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SOME POLYNOMIAL INEQUALITIES IN THE COMPLEX DOMAIN

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Abstract. Let

$$P(z)=\sum_{\nu=0}^{n}c_\nu z^\nu$$

be a polynomial of degree n and let $M(f,r)=\max\limits_{|z|=r}|f(z)|$ for an arbitrary entire function $f(z)$. If $P(z)$ has no zeros in $|z|<1$ with $M(P,1)=1$, then for $|\alpha|\le 1$, it is proved by Jain[Glasnik Matematički, 32(52) (1997), 45−51] that

$$\begin{aligned}\left|P(Rz)+\alpha\left(\frac{R+1}{2}\right)^n P(z)\right| \quad\le\quad & \frac{1}{2}\left\{\left|1+\alpha\left(\frac{R+1}{2}\right)^n\right|\right.\\ & \left.+\left|R^n+\alpha\left(\frac{R+1}{2}\right)^n\right|\right\},\quad R\ge 1,|z|=1.\end{aligned}$$

In this paper, we shall first obtain a result concerning minimum modulus of polynomials and next improve the above inequality for polynomials with restricted zeros. Our result improves the well known inequality due to Ankeny and Rivlin[1] and besides generalizes some well known polynomial inequalities proved by Aziz and Dawood[3].

Key words: *polynomial, inequality, restricted zeros*

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1 Introduction and Statement of Results

If $P(z)$ is a polynomial of degree n, then[6, p.158, problem III, p.269]

$$M(P,R)\le R^n M(P,1)\ \text{ for }\ R\ge 1. \tag{1.1}$$

The result is best possible and the equality holds for polynomials having zeros at the origin.

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Inequality (1.1) was generalized by Jain[4] who proved that if $P(z)$ is a polynomial of degree n, then for $|z|=1$ and $|\alpha|\leq 1$,

$$\left|P(Rz)+\alpha\left(\frac{R+1}{2}\right)^{n}P(z)\right|\leq\left|R^{n}+\alpha\left(\frac{R+1}{2}\right)^{n}\right|M(P,1),\ \ R\geq 1. \tag{1.2}$$

It was shown by Ankeny and Rivlin[1] that if $P(z)\neq 0$ in $|z|<1$, then (1.1) can be replaced by

$$M(P,R)\leq\frac{R^{n}+1}{2}M(P,1)\ \text{ for }\ R\geq 1. \tag{1.3}$$

Inequality (1.3) is sharp and the equality holds for $P(z)=\beta+\gamma z^{n}$, where $|\beta|=|\gamma|=1/2$.

In the same manner the inequality (1.3) was generalized by Jain[5] for polynomials having no zeros in $|z|<1$ with $|\alpha|\leq 1$ and $|z|=1$,

$$\begin{aligned}&\left|P(Rz)+\alpha\left(\frac{R+1}{2}\right)^{n}P(z)\right|\\&\leq\frac{1}{2}\left\{\left|1+\alpha\left(\frac{R+1}{2}\right)^{n}\right|+\left|R^{n}+\alpha\left(\frac{R+1}{2}\right)^{n}\right|\right\}M(P,1),\ \ R\geq 1.\end{aligned} \tag{1.4}$$

The result is best possible and the equality holds for $P(z)=\beta+\gamma z^{n}$, where $|\beta|=|\gamma|=1/2$.

In this paper, we firstly obtain an interesting result concerning minimum modulus of polynomials $P(z)$ which is analogous to the inequality (1.2).

Theorem 1. *If $P(z)$ is a polynomial of degree n, having all its zeros in $|z|<1$, then for every real or complex number α with $|\alpha|\leq 1$ and $R\geq 1$,*

$$\min_{|z|=1}\left|P(Rz)+\alpha\left(\frac{R+1}{2}\right)^{n}P(z)\right|\geq\left|R^{n}+\alpha\left(\frac{R+1}{2}\right)^{n}\right|\min_{|z|=1}|P(z)|. \tag{1.5}$$

The result is best possible and the equality holds for $P(z)=me^{i\beta}z^{n}, m>0$.

If we take $\alpha=0$ in Theorem 1, then the inequality (1.5) reduces to the following result proved by Aziz and Dawood[3].

Corollary 1. *Let $P(z)$ be a polynomial of degree n, having all its zeros in $|z|<1$, then for* $|z|=1$

$$|P(Rz)|\geq R^{n}\min_{|z|=1}|P(z)|\ \ for\ \ R\geq 1.$$

We next improve the inequality (1.4), by using Theorem 1. More precisely, we prove the following

Theorem 2. *If $P(z)$ is a polynomial of degree n, having no zeros in $|z|<1$, then for every real or complex number α with $|\alpha|\leq 1$, $R\geq 1$ and $|z|=1$,*

$$\begin{aligned}\left|P(Rz)+\alpha\left(\frac{R+1}{2}\right)^{n}P(z)\right|\leq\frac{1}{2}\Bigg[&\left\{\left|R^{n}+\alpha\left(\frac{R+1}{2}\right)^{n}\right|+\left|1+\alpha\left(\frac{R+1}{2}\right)^{n}\right|\right\}M(P,1)\\&-\left\{\left|R^{n}+\alpha\left(\frac{R+1}{2}\right)^{n}\right|-\left|1+\alpha\left(\frac{R+1}{2}\right)^{n}\right|\right\}\min_{|z|=1}|P(z)|\Bigg].\end{aligned} \tag{1.6}$$

Inequality (1.6) *is sharp and the equality holds for* $P(z) = \beta + \gamma z^n$, *where* $|\beta| = |\gamma| = 1/2$.

For $\alpha = 0$, Theorem 2 reduces to the following result proved by Aziz and Dawood[3].

Corollary 2. *Let* $P(z)$ *is a polynomial of degree n, having no zeros in* $|z| < 1$, *then for* $|z| = 1$

$$|P(Rz)| \le \left(\frac{R^n+1}{2}\right) M(P,1) - \left(\frac{R^n-1}{2}\right) \min_{|z|=1} |P(z)|.$$

2 Lemmas

For the proof of these theorems, we need the following lemmas.

Lemma 1. *If* $P(z)$ *is a polynomial of degree n, having all its zeros in the disk* $|z| \le k$, $k \le 1$, *then for* $R \ge 1$

$$|P(Rz)| \ge \left(\frac{R+k}{1+k}\right)^n |P(z)| \quad for \quad |z| = 1. \tag{2.1}$$

The above Lemma is due to Aziz[2].

Lemma 2. *Let* $F(z)$ *be a polynomial of degree n, having all its zeros in the disk* $|z| \le 1$. *If* $P(z)$ *is a polynomial of degree at most n such that*

$$|P(z)| \le |F(z)| \quad for \quad |z| = 1,$$

then for $|\alpha| \le 1$ *and* $R \ge 1$,

$$\left|P(Rz) + \alpha\left(\frac{R+1}{2}\right)^n P(z)\right| \le \left|F(Rz) + \alpha\left(\frac{R+1}{2}\right)^n F(z)\right| \quad for \quad |z| = 1. \tag{2.2}$$

The above Lemma is due to Jain[4].

Lemma 3. *If* $P(z)$ *is a polynomial of degree n, then for* $|z| = 1$ *and* $|\alpha| \le 1$,

$$\begin{aligned}&\left|P(Rz) + \alpha\left(\frac{R+1}{2}\right)^n P(z)\right| + \left|Q(Rz) + \alpha\left(\frac{R+1}{2}\right)^n Q(z)\right| \\ &\quad \le \left\{\left|R^n + \alpha\left(\frac{R+1}{2}\right)^n\right| + \left|1 + \alpha\left(\frac{R+1}{2}\right)^n\right|\right\} M(P,1).\end{aligned} \tag{2.3}$$

where $Q(z) = z^n \overline{P(1/\overline{z})}$.

The above Lemma is due to Jain[4].

3 Proof of Theorems

Proof of Theorem 1. For $R = 1$ the result is obvious. Therefore we shall prove the result for $R > 1$. If $P(z)$ has a zero on $|z| = 1$, then the inequality (1.5) is trivial. So we suppose that $P(z)$ has all its zeros in $|z| < 1$. If $m = \min_{|z|=1} |P(z)|$, then $0 < m \le |P(z)|$ for $|z| = 1$.

Therefore, if λ is a complex number such that $|\lambda| < 1$, then it follows by Rouche's Theorem that the polynomial $G(z) = P(z) - \lambda m z^n$ of degree n, has all its zeros in $|z| < 1$. Applying Lemma 1 to the polynomial $G(z)$ with $k = 1$ and $R > 1$, we get

$$|G(Rz)| \geq \left(\frac{R+1}{2}\right)^n |G(z)| \quad \text{for} \quad |z| = 1.$$

Since $G(Rz)$ has all its zeros in $|z| \leq 1/R < 1$, again applying Rouche's Theorem for real or complex number α with $|\alpha| \leq 1$, one can show that the polynomial $T(z) = G(Rz) + \alpha \left(\frac{R+1}{2}\right)^n G(z)$ has all its zeros in $|z| < 1$. That is,

$$T(z) = G(Rz) + \alpha \left(\frac{R+1}{2}\right)^n G(z) \neq 0 \quad \text{for} \quad |z| \geq 1, \ R > 1.$$

Substituting for $G(z)$, we conclude that for every α, λ with $|\lambda| < 1$, $|\alpha| \leq 1$, $|z| \geq 1$ and $R > 1$

$$T(z) = \left[P(Rz) + \alpha \left(\frac{R+1}{2}\right)^n P(z)\right] - \lambda \left[mR^n z^n + \alpha \left(\frac{R+1}{2}\right)^n m z^n\right] \neq 0. \tag{3.1}$$

This implies for every α with $|\alpha| \leq 1$, $|z| \geq 1$ and $R > 1$,

$$\left|P(Rz) + \alpha \left(\frac{R+1}{2}\right)^n P(z)\right| \geq \left|R^n + \alpha \left(\frac{R+1}{2}\right)^n\right| m |z|^n. \tag{3.2}$$

If this inequality is not true , then there is a point $z = z_0$ with $|z_0| \geq 1$ such that for $R > 1$

$$\left|P(Rz) + \alpha \left(\frac{R+1}{2}\right)^n P(z)\right| < \left|R^n + \alpha \left(\frac{R+1}{2}\right)^n\right| m |z|^n.$$

We take

$$\lambda = \frac{P(Rz_0) + \alpha \left(\frac{R+1}{2}\right)^n P(z_0)}{\left[R^n + \alpha \left(\frac{R+1}{2}\right)^n\right] m z_0^n},$$

then $|\lambda| < 1$ and with this choice of λ, we have from (3.1), $T(z_0) = 0$ for $|z_0| \geq 1$. But this contradicts the fact that $T(z) \neq 0$ for $|z| \geq 1$. Hence in particular, (3.2) gives for every α with $|\alpha| \leq 1$ and $R > 1$,

$$\min_{|z|=1} \left|P(Rz) + \alpha \left(\frac{R+1}{2}\right)^n P(z)\right| \geq \left|R^n + \alpha \left(\frac{R+1}{2}\right)^n\right| \min_{|z|=1} |P(z)|.$$

This completes the proof of Theorem 1.

Proof of Theorem 2. For $R = 1$ there is nothing to prove. Therefore we assume that $R > 1$. By hypothesis,the polynomial $P(z) \neq 0$ in $|z| < 1$, therefore if $m = \min_{|z|=1} |P(z)|$, then $m \leq |P(z)|$ for $|z| \leq 1$. Therefore, for a given complex number β with $|\beta| \leq 1$, it follows by Rouche's Theorem that the polynomial $G(z) = P(z) - \beta m$ has no zero in $|z| < 1$. Now if

$$H(z) = z^n \overline{G(1/\overline{z})} = Q(z) - m\overline{\beta} z^n,$$

then all the zeros $H(z)$ lie in $|z|<1$ and $|G(z)|=|F(z)|$ for $|z|=1$. Therefore by Lemma 2, we have for $|\alpha|\le 1$ and $|z|=1$,

$$\begin{aligned}&\left|\{P(Rz)-\beta m\}+\alpha\left(\frac{R+1}{2}\right)^n\{P(z)-\beta m\}\right|\\ &\quad\le\left|\left\{Q(Rz)-\overline{\beta}mR^nz^n\right\}+\alpha\left(\frac{R+1}{2}\right)^n\left\{Q(z)-\overline{\beta}mz^n\right\}\right|.\end{aligned}$$

This implies

$$\begin{aligned}&\left|\left\{P(Rz)+\alpha\left(\frac{R+1}{2}\right)^nP(z)\right\}-\beta m\left\{1+\alpha\left(\frac{R+1}{2}\right)^n\right\}\right|\\ &\quad\le\left|\left\{Q(Rz)+\alpha\left(\frac{R+1}{2}\right)^nQ(z)\right\}-\overline{\beta}mz^n\left\{R^n+\alpha\left(\frac{R+1}{2}\right)^n\right\}\right|.\end{aligned}\tag{3.3}$$

Since all zeros of $Q(z)$ lie in $|z|<1$, we have by Theorem 1 for $|z|=1$ and $|\alpha|\le 1$

$$\begin{aligned}\left|Q(Rz)+\alpha\left(\frac{R+1}{2}\right)^nQ(z)\right| &\ge \left|R^n+\alpha\left(\frac{R+1}{2}\right)^n\right|\min_{|z|=1}|Q(z)|,\\ &= \left|R^n+\alpha\left(\frac{R+1}{2}\right)^n\right|m.\end{aligned}$$

Now choosing the argument of β in (3.3) and letting $|\beta|\to 1$, we get for $|z|=1$ and $|\alpha|\le 1$,

$$\begin{aligned}&\left|P(Rz)+\alpha\left(\frac{R+1}{2}\right)^nP(z)\right|-m\left|1+\alpha\left(\frac{R+1}{2}\right)^n\right|\\ &\quad\le\left|Q(Rz)+\alpha\left(\frac{R+1}{2}\right)^nQ(z)\right|-m\left|R^n+\alpha\left(\frac{R+1}{2}\right)^n\right|.\end{aligned}$$

Equivalently

$$\begin{aligned}&\left|P(Rz)+\alpha\left(\frac{R+1}{2}\right)^nP(z)\right|\\ &\quad\le\left|Q(Rz)+\alpha\left(\frac{R+1}{2}\right)^nQ(z)\right|-\left\{\left|R^n+\alpha\left(\frac{R+1}{2}\right)^n\right|-\left|1+\alpha\left(\frac{R+1}{2}\right)^n\right|\right\}m,\end{aligned}$$

which implies for every real or complex number α with $|\alpha|\le 1$, $R>1$ and $|z|=1$,

$$\begin{aligned}2\left|P(Rz)+\alpha\left(\frac{R+1}{2}\right)^nP(z)\right| &\le\left|P(Rz)+\alpha\left(\frac{R+1}{2}\right)^nP(z)\right|\\ &\quad+\left|Q(Rz)+\alpha\left(\frac{R+1}{2}\right)^nQ(z)\right|-\left\{\left|R^n+\alpha\left(\frac{R+1}{2}\right)^n\right|-\left|1+\alpha\left(\frac{R+1}{2}\right)^n\right|\right\}m.\end{aligned}$$

This in conjunction with Lemma 3 gives for $|\alpha|\le 1$, $R>1$ and $|z|=1$,

$$\begin{aligned}2\left|P(Rz)+\alpha\left(\frac{R+1}{2}\right)^nP(z)\right| &\le\left\{\left|R^n+\alpha\left(\frac{R+1}{2}\right)^n\right|+\left|1+\alpha\left(\frac{R+1}{2}\right)^n\right|\right\}M(P,1)\\ &\quad-\left\{\left|R^n+\alpha\left(\frac{R+1}{2}\right)^n\right|-\left|1+\alpha\left(\frac{R+1}{2}\right)^n\right|\right\}m,\end{aligned}$$

and the theorem follows.

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