Analysis in Theory and Applications
Volume 24 Number 1 (2008), 1–17
DOI10.1007/s10496-008-0001-z

NORM SUMMABILITY OF NÖRLUND LOGARITHMIC MEANS ON UNBOUNDED VILENKIN GROUPS *

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Received Jan. 23, 2006

Abstract. The (Nörlund) logarithmic means of the Fourier series is:

$$t_n f = \frac{1}{l_n}\sum_{k=1}^{n-1}\frac{S_k f}{n-k}, \qquad \text{where } l_n = \sum_{k=1}^{n-1}\frac{1}{k}.$$

In general, the Fejér $(C,1)$ means have better properties than the logarithmic ones. We compare them and show that in the case of some unbounded Vilenkin systems the situation changes.

Key words: *unbounded Vilenkin group, Fejér means, logarithmic means, norm convergence*

AMS (2000) subject classification: 42C10

1 Introduction

Introduce the necessary notations and definitions.

Let $\mathbb{P} = \mathbb{N}\setminus\{0\}$, and let $m := (m_0, m_1, \cdots)$ denote a sequence of positive integers not less than 2. Denote by $Z_{m_k} := \{0, 1, \cdots, m_k - 1\}$ the additive group of integers modulo m_k.

Define the set G_m as the complete direct product of the sets Z_{m_j}, with the product of the discrete topologies of Z_{m_j}'s.

The direct product μ of the measures

$$\mu_k(\{j\}) := \frac{1}{m_k}, \qquad j \in Z_{m_k},$$

is the Haar measure on G_m with $\mu(G_m) = 1$.

*The first author is supported by the Békésy Postdoctoral fellowship of the Hungarian Ministry of Education Bö 91/2003, the second author is supported by the Hungarian National Foundation for Scientific Research (OTKA), grant no. M 36511/2001, T 048780 and by the Széchenyi fellowship of the Hungarian Ministry of Education Szö 184/2003.

If the sequence m is bounded, then G_m is called a bounded Vilenkin group, else its name is an unbounded one. The elements of G_m can be represented by sequences $x := (x_0, x_1, \cdots, x_j, \cdots)$ $(x_j \in Z_{m_j})$. It is easy to give a base for the neighborhoods of G_m :

$$I_0(x) := G_m,$$
$$I_n(x) := \{y \in G_m | y_0 = x_0, \cdots, y_{n-1} = x_{n-1}\}$$

for $x \in G_m,\ n \in \mathbb{N}$. Define $I_n := I_n(0)$ for $n \in \mathbb{P}$.

If we define the so-called generalized number system based on m in the following way: $M_0 := 1, M_{k+1} := m_k M_k (k \in \mathbb{N})$, then every $n \in \mathbb{N}$ can be uniquely expressed as $n = \sum_{j=0}^{\infty} n_j M_j$, where $n_j \in G_{m_j}$ $(j \in \mathbb{P})$ and only a finite number of n_j's differ from zero. We use the following notations. Let (for $n > 0$) $|n| := \max\{k \in \mathbb{N} : n_k \neq 0\}$ (that is, $M_{|n|} \leq n < M_{|n|+1}$), $n^{(k)} = \sum_{j=k}^{\infty} n_j M_j$ and $n_{(k)} := n - n^{(k)}$.

Denote by $L^p(G_m)$ the usual (one dimensional) Lebesgue spaces ($\|.\|_p$ the corresponding norms) $(1 \leq p \leq \infty)$.

Next we introduce on G_m an orthonormal system which is called Vilenkin system. At first define the complex valued functions $r_k(x) : G_m \to \mathbb{C}$, the Generalized Rademacher functions in this way

$$r_k(x) := \exp \frac{2\pi i x_k}{m_k}, \qquad ß^2 := -1,\ x \in G_m,\ k \in \mathbb{N}.$$

Now define the Vilenkin system $\psi := (\psi_n : n \in \mathbb{N})$ on G_m as follows.

$$\psi_n(x) := \prod_{k=0}^{\infty} r_k^{n_k}(x), \qquad n \in \mathbb{N}.$$

Specially, we call this system the Walsh-Paley one if $m \equiv 2$.

The Vilenkin system is orthonormal and complete in $L^2(G_m)$[10].

Now, introduce the usual definitions from Fourier-analysis. If $f \in L^1(G_m)$ we can define the following definitions in the usual way.

Fourier coefficients:

$$\widehat{f}(k) := \int_{G_m} f \overline{\psi}_k \mathrm{d}\mu, \qquad k \in \mathbb{N},$$

partial sums:

$$S_n f := \sum_{k=0}^{n-1} \widehat{f}(k) \psi_k, \qquad n \in \mathbb{P},$$
$$S_0 f := 0,$$

Fejér means:

$$\sigma_n f := \frac{1}{n} \sum_{k=0}^{n-1} S_n f, \qquad n \in \mathbb{P},$$

Dirichlet kernels:

$$D_n := \sum_{k=0}^{n-1} \psi_k, \qquad n \in \mathbb{P},$$

Fejér kernels:

$$K_n := \frac{1}{n} \sum_{k=0}^{n-1} D_k, \qquad n \in \mathbb{P}.$$

We use a further notation:

$$K_{a,b} := \sum_{j=a}^{a+b-1} D_j, \qquad a,b \in N.$$

2 Approximation in Norm

Review some approximation results with respect to the Vilenkin system. It is well-known that the partial sums converge to the function in norm. That is,

$$\|S_n f - f\|_p \to 0$$

for all $f \in L^p$, where $1 < p < \infty$[8]. This is not the case if $p = 1$ or $p = \infty$.

Moreover, if we use the partial sequence M_n it is also well-known ([1]) the convergence in the supremum norm for continuous functions, and in the Lebesgue norm L^1 for functions $f \in L^1$

$$\|S_{M_n} f - f\| \to 0.$$

The properties of the convergence are better using the Fejér means on bounded Vilenkin system[9]

$$\|\sigma_n f - f\|_p \to 0, \quad \forall f \in L^p, \qquad 1 \leq p < \infty$$

and in the supremum norm for continuous functions. On arbitrary Vilenkin system (it is a trivial consequence of the convergence of the partial sums)

$$\|\sigma_n f - f\|_p \to 0, \quad \forall f \in L^p, \qquad 1 < p < \infty.$$

On the other hand, in the case of unbounded Vilenkin systems and $p = 1$ the theorem above does not hold. We have the same situation in the case of the supremum norm and continuous functions.

What is the situation regarding the Fejér means in the case of special subsequence M_n? It is interesting, that every unbounded sequence m there is $f \in L^1$ such that[7]

$$\|\sigma_{M_n} f - f\|_1 \not\to 0,$$

but for every Vilenkin system

$$\sigma_{M_n} f \to f \quad \text{a.e.} \quad \forall f \in L^1,$$

see [2]. Moreover, there exists a continuous function such that $\sigma_{M_n} f$ does not converge to f in the supremum norm[7].

3 Nörlund Logarithmic Means

Define the Nörlund means in general. If $q_k \geq 0$ $(k \in \mathbb{N})$,

$$\frac{1}{Q_n}\sum_{k=1}^{n-1} q_{n-k}S_kf, \qquad \text{where } Q_n = \sum_{k=1}^{n-1} q_k.$$

Móricz discussed in [6] the norm convergence for Walsh-Paley system with respect to some sequence q. The case $q_k = \frac{1}{k}$ is excluded in this paper. Among others, that is why this case, the so-called Nörlund logarithmic means is also interesting to be discussed. The concept of Nörlund logarithmic means is as follows

$$t_nf := \frac{1}{l_n}\sum_{k=1}^{n-1} \frac{S_kf}{n-k}, \qquad \text{where } l_n := \sum_{k=1}^{n-1} \frac{1}{k}.$$

It is evident that

$$t_nf = f * F_n,$$

where

$$F_n = \frac{1}{l_n}\sum_{k=1}^{n-1} \frac{D_k}{n-k}.$$

For further information with respect to Nörlund logarithmic means for Walsh-Paley system see [5].

In their paper [4] Gát, G. and Goginava, U. proved (for Walsh-Paley system) that there is $f \in L^1$ such that

$$\|t_nf - f\|_1 \not\to 0.$$

On the other hand, the main aim of this paper is to prove that the Nörlund logarithmic means have better apprioximation properties on some unbounded Vilenkin groups, than the Fejér means.

4 New Results on Vilenkin Systems

Lemma 1. *Let* $1 \leq j \leq M_k - 1$. *Then*

$$D_{M_k-j}(x) = D_{M_k}(x) - \overline{\psi}_{M_k-1}(-x)D_j(-x).$$

Proof. Using the well known equalities $\psi_{n\oplus m} = \psi_n\psi_m$, $\psi_{n\ominus m} = \psi_n/\psi_m$, $\psi_n(x) = \overline{\psi}_n(-x)$ and $\psi_n\overline{\psi}_n = 1$ as well as the simple fact $(M_k-1)\ominus\nu = M_k-1-\nu$ for $\nu = 0,1,\cdots,M_k-1$ we can get

$$\begin{aligned}
D_{M_k-j}(x) &= D_{M_k}(x)-\sum_{\nu=M_k-j}^{M_k-1}\psi_\nu(x)\\
&= D_{M_k}(x)-\overline{\psi}_{M_k-1}(-x)\sum_{\nu=M_k-j}^{M_k-1}\psi_{M_k-1}(-x)\overline{\psi}_\nu(-x)\\
&= D_{M_k}(x)-\overline{\psi}_{M_k-1}(-x)\sum_{\nu=M_k-j}^{M_k-1}\psi_{M_k-1}(-x)/\psi_\nu(-x)\\
&= D_{M_k}(x)-\overline{\psi}_{M_k-1}(-x)\sum_{\nu=M_k-j}^{M_k-1}\psi_{(M_k-1)\ominus\nu}(-x)\\
&= D_{M_k}(x)-\overline{\psi}_{M_k-1}(-x)\sum_{\nu=M_k-j}^{M_k-1}\psi_{M_k-1-\nu}(-x)\\
&= D_{M_k}(x)-\overline{\psi}_{M_k-1}(-x)\sum_{\nu=0}^{j-1}\psi_\nu(-x)\\
&= D_{M_k}(x)-\overline{\psi}_{M_k-1}(-x)D_j(-x).
\end{aligned}$$

The next two lemmas are interesting even in themself, because it is not known any general estimate of the Fejér kernel functions before for unbounded Vilenkin systems.

Lemma 2. *If* $\log m_n = O(n^\delta)$, *then* $\|K_n\|_1 = O(|n|^\delta)$, *for any* $\delta > 0$.

Proof. It is known[3], that

$$nK_n=\sum_{s=0}^{|n|}\sum_{j=0}^{n_s-1}K_{n^{(s+1)}+jm_s,M_s}.$$

Let $z\in I_t\backslash I_{t+1}$.

At this point divide the examination into four parts.

1. The case $|n|\geq s>t$. Then

$$K_{n^{(s)},M_s}(z)=\begin{cases} M_tM_s\psi_{n^{(s)}}(z)\dfrac{1}{1-r_t(z)}, & \text{if } z-z_te_t\in I_s,\\ 0, & \text{otherwise.}\end{cases}$$

So we obtain

$$\sum_{j=1}^{n_s-1}K_{n^{(s+1)}+jM_s,M_s}(z) \quad = \begin{cases} \psi_{n^{(s+1)}}(z)M_tM_s\dfrac{1}{1-r_t(z)}\dfrac{r_s^{n_s}(z)-1}{r_s(z)-1}, & \text{if } z-z_te_t\in I_s \text{ and } z_s\neq 0,\\ \psi_{n^{(s+1)}}(z)M_tM_s\dfrac{n_s}{1-r_t(z)}, & \text{if } z_s=0,\\ 0, & \text{otherwise.}\end{cases}$$

Using this equality we have

$$\int_{I_t\setminus I_{t+1}}\left|\sum_{j=1}^{n_s-1}K_{n^{(s+1)}+jM_s,M_s}(z)\right|\mathrm{d}\mu(z)=M_tM_s\sum_{z_t=1}^{m_t-1}\int_{I_s(e_tz_t)}\frac{1}{|1-r_t(z)|}\left|\frac{r_s^{n_s}(z)-1}{r_s(z)-1}\right|\mathrm{d}\mu(z)$$
$$\leq M_tM_sm_t\log m_t\frac{1}{M_s}\log n_s=M_{t+1}\log m_t\log n_s.$$

Let $A:=|n|$. We would like to give an upper estimate for the next expression

$$\frac{1}{n_AM_A}\sum_{t=0}^{A-1}\int_{I_t\setminus I_{t+1}}\left|\sum_{s=t+1}^{A}\sum_{j=0}^{n_s-1}K_{n^{(s+1)}+jM_s,M_s}(z)\right|\mathrm{d}\mu(z)$$
$$\leq\frac{1}{n_AM_A}\sum_{t=0}^{A-1}\sum_{s=t+1}^{A}M_{t+1}\log m_t\log n_s$$
$$=\frac{1}{n_AM_A}\sum_{s=1}^{A}\log n_s\sum_{t=0}^{s-1}M_{t+1}\log m_t:=\frac{1}{n_AM_A}\sum_{s=1}^{A}\Gamma_s.$$

Observe the addends of this sum separately. At first let $s=A$.

$$\frac{1}{n_AM_A}\Gamma_A=\frac{1}{n_AM_A}\log n_A\sum_{t=0}^{A-1}M_{t+1}\log m_t\leq c\frac{1}{M_A}\sum_{t=0}^{A-1}M_{t+1}t^{\delta}\leq A^{\delta}.$$

Now let $s=A-1$.

$$\frac{1}{n_AM_A}\Gamma_{A-1}\leq\frac{1}{M_A}\log m_{A-1}\sum_{t=0}^{A-2}M_{t+1}t^{\delta}\leq A^{\delta}\frac{\log m_{A-1}}{m_{A-1}}\leq A^{\delta}.$$

If $s=A-2$, then

$$\frac{1}{n_AM_A}\Gamma_{A-2}\leq\frac{1}{M_A}\log m_{A-2}\sum_{t=0}^{A-3}M_{t+1}t^{\delta}\leq A^{\delta}\frac{\log m_{A-2}}{m_{A-1}m_{A-2}}\leq\frac{1}{2}A^{\delta},$$

and so on

$$\frac{1}{n_AM_A}\Gamma_{A-3}\leq\frac{1}{2^2}A^{\delta},\cdots,\frac{1}{n_AM_A}\Gamma_1\leq\frac{1}{2^{A-2}}A^{\delta}.$$

To sum up these we obtain

$$\frac{1}{n_AM_A}\sum_{s=1}^{A}\Gamma_s\leq cA^{\delta}.$$

2. The case $|n|>s=t$. From [3]

$$\sum_{j=0}^{n_t-1}K_{n^{(t+1)}+jM_t,M_t}(z)=\frac{M_t(M_t-1)}{2}\psi_{n^{(t+1)}}(z)\frac{r_t^{n_t}(z)-1}{r_t(z)-1}$$
$$+M_t^2\psi_{n^{(t+1)}}(z)\frac{r_t^{n_t}(z)-1}{(r_t(z)-1)^2}-M_t^2\psi_{n^{(t+1)}}(z)n_t\frac{1}{r_t(z)-1}$$
$$=:\Theta_1(z)+\Theta_2(z)+\Theta_3(z).$$

Sonsider the integral of the absolute value of these three functions on $I_t \backslash I_{t+1}$. At first

$$\int_{I_t \backslash I_{t+1}} |\Theta_1(z)| \mathrm{d}\mu(z) \le cM_t^2 \frac{1}{M_{t+1}} \sum_{z_t=1}^{m_t-1} \frac{|\sin(\pi n_t z_t/m_t)|}{|\sin(\pi z_t/m_t)|} \le cM_t \log n_t \le cM_{t+1}.$$

After

$$\int_{I_t \backslash I_{t+1}} |\Theta_2(z)| \mathrm{d}\mu(z) \le cM_t^2 \frac{1}{M_{t+1}} \sum_{z_t=1}^{m_t-1} \frac{m_t^2}{z_t^2} \le cM_{t+1},$$

and at the end

$$\int_{I_t \backslash I_{t+1}} |\Theta_3(z)| \mathrm{d}\mu(z) \le M_t^2 n_t \frac{1}{M_{t+1}} \sum_{z_t=1}^{m_t-1} \frac{1}{|\sin(\pi z_t/m_t)|} \le cM_t n_t \log m_t \le cM_{t+1} \log m_t.$$

From these three inequalities

$$\frac{1}{n_A M_A} \sum_{t=0}^{A-1} \int_{I_t \backslash I_{t+1}} \left| \sum_{j=0}^{n_t-1} K_{n^{(t+1)}+jM_t, M_t}(z) \right| \mathrm{d}\mu(z) \le \frac{1}{n_A M_A} \sum_{t=0}^{A-1} cM_{t+1} \log m_t \le c \frac{1}{M_A} \sum_{t=0}^{A-1} M_{t+1} t^{\delta} \le cA^{\delta}.$$

3. The case $|n| \ge t > s$. It is easy to see that if $z \in I_t \backslash I_{t+1}$, then

$$D_k(z) = \psi_k(z) \left(\sum_{j=0}^{t-1} k_j M_j + M_t \sum_{s=m_t-k_t}^{m_t-1} r_t^s \right),$$

so $|D_k(z)| \le (k_t+1)M_t$. It yields

$$\int_{I_t \backslash I_{t+1}} |D_k(z)| \mathrm{d}\mu(z) \le k_t + 1 \le \begin{cases} 2m_t, & \text{if } t < A, \\ 2n_t, & \text{if } t = A \end{cases} =: \alpha(n,t,A).$$

From this we get

$$\int_{I_t \backslash I_{t+1}} \left| K_{n^{(s+1)}+jM_s, M_s}(z) \right| \mathrm{d}\mu(z) \le M_s \alpha(n,t,A),$$

and therefore

$$\int_{I_t \backslash I_{t+1}} \left| \sum_{j=0}^{n_s-1} K_{n^{(s+1)}+jM_s, M_s}(z) \right| \mathrm{d}\mu(z) \le M_{s+1} \alpha(n,t,A).$$

Using this result we get

$$\frac{1}{n_A M_A} \sum_{t=1}^{A} \int_{I_t \backslash I_{t+1}} \left| \sum_{s=0}^{t-1} \sum_{j=0}^{n_s-1} K_{n^{(s+1)}+jM_s, M_s}(z) \right| \mathrm{d}\mu(z) \le \frac{1}{n_A M_A} \sum_{t=1}^{A-1} \sum_{s=0}^{t-1} 2M_{s+1} m_t + \frac{1}{n_A M_A} \sum_{s=0}^{A-1} 2M_{s+1} n_A \le c.$$

4. The case $|n| = t = s$. It is known from [3]: if $z \in I_t \setminus I_{t+1}$, then

$$K_{n^{(t+1)}+jM_t,M_t}(z) = K_{jM_A,M_A}(z) =: \Theta_4(z) + \Theta_5(z),$$

where $\Theta_4(z) \le M_A^2$, so

$$\frac{1}{n_A M_A} \int_{I_A \setminus I_{A+1}} \sum_{j=0}^{n_A - 1} |\Theta_4(z)| \,\mathrm{d}\mu(z) \le c.$$

On the other hand[3]

$$\begin{aligned}\sum_{j=0}^{n_A-1} \Theta_5(z) &= M_A^2 \frac{r_A^{n_A}(z) - 1}{(r_A(z)-1)^2} + M_A^2 n_A \frac{1}{1 - r_A(z)} \\ &=: \Theta_{5,1}(z) + \Theta_{5,2}(z).\end{aligned}$$

At first observe $\Theta_{5,2}(z)$.

$$\begin{aligned}&\frac{1}{n_A M_A} \int_{I_A \setminus I_{A+1}} |\Theta_{5,2}(z)| \,\mathrm{d}\mu(z) \\ &\quad = M_A \int_{I_A \setminus I_{A+1}} \frac{1}{|1 - r_A(z)|} \mathrm{d}\mu(z) \le c \log m_A \le cA^{\delta}.\end{aligned}$$

Then

$$|\Theta_{5,1}(z)| \le 2M_A^2 \frac{|\sin(\pi n_A z_A / m_A)|}{|\sin(\pi z_A / m_A)|^2} \le c M_A^2 \frac{n_A z_A}{m_A} \frac{1}{\sin^2(\pi z_A / m_A)},$$

so

$$\begin{aligned}\frac{1}{n_A M_A} \int_{I_A \setminus I_{A+1}} |\Theta_{5,1}(z)| \,\mathrm{d}\mu(z) &\le c \frac{1}{n_A M_A} \sum_{z_A=1}^{m_A-1} \frac{1}{M_{A+1}} M_A^2 n_A \frac{z_A}{m_A} \frac{m_A^2}{z_A^2} \\ &= c \sum_{z_A=1}^{m_A-1} \frac{1}{z_A} = c \log m_A \le cA^{\delta}.\end{aligned}$$

Summarizing the reasons of the last four cases

$$\sum_{t=0}^{A} \int_{I_t \setminus I_{t+1}} |K_n(z)| \,\mathrm{d}\mu(z) \le cA^{\delta}.$$

At the end, notting that $z \in I_{A+1}$ implies $|K_n(z)| \le cn$, we get

$$\int_{I_{A+1}} |K_n(z)| \,\mathrm{d}\mu(z) \le \frac{1}{M_{A+1}} n \le c.$$

Consequently

$$\|K_n(z)\|_1 \le cA^{\delta} = c|n|^{\delta}.$$

From now we use the following notation.

$$\alpha_A := \frac{1}{M_A} \sum_{t=0}^{A-1} M_{t+1} \log m_t,$$

where $M_A \leq n < M_{A+1}$ (it means $|n|$=A).

Lemma 3.

$$\|K_n\|_1 \leq c\sum_{i=0}^{A+1}\frac{\alpha_{A+1-i}}{2^i}.$$

Proof. During the proof we use some method and result from the proof of Lemma 2. Analogously, we have four cases.

1. The case $|n| \geq s > t$. With the notations of Lemma 2. For the case $s = A$

$$\frac{\Gamma_A}{n_A M_A} = \frac{1}{n_A M_A}\log n_A \sum_{t=0}^{A-1} M_{t+1}\log m_t \leq c\alpha_A.$$

Similarly, for $s = A-1,\ s = A-2$

$$\frac{\Gamma_{A-1}}{n_A M_A} \leq \frac{1}{M_A}\log m_{A-1}\sum_{t=0}^{A-2} M_{t+1}\log m_t \leq \alpha_{A-1},$$
$$\frac{\Gamma_{A-2}}{n_A M_A} \leq \alpha_{A-2}2^{-1},\ \ \frac{\Gamma_{A-3}}{n_A M_A} \leq \alpha_{A-3}2^{-2},$$

and so on. In this way we obtain

$$\frac{1}{n_A M_A}\sum_{i=1}^{A}\Gamma_i \leq c\sum_{i=1}^{A}\frac{\alpha_{A-i}}{2^i}.$$

2. The case $|n| > s = t$. We use the same estimation as the 2nd case for the proof of Lemma 2.

$$\frac{1}{n_A M_A}\sum_{t=0}^{A-1}\int_{I_t\setminus I_{t+1}}\left|\sum_{j=0}^{n_t-1} K_{n^{(t+1)}+jM_t,M_t}(z)\right|\mathrm{d}\mu(z)$$
$$\leq \frac{1}{n_A M_A}\sum_{t=0}^{A-1} cM_{t+1}\log m_t \leq c\alpha_A.$$

3. The case $|n| \geq t > s$. As in the 3rd case for the proof of Lemma 2,

$$\frac{1}{n_A M_A}\sum_{t=1}^{A}\int_{I_t\setminus I_{t+1}}\left|\sum_{s=0}^{t-1}\sum_{j=0}^{n_s-1} K_{n^{(s+1)}+jM_s,M_s}(z)\right|\mathrm{d}\mu(z) \leq c.$$

4. The case $|n| = t = s$. Lemma 2 yields that

$$\int_{I_{A+1}}|K_n| \leq c.$$

If we summarize the results of the last four cases we get

$$\|K_n\|_1 \leq c\sum_{i=0}^{A}\frac{\alpha_{A-i}}{2^i} + c + c\alpha_{A+1} \leq c\sum_{i=0}^{A+1}\frac{\alpha_{A+1-i}}{2^i},$$

where $|n| = A$.

Lemma 4.

$$\|F_{M_n}\|_1 \le c\frac{\sum_{j=0}^{n-1}\log^2 m_j}{\log M_n}.$$

Proof. From Lemma 1 we get

$$\begin{aligned}\sum_{k=1}^{M_n-1}\frac{D_k(x)}{M_n-k} &= \sum_{k=1}^{M_n-1}\frac{1}{k}\left(D_{M_n}(x)-\overline{\psi}_{M_n-1}(-x)D_k(-x)\right)\\ &= D_{M_n}(x)l_{M_n}-\overline{\psi}_{M_n-1}(-x)\sum_{k=1}^{M_n-1}\frac{1}{k}D_k(-x) =: \Lambda_1-\Lambda_2.\end{aligned}$$

Observe the case Λ_2. Since

$$\|\Lambda_2\|_1 = \left\|\overline{\psi}_{M_n-1}(-x)\sum_{k=1}^{M_n-1}\frac{1}{k}D_k(-x)\right\|_1 = \left\|\sum_{k=1}^{M_n-1}\frac{1}{k}D_k(x)\right\|_1,$$

we consider this expression instead of the origial one.

Using Abel transfomation we obtain

$$\begin{aligned}\left|\sum_{k=1}^{M_n-1}\frac{1}{k}D_k\right| &= \left|\sum_{k=1}^{M_n-2}\left(\frac{1}{k}-\frac{1}{k+1}\right)\sum_{j=1}^{k}D_j+\frac{1}{M_n-1}\sum_{j=1}^{M_n-1}D_j\right|\\ &\le \sum_{k=1}^{M_n-2}\frac{|K_{k+1}|}{k}+c|K_{M_n}| =: \Lambda_3+\Lambda_4.\end{aligned}$$

Start with Λ_3.

$$\begin{aligned}\|\Lambda_3\|_1 &\le c\sum_{k=1}^{M_n-1}\frac{1}{k}\|K_k\|_1\\ &\le c\sum_{j=0}^{n-1}\sum_{a=1}^{m_j-1}\sum_{s=aM_j}^{(a+1)M_j-1}\frac{1}{aM_j}(\alpha_{j+1}+\alpha_j2^{-1}+\cdots+\alpha_0 2^{-j-1})\\ &\le c\sum_{j=0}^{n-1}\log m_j(\alpha_{j+1}+\alpha_j2^{-1}+\cdots+\alpha_0 2^{-j-1}).\end{aligned}$$

It is easy to see that

$$\log m_A \le \alpha_{A+1} \le \log m_A + 2^{-1}\log m_{A-1}+\cdots+2^{-A}\log m_0.$$

Since

$$\begin{aligned}&\alpha_{j+1}+\alpha_j2^{-1}+\cdots+\alpha_0 2^{-j-1} = \sum_{b=0}^{j+1}2^{b-j-1}\alpha_b\\ &\quad\le \sum_{b=0}^{j+1}2^{b-j-1}(\log m_{b-1}+2^{-1}\log m_{b-2}+\cdots+2^{-b+1}\log m_0)\end{aligned}$$

$$= \sum_{b=0}^{j+1} 2^{b-j-1} \sum_{u=0}^{b-1} 2^{-b+u+1} \log m_u = 2^{-j} \sum_{b=0}^{j+1} \sum_{u=0}^{b-1} 2^u \log m_u$$
$$= 2 \sum_{u=0}^{j} \frac{j+1-u}{2^{j+1-u}} \log m_u = 2 \sum_{u=1}^{j+1} \frac{u}{2^u} \log m_{j+1-u},$$

so, using $\sum_{u=1}^{\infty} \frac{u}{2^u} < \infty$

$$\begin{aligned}
\|\Lambda_3\|_1 &\le c \sum_{j=0}^{n-1} \log m_j \sum_{u=1}^{j+1} \frac{u}{2^u} \log m_{j+1-u} = c \sum_{j=0}^{n-1} \sum_{u=1}^{j+1} \frac{u}{2^u} \log m_j \log m_{j+1-u} \\
&\le c \sum_{j=0}^{n-1} \left(\sum_{u=1}^{j+1} \frac{u}{2^u} \log^2 m_j + \sum_{u=1}^{j+1} \frac{u}{2^u} \log^2 m_{j+1-u} \right) \\
&\le c \sum_{j=0}^{n-1} \log^2 m_j + c \sum_{j=0}^{n-1} \sum_{u=1}^{j+1} \frac{u}{2^u} \log^2 m_{j+1-u} \\
&= c \sum_{j=0}^{n-1} \log^2 m_j + c \sum_{j=0}^{n-1} \log^2 m_j \sum_{u=1}^{n-j} \frac{u}{2^u} \\
&\le c \sum_{j=0}^{n-1} \log^2 m_j.
\end{aligned}$$

Proceed to Λ_4.

$$\|\Lambda_4\|_1 \le \left\| \frac{1}{M_n - 1} \sum_{j=0}^{M_n-2} D_j \right\|_1 + \frac{1}{M_n - 1} \|D_{M_n-1}\|_1.$$

It is known that $\|D_{M_n-1}\|_1 \le c$. On the other hand,

$$\begin{aligned}
\left\| \frac{1}{M_n - 1} \sum_{j=0}^{M_n-2} D_j \right\|_1 &= \|K_{M_n-1}\|_1 \le \alpha_n + 2^{-1}\alpha_{n-1} + \cdots + 2^{-n}\alpha_0 \\
&\le (\log m_{n-1} + 2^{-1} \log m_{n-2} + \cdots + 2^{-n+1} \log m_0) \\
&\quad + (2^{-1} \log m_{n-2} + 2^{-2} \log m_{n-3} + \cdots + 2^{-n+1} \log m_0) \\
&\quad + \cdots + (2^{-n+2} \log m_1 + 2^{-n+1} \log m_0) + (2^{-n+1} \log m_0) \\
&= \sum_{j=0}^{n-1} \frac{n-j}{2^{n-1-j}} \log m_j \le \sum_{j=0}^{n-1} \log m_j \\
&= \log M_n.
\end{aligned}$$

Originally, we started with

$$\begin{aligned}
\|F_{M_n}\|_1 &= \frac{1}{l_{M_n}} \left\| \sum_{k=1}^{M_n-1} \frac{D_k}{M_n - k} \right\|_1 \le \frac{\|\Lambda_1\|_1 + c \sum_{j=0}^{n-1} \log^2 m_j + c + \log M_n}{\log M_n} \\
&\le c \frac{\sum_{j=0}^{n-1} \log^2 m_j}{\log M_n}.
\end{aligned}$$

Theorem 1. *If* $f \in L^p$ $(1 \leq p < \infty)$ *and*

$$\limsup_{n \in \mathbb{N}} \frac{\sum_{k=0}^{n-1} \log^2 m_k}{\log M_n} < \infty,$$

then

$$\|t_{M_n} f - f\|_p \to 0.$$

In the case $f \in C$ *the convergence holds in the supremum norm.*

Proof. Using Fubini's theorem and Lemma 4 we obtain

$$\begin{aligned}
\|t_{M_n} f\|_1 &= \int_{G_m} |t_{M_n} f(y)| \mathrm{d}\mu(y) \\
&= \int_{G_m} \left| \int_{G_m} f(x) F_{M_n}(y-x) d\mu(x) \right| \mathrm{d}\mu(y) \\
&\leq \int_{G_m} \int_{G_m} |f(x)||F_{M_n}(y-x)| \mathrm{d}\mu(y) \mathrm{d}\mu(x) \\
&\leq \int_{G_m} |f(x)| \|F_{M_n}\|_1 d\mu(x) \leq c\|f\|_1,
\end{aligned}$$

that is, the operator t_{M_n} is of type $(1,1)$, uniformly in n. Similarly,

$$\begin{aligned}
\|t_{M_n} f\|_\infty &= \sup_{y \in G_m} |t_{M_n} f(y)| \\
&= \sup_{y \in G_m} \left| \int_{G_m} f(x) F_{M_n}(y-x) \mathrm{d}\mu(x) \right| \\
&\leq \sup_{y \in G_m} \int_{G_m} |f(x)||F_{M_n}(y-x)| \mathrm{d}\mu(y) \\
&\leq \sup_{y \in G_m} \int_{G_m} \|f(x)\|_1 \||F_{M_n}(y-x)| d\mu(x) \leq \|f\|_\infty \|F_{M_n}\|_1 \leq c\|f\|_\infty,
\end{aligned}$$

that is, the operator t_{M_n} is of type (∞, ∞) (uniformly in n).

The Marcinkiewicz interpolation theorem implies that the operator t_{M_n} is of type (p,p) $(1 \leq p \leq \infty)$. From this we prove of the theorem in the usual way.

Lemma 5. *Let* $p_A = M_{2A} + M_{2A-2} + \cdots + M_0$. *If* $\log m_n = O(n^\delta)$ *for some* $0 < \delta < 1/2$, *then*

$$\|F_{p_A}\|_1 \geq c_\beta A^\beta$$

for every $0 < \beta < 1 - \delta$.

Proof. Introduce the following notation

$$G_n := \sum_{k=1}^{n-1} \frac{D_{n-k}}{k}.$$

We use this decomposition of G_{p_A}

$$G_{p_A} = \sum_{k=1}^{M_{2A-2}+\cdots+M_0-1} \frac{1}{k} D_{p_A-k} + \sum_{k=M_{2A-2}+\cdots+M_0}^{p_A-1} \frac{1}{k} D_{p_A-k} =: B_1 + B_2.$$

Start with B_2. Let $k' := k - (M_{2A-2} + \cdots + M_0)$. In this case $0 \le k' \le M_{2A} - 1$. Consequently we can use Lemma 1

$$D_{p_A-k}(x) = D_{M_{2A}-k'}(x) = D_{M_{2A}}(x) - \overline{\psi}_{M_{2A}-1}(-x) D_{k'}(-x).$$

So,

$$\begin{aligned}
B_2(x) &= \sum_{k=M_{2A-2}+\cdots+M_0}^{p_A-1} \frac{1}{k} (D_{M_{2A}}(x) - \overline{\psi}_{M_{2A}-1}(-x) D_{k'}(-x)) \\
&= \sum_{k'=0}^{M_{2A}-1} \frac{1}{k} (D_{M_{2A}}(x) - \overline{\psi}_{M_{2A}-1}(-x) D_{k'}(-x)) \\
&= \sum_{k'=0}^{M_{2A}-1} \frac{1}{M_{2A-2} + \cdots + M_0 + k'} (D_{M_{2A}}(x) - \overline{\psi}_{M_{2A}-1}(-x) D_{k'}(-x)) \\
&= D_{M_{2A}}(x) c_2(A) - \overline{\psi}_{M_{2A}-1}(-x) \sum_{k'=0}^{M_{2A}-1} \frac{1}{M_{2A-2} + \cdots + M_0 + k'} D_{k'}(-x) \\
&=: B_{2,1}(x) + B_{2,2}(x).
\end{aligned}$$

Investigate $c_2(A)$

$$0 < c_2(A) = \sum_{k'=0}^{M_{2A}-1} \frac{1}{M_{2A-2} + \cdots + M_0 + k'} < \log\left(\frac{M_{2A}}{M_{2A-2}}\right) = \log(m_{2A-1} m_{2A-2}) \le cA^{\delta}.$$

Consequently $c_2(A)/A \to 0$ if $A \to \infty$, that is $c_2(A) = o(A)$. This gives

$$\|B_{2,1}\|_1 = \int_{G_m} |D_{M_{2A}}| o(A) = o(A).$$

Using Abel transform and bringing Fejér kernels into $B_{2,2}$, we have

$$\begin{aligned}
|B_{2,2}(x)| &\le \sum_{k'=0}^{M_{2A}-2} \left(\frac{1}{M_{2A-2} + \cdots + M_0 + k'} - \frac{1}{M_{2A-2} + \cdots + M_0 + k' + 1} \right) \\
&\quad \cdot (k'+1) |K_{k'+1}(-x)| + \frac{1}{M_{2A} + \cdots + M_0} (M_{2A}) |K_{M_{2A}}(-x)|.
\end{aligned}$$

So using Lemma 2 we can obtain

$$\begin{aligned}
\|B_{2,2}\|_1 &\le \sum_{k'=0}^{M_{2A}-2} \frac{1}{(M_{2A-2} + \cdots + M_0 + k')^2} (k'+1) \|K_{k+1}\|_1 + \|K_{M_{2A}}\|_1 \\
&\le \sum_{k'=0}^{M_{2A}-2} \frac{1}{M_{2A-2} + \cdots + M_0 + k'} \|K_{k+1}\|_1 + \|K_{M_{2A}}\|_1 \\
&\le cA^{\delta} \log\left(\frac{M_{2A}}{M_{2A-2}}\right) = cA^{\delta} \log(m_{2A-1} m_{2A-2}) \le cA^{2\delta}.
\end{aligned}$$

This gives

$$\|B_{2,2}\|_1 = o(A),$$

consequently we have

$$\|B_2\|_1 = o(A).$$

Focus on B_1. Easy to see that in case of $0 \le j < M_n$,

$$\begin{aligned} D_{M_n+j} &= D_{M_n} + \sum_{k=M_n}^{M_n+j-1} \psi_k = D_{M_n} + \sum_{l=0}^{j-1} \psi_{l+M_n} \\ &= D_{M_n} + \sum_{l=0}^{j-1} \psi_{l\oplus M_n} = D_{M_n} + \psi_{M_n} \sum_{l=0}^{j-1} \psi_l = D_{M_n} + r_n D_j \end{aligned}$$

holds. Thus,

$$\begin{aligned} B_1 &= \sum_{k=1}^{M_{2A-2}+\cdots+M_0-1} \frac{1}{k} D_{p_A-k} \\ &= \sum_{k=1}^{M_{2A-2}+\cdots+M_0-1} \frac{1}{k} \left(D_{M_{2A}} + r_{2A} D_{M_{2A-2}+\cdots+M_0-k}\right) \\ &= D_{M_{2A}}(\log(p_{A-1}) + o(A)) + r_{2A} \sum_{k=1}^{M_{2A-2}+\cdots+M_0-1} \frac{1}{k} D_{M_{2A-2}+\cdots+M_0-k} \\ &= D_{M_{2A}}(\log(p_{A-1}) + o(A)) + r_{2A} G_{M_{2A-2}+\cdots+M_0}. \end{aligned}$$

From B_1 and B_2 we get

$$\|G_{M_{2A}+\cdots+M_0}\|_1 \ge \|D_{M_{2A}} \log(p_{A-1}) + r_{2A} G_{M_{2A-2}+\cdots+M_0}\|_1 - o(A).$$

Let us devide this norm of the sum into two pieces: $\|.\|_1 = \int_{I_{2A}} |.| + \int_{I\setminus I_{2A}} |.|$. First

$$\int_{I_{2A}} |D_{M_{2A}} \log(p_{A-1}) + r_{2A} G_{M_{2A-2}+\cdots+M_0}| \ge \log(p_{A-1}) - \frac{1}{M_{2A}} G_{M_{2A-2}+\cdots+M_0}(0).$$

After

$$\begin{aligned} \int_{I\setminus I_{2A}} |D_{M_{2A}} \log(M_{2A}) + r_{2A} G_{M_{2A-2}+\cdots+M_0}| &= \int_{I\setminus I_{2A}} |G_{M_{2A-2}+\cdots+M_0}| \\ &= \|G_{M_{2A-2}+\cdots+M_0}\|_1 - \int_{I_{2A}} |G_{M_{2A-2}+\cdots+M_0}| \\ &= \|G_{M_{2A-2}+\cdots+M_0}\|_1 - \frac{1}{M_{2A}} G_{M_{2A-2}+\cdots+M_0}(0). \end{aligned}$$

These inequalities yield

$$\|G_{M_{2A}+\cdots+M_0}\|_1 \ge \log(p_{A-1}) + \|G_{M_{2A-2}+\cdots+M_0}\|_1 - 2\frac{1}{M_{2A}} G_{M_{2A-2}+\cdots+M_0}(0) - o(A).$$

We will use the estimation

$$G_n(0) = \sum_{k=1}^{n} \frac{n-k}{k} = n\sum_{k=1}^{n} \frac{1}{k} - n + 1 = n\log(n) + O(n).$$

By simple calculation we get

$$M_{2A} < p_A = M_{2A} + \cdots + M_0 < M_{2A} + M_{2A-1} \le \frac{3}{2} M_{2A}.$$

Since

$$\begin{aligned} 2\frac{1}{M_{2A}} G_{M_{2A-2}+\cdots+M_0}(0) &= 2\frac{1}{M_{2A}}(M_{2A-2}+\cdots+M_0)\log(M_{2A-2}+\cdots+M_0)+c \\ &\le 2\frac{1}{M_{2A}}\frac{3}{2}M_{2A-2}\log(p_{A-1})+c \\ &= \frac{3\log(p_{A-1})}{m_{2A-2}m_{2A-1}}+c \le \frac{3\log(p_{A-1})}{4}+c, \end{aligned}$$

we have

$$\begin{aligned} \|G_{M_{2A}+\cdots+M_0}\|_1 &\ge \|G_{M_{2A-2}+\cdots+M_0}\|_1 + \log(p_{A-1}) - \frac{3\log(p_{A-1})}{4} - c \\ &= \|G_{M_{2A-2}+\cdots+M_0}\|_1 + \frac{\log(p_{A-1})}{4} - c. \end{aligned}$$

This yields

$$\|G_{M_{2A}+\cdots+M_0}\|_1 \ge \frac{1}{4}\cdot\frac{3}{2}\sum_{k=0}^{A-1}\log(M_{2k}) - O(A).$$

Easy to see that

$$\begin{aligned} \sum_{k=1}^{K}\log M_{2A-2k} &= \sum_{k=1}^{K}\left(\log M_{2A-2} - \log\left(\prod_{h=1}^{k-1} m_{2A-2h-1}m_{2A-2h-2}\right)\right) \\ &\ge K\log M_{2A} - K\log(m_{2A-1}m_{2A-2}) \\ &\quad - \sum_{k=1}^{K}\sum_{h=1}^{k-1}(\log m_{2A-2h-1} + \log m_{2A-2h-2}) \\ &\ge K\log M_{2A} - cA^{\delta}K^2, \end{aligned}$$

if $K \in \{1,\cdots,A\}$. Since $\|F_{p_A}\|_1 \ge c\dfrac{\|G_{p_A}\|_1}{\log p_A}$, we have

$$\begin{aligned} \frac{\sum_{k=0}^{A-1}\log M_{2k}}{\log p_A} &\ge \frac{2\sum_{k=1}^{K}\log M_{2A-2k}}{3\log M_{2A}} \ge \frac{2}{3}K - \frac{2cA^{\delta}K^2}{3A\log 4} \\ &\ge c\left(K - \frac{K^2}{A^{1-\delta}}\right). \end{aligned}$$

Finally, let us choose $0 < \beta < 1-\delta$ and $A^\beta - 1 \le K < A^\beta$. In this case $\lim_{A\to\infty} K = \infty$ and $K - \frac{K^2}{A^{1-\delta}} = O(A^\beta)$, so

$$\|F_{p_A}\|_1 \ge c_\beta A^\beta.$$

Theorem 2. *If* $\log m_n = O(n^\delta)$ *for some* $0 < \delta < 1/2$, *then there exists* $f \in L^1$ *such that*

$$\|t_n f - f\|_1 \not\to 0.$$

Proof. The proof of this theorem is based on Lemma 5 and some standard method. See for instance [7].

Some conditions in our theorems can be a bit strange, which we could demonstrate by a simple example.

Example. *Let*

$$m_k = \begin{cases} \left[\exp(k^{\frac{1}{4}})\right], & \text{if } k = j^2 \text{ for any } j \in \mathbb{N}^+, \\ 2, & \text{otherwise.} \end{cases}$$

In this case it is true that

a) $\limsup_{k\in\mathbb{N}} m_k = \infty$,

b) $\log m_k = O(k^{\frac{1}{4}})$,

c) $\limsup_{n\in\mathbb{N}} \frac{\sum_{k=0}^{n-1} \log^2 m_k}{\log M_n} < \infty$. (It implies $\|F_{M_n}\|_1 \le c$).

Proof. It is easy to see that a) and b) hold. Investigate the case c). Since

$$\sum_{k=0}^{n-1} \log^2 m_k \le \sum_{k\neq j^2} \log^2 2 + \sum_{k=j^2} k^{\frac{1}{2}} \le (n - [\sqrt{n}]) \log^2 2 + \sum_{j=1}^{[\sqrt{n}]} j \le n$$

and $\log M_n \ge n \log 2$, we obtain

$$\frac{\sum\limits_{k=0}^{n-1} \log^2 m_k}{\log M_n} < \frac{3}{2}.$$

Remark. We formerly thought, that the Fejér means have "better properties", than the logarithmic ones. Although, what is the situation in case of unbounded Vilenkin system? For every m unbounded sequence there is $f \in L^1$ such that

$$\|\sigma_{M_n} f - f\|_1 \not\to 0,$$

(it was known before [7]), but we realized: there is m unbounded sequence, such that for every $f \in L^1$

$$\|t_{M_n} f - f\|_1 \to 0.$$

We have the same situation with respect to the space of continuous functions and the supremum norm.

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