_n+1)!} \quad \text{for all } x \in [-1,1].$$

Hence, we obtain

$$\lim_{n\to\infty} \|f - p_{f,X_n}\|_\infty = 0,$$

where $\|\cdot\|_\infty$ denotes the supremum norm on $[-1,1]$.

Example 2. Let $g(x) = \dfrac{1}{1+25x^2}$, $x \in [-1,1]$. Let

$$X_n : x_0^{(n)} = -1, x_1^{(n)} = -1 + \frac{1}{n}, \cdots, x_n^{(n)} = 0, \cdots, x_{2n}^{(n)} = 1, \qquad n \in \mathbf{N}$$

be the system of equally spaced nodes and let $p_{g,\,X_n}(x), n \in \mathbf{N}$ be the polynomials of degree at most $2n$ that interpolates the function g at nodes of X_n. Though g is an analytic function on $[-1,1]$, it is known that

$$\overline{\lim_{n\to\infty}} \|f - p_{f,\,X_n}\|_\infty = +\infty.$$

This example is called Runge example.

In this note, we seek for functions which have the same property as the function of Example 1. Before precisely stating the purpose of this note, we shall explain some definitions and notations.

Let $M[-1,1]$ be the space of all real-valued bounded functions on the interval $[-1,1]$ of $\mathbf{R}$. $M[-1,1]$ is endowed with the supremum norm $\|\cdot\|_\infty$. If

$$X_n : (-1 \leqq) x_0^{(n)} < x_1^{(n)} < \cdots < x_{k_n}^{(n)} (\leqq 1), \qquad n \in \mathbf{N}$$

is any prescribed system of nodes, where $\lim_{n\to\infty} k_n = \infty$, then it is said that the system has $(*)$-property.

In this note, we consider a class of functions f in $M[-1,1]$ such that $\lim_{n\to\infty} \|f - p_{f,\,X_n}\|_\infty = 0$ for any system $X_n, n \in \mathbf{N}$ of nodes with $(*)$-property. For convenience, we write $\mathcal{A}$ for this class of functions.

Definition. Let n be a nonnegative integer. For $n+1$ distinct nodes $X : x_0, \ldots, x_n$ in $[-1,1]$ and $f \in M[-1,1]$, we put

$$p_{f,X}(x) = a_0 + a_1 x + \cdots + a_n x^n.$$

Then we denote by $f[x_0, \ldots, x_n]$ the coefficient a_n.

Remark 1. $f[x_0,\dots,x_n]$ is obtained by the equations

$$f[x_i]=f(x_i)\ ,\ \ f[x_i,x_{i+1},\dots,x_{i+j}]=\frac{f[x_{i+1},\dots,x_{i+j}]-f[x_i,\dots,x_{i+j-1}]}{x_{i+j}-x_i}.$$

Hence, we see that $f[x_0,\dots,x_n]$ is determined only by the values $f(x_0),f(x_1),\dots,f(x_n)$ (see pp.353-354 in Kincaid and Cheney[2]).

We need the following two lemmas to proceed our argument.

Lemma 1. (cf. Theorem 4 in p.357 in Kincaid and Cheney[2]) *If f is n times continuously differentiable on $[-1,1]$ and if $x_0,\dots,x_n$ are distinct nodes, then $f[x_0,\dots,x_n]$ converges to $\dfrac{f^{(n)}(c)}{n!}$ as each $x_i, i=0,\dots,n$ converges to c.*

Lemma 2. (see Markoff's inequality in p.91 in Cheney[1]) *Let k,n be nonnegative integers with $k\leqq n$. For any polynomial p on $[-1,1]$ of degree at most n, it holds that*

$$\|p^{(k)}\|_\infty\leqq M_{n,k}\|p\|_\infty,$$

where $M_{n,k}=(n\cdots(n-k+1))^2$ and $p^{(0)}=p$.

Now we state the following

Theorem. *If a function $f\in M[-1,1]$ belongs to $\mathcal{A}$, then f is an analytic function on $[-1,1]$ and the radiuses of convergence of the Taylor series about -1, 1 for f are at least* 2.

Proof. Let f be a function in $\mathcal{A}$. For a system $X_n, n\in\mathbf{N}$ of nodes with $(*)$-property, since the sequence $p_{f,X_n}, n\in\mathbf{N}$ uniformly converges to f, f is a continuous function on $[-1,1]$.

We shall verify that f is continuously differentiable on $[-1,1]$. First we show what to prove in order to show that f is differentiable on $[-1,1]$ and f' is continuous on $[-1,1]$.

Suppose on the contrary that f is not differentiable at a point $a\in[-1,1]$. Then, there exist a positive number ε_1 and two sequences $\{\alpha_n\}$, $\{\beta_n\}$ which converge to a such that

$$\left|\frac{f(\alpha_n)-f(a)}{\alpha_n-a}-\frac{f(\beta_n)-f(a)}{\beta_n-a}\right|>\varepsilon_1,\quad n=1,2,\dots, \tag{1}$$

where a, α_n, β_n, $n=1,2,\dots$ are distinct from each other.

Suppose that f is differentiable on $[-1,1]$ and suppose on the contrary that f' is not continuous at a point $a\in[-1,1]$. Then, there exist a positive number δ_1 and three sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{z_n\}$ which converge to a such that

$$\left|\frac{f(\alpha_n)-f(z_n)}{\alpha_n-z_n}-\frac{f(\beta_n)-f(a)}{\beta_n-a}\right|>\delta_1,\quad n=1,2,\dots, \tag{2}$$

where a, α_n, β_n, z_n, $n=1,2,\dots$ are distinct from each other.

Since any condition of (1) and (2) analogously lead to a contradiction, we show a contradiction under the condition (2). Without loss of generality, we can assume that

$$\max\{a,\alpha_n,\beta_n,z_n\} - \min\{a,\alpha_n,\beta_n,z_n\} < \frac{\delta_1}{n\cdot M_{n+2,2}}, \quad n = 1,2,\dots \tag{3}$$

and set a system $X_n, n \in \mathbf{N}$ of nodes such that

$$X_n : a,\alpha_n,\beta_n,z_1,\dots,z_n.$$

Since f belongs to $\mathcal{A}$, $\lim_{n\to\infty} \|f - p_{f,X_n}\|_\infty = 0$ and we put a positive number L_1 such that

$$\sup_{n\in\mathbf{N}} \|p_{f,X_n}\|_\infty < L_1.$$

From Mean Value Theorem and (2), we have

$$\begin{aligned}\delta_1 &< \left|\frac{f(\alpha_n)-f(z_n)}{\alpha_n - z_n} - \frac{f(\beta_n)-f(a)}{\beta_n - a}\right| = \left|\frac{p_{f,X_n}(\alpha_n)-p_{f,X_n}(z_n)}{\alpha_n - z_n} - \frac{p_{f,X_n}(\beta_n)-p_{f,X_n}(a)}{\beta_n - a}\right| \\ &= \left|p'_{f,X_n}(\sigma_n^{(1)}) - p'_{f,X_n}(\tau_n^{(1)})\right|,\end{aligned}$$

where $\sigma_n^{(1)}$ is a point between α_n and z_n, and $\tau_n^{(1)}$ is a point between β_n and a. Again from Mean Value Theorem and (3), since $|\sigma_n^{(1)} - \tau_n^{(1)}| < \dfrac{\delta_1}{n\cdot M_{n+2,2}}$, we obtain

$$|p''_{f,X_n}(\xi_n^{(1)})| = \left|\frac{p'_{f,X_n}(\sigma_n^{(1)}) - p'_{f,X_n}(\tau_n^{(1)})}{\sigma_n^{(1)} - \tau_n^{(1)}}\right| > n\cdot M_{n+2,2},$$

where $\xi_n^{(1)}$ is a point between $\sigma_n^{(1)}$ and $\tau_n^{(1)}$. For sufficiently large n it holds that

$$\|p''_{f,X_n}\|_\infty \geqq |p''_{f,X_n}(\xi_n^{(1)})| > n\cdot M_{n+2,2} \geqq L_1\cdot M_{n+2,2},$$

which contradicts Lemma 2. Hence, f is continuously differentiable on $[-1,1]$.

Next we shall verify that f is $k(\geqq 2)$ times continuously differentiable on $[-1,1]$ under the condition that f is $k-1$ times continuously differentiable on $[-1,1]$. To do this, we have to prove that f is k times differentiable on $[-1,1]$ and further $f^{(k)}$ is continuous on $[-1,1]$. By the same reason as the proof of continuously differentiability of f, we devote ourself to proving that $f^{(k)}$ is continuous on $[-1,1]$.

Suppose that $f^{(k-1)}$ is differentiable on $[-1,1]$ and suppose on the contrary that $f^{(k)}$ is not continuous at a point $a \in [-1,1]$. Then, there exist a positive number δ_k and three sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{z_n\}$ which converge to a such that

$$\left|\frac{f^{(k-1)}(\alpha_n)-f^{(k-1)}(z_n)}{\alpha_n - z_n} - \frac{f^{(k-1)}(\beta_n)-f^{(k-1)}(a)}{\beta_n - a}\right| > \delta_k, \quad n = 1,2,\dots. \tag{4}$$

For each $\alpha_n, \beta_n, z_n, n = 1, 2, \ldots$ and a, by Lemma 1, there exist $\alpha_1^{(n)}, \ldots, \alpha_k^{(n)}$, $\beta_1^{(n)}, \ldots, \beta_k^{(n)}$, $z_1^{(n)}, \ldots, z_k^{(n)}$ and $a_1^{(n)}, \ldots, a_k^{(n)}$ satisfying that

$$D := \left| \frac{f[\alpha_1^{(n)}, \ldots, \alpha_k^{(n)}] - f[z_1^{(n)}, \ldots, z_k^{(n)}]}{\alpha_n - z_n} - \frac{f[\beta_1^{(n)}, \ldots, \beta_k^{(n)}] - f[a_1^{(n)}, \ldots, a_k^{(n)}]}{\beta_n - a} \right| > \frac{\delta_k}{2(k-1)!} \quad (5)$$

Furthermore, without loss of generality, we assume that $4k + n$ points $\alpha_i^{(n)}$, $\beta_i^{(n)}$, $z_i^{(n)}$, $a_i^{(n)}$, $i = 1, \cdots, k$ and $z_1, \cdots, z_n$ are distinct from each other, and assume that

$$\begin{aligned} &\max\{\alpha_1^{(n)}, \cdots, \alpha_k^{(n)}, \beta_1^{(n)}, \cdots, \beta_k^{(n)}, z_1^{(n)}, \cdots, z_k^{(n)}, a_1^{(n)}, \cdots, a_k^{(n)}\} \\ &\quad - \min\{\alpha_1^{(n)}, \cdots, \alpha_k^{(n)}, \beta_1^{(n)}, \cdots, \beta_k^{(n)}, z_1^{(n)}, \cdots, z_k^{(n)}, a_1^{(n)}, \cdots, a_k^{(n)}\} < \frac{\delta_k}{4n \cdot M_{n+4k-1,k+1}}. \quad (6) \end{aligned}$$

We set a system $X_n, n \in \mathbf{N}$ of nodes such that

$$X_n : \alpha_1^{(n)}, \ldots, \alpha_k^{(n)}, \beta_1^{(n)}, \ldots, \beta_k^{(n)}, z_1^{(n)}, \ldots, z_k^{(n)}, a_1^{(n)}, \ldots, a_k^{(n)}, z_1, \ldots, z_n.$$

Since f belongs to $\mathcal{A}$ and the system $X_n, n \in \mathbf{N}$ has $(*)$-property, $\lim_{n\to\infty} \|f - p_{f,X_n}\|_\infty = 0$ and we put a positive number L_k such that $\sup_{n\in\mathbf{N}} \|p_{f,X_n}\|_\infty < L_k$. From Lemma 1, we have

$$\begin{aligned} D &= \left| \frac{p_{f,X_n}[\alpha_1^{(n)}, \ldots, \alpha_k^{(n)}] - p_{f,X_n}[z_1^{(n)}, \ldots, z_k^{(n)}]}{\alpha_n - z_n} - \frac{p_{f,X_n}[\beta_1^{(n)}, \ldots, \beta_k^{(n)}] - p_{f,X_n}[a_1^{(n)}, \ldots, a_k^{(n)}]}{\beta_n - a} \right| \\ &= \frac{1}{(k-1)!} \left| \frac{p_{f,X_n}^{(k-1)}(\alpha_n') - p_{f,X_n}^{(k-1)}(z_n')}{\alpha_n - z_n} - \frac{p_{f,X_n}^{(k-1)}(\beta_n') - p_{f,X_n}^{(k-1)}(a')}{\beta_n - a} \right| > \frac{\delta_k}{2(k-1)!}. \end{aligned}$$

Here we note that $\max\{|p_{f,X_n}^{(k-1)}(\alpha_n')|, |p_{f,X_n}^{(k-1)}(z_n')|, |p_{f,X_n}^{(k-1)}(\beta_n')|, |p_{f,X_n}^{(k-1)}(a')|\} \leqq \|f^{(k-1)}\|_\infty$. Moreover, we assume that $\alpha_i^{(n)}$, $z_i^{(n)}$, $\beta_i^{(n)}$ and $a_i^{(n)}$, $i = 1, \ldots, k$ are sufficiently close to α_n, z_n, β_n and a respectively such that

$$\left| \frac{p_{f,X_n}^{(k-1)}(\alpha_n') - p_{f,X_n}^{(k-1)}(z_n')}{\alpha_n' - z_n'} - \frac{p_{f,X_n}^{(k-1)}(\beta_n') - p_{f,X_n}^{(k-1)}(a')}{\beta_n' - a'} \right| > \frac{\delta_k}{4}.$$

From Mean Value Theorem, we have

$$\left| \frac{p_{f,X_n}^{(k-1)}(\alpha_n') - p_{f,X_n}^{(k-1)}(z_n')}{\alpha_n' - z_n'} - \frac{p_{f,X_n}^{(k-1)}(\beta_n') - p_{f,X_n}^{(k-1)}(a')}{\beta_n' - a'} \right| = \left| p_{f,X_n}^{(k)}(\sigma_n^{(k)}) - p_{f,X_n}^{(k)}(\tau_n^{(k)}) \right| > \frac{\delta_k}{4},$$

where $\sigma_n^{(k)}$ is a point between α_n' and z_n', and $\tau_n^{(k)}$ is a point between β_n' and a'. Again from Mean Value Theorem and (6), it follows that there exists a point $\xi_n^{(k)}$ between $\sigma_n^{(k)}$ and $\tau_n^{(k)}$ and

$$|p_{f,X_n}^{(k+1)}(\xi_n^{(k)})| = \left| \frac{p_{f,X_n}^{(k)}(\sigma_n^{(k)}) - p_{f,X_n}^{(k)}(\tau_n^{(k)})}{\sigma_n^{(k)} - \tau_n^{(k)}} \right| > \frac{\delta_k}{4(\sigma_n - \tau_n)} > n \cdot M_{n+4k-1,k+1}.$$

Hence, for sufficiently large n it holds that

$$\|p_{f,X_n}^{(k+1)}\|_\infty \geqq |p_{f,X_n}^{(k+1)}(\xi_n)| > n \cdot M_{n+4k-1,k+1} \geqq L_k \cdot M_{n+4k-1,k+1},$$

which contradicts Lemma 2. Hence, f is k times continuously differentiable on $[-1,1]$.

Now we are in position to state that any $f \in \mathcal{A}$ is infinitely differentiable on $[-1,1]$. For any $f \in \mathcal{A}$ and any $c \in [-1,1]$, we can consider a system $X_n : x_0^{(n)}, \ldots, x_n^{(n)}, n \in \mathbf{N}$ with $(*)$-property such that

$$\left\| p_{f,X_n} - \sum_{p=0}^{n} \frac{f^{(p)}(c)}{p!}(x-c)^p \right\|_\infty < \frac{1}{n}, \quad n = 1, 2, \ldots. \tag{7}$$

Since $\lim_{n\to\infty} \|f - p_{f,X_n}\|_\infty = 0$, by (7), f is expressed as

$$f(x) = \sum_{p=0}^{\infty} \frac{f^{(p)}(c)}{p!}(x-c)^p, \quad x \in [-1,1].$$

This means that the radiuses of the Taylor series about -1, 1 for f are at least 2.

Remark 2. Let f be an analytic function on $[-1,1]$ and suppose that the radiuses of convergence of the Taylor series about -1, 1 for f are more than 2. Then we can regard $f(z)$ as a regular function on a simply connected region $D(\supset [-1,1])$ in the complex plane. Furthermore, D contains a simple, closed, rectifiable curve C such that

$$|z-a| > |x-a| \quad \text{for all } z \in C \text{ and all } a, x \in [-1,1].$$

Let $X : x_0, x_1, \ldots, x_n$ be any distinct nodes in $[-1,1]$. Then, from the formula $(4.10.6)$ in p.165 in Mori[3], we have

$$f(x) - p_{f,X}(x) = \frac{1}{2\pi i} \oint_C \frac{(x-x_0)\cdots(x-x_n)}{(z-x)(z-x_0)\cdots(z-x_n)} f(z)\mathrm{d}z, \quad x \in [-1,1].$$

Hence, we see $f \in \mathcal{A}$ without difficulty.

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