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L^p BOUNDS FOR SINGULAR INTEGRALS ASSOCIATED TO SURFACES OF REVOLUTION ON PRODUCT DOMAINS*

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Abstract. In this paper, the authors establish L^p boundedness for several classes of multiple singular integrals along surfaces of revolution with kernels satisfying rather weak size condition. The results of the corresponding maximal truncated singular integrals are also obtained. The main results essentially improve and extend some known results.

Key words: *singular integral, product domain, surfaces of revolution, rough kernel*

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1 Introduction

Let $\mathbb{R}^N (N = m \text{ or } n)$, $N \geq 2$, be the N-dimensional Euclidean space and S^{N-1} be the unit sphere in $\mathbb{R}^N$ equipped with normalized Lebesgue measure $d\sigma = d\sigma(\cdot)$. For a nonzero point $w \in \mathbb{R}^N$, we denote $w' = w/|w|$. For $m \geq 2$, $n \geq 2$, let Ω be a homogeneous function of degree zero, integrable on $S^{m-1} \times S^{n-1}$ and satisfy

$$\int_{S^{m-1}} \Omega(u', v') \mathrm{d}\sigma(u') = \int_{S^{n-1}} \Omega(u', v') \mathrm{d}\sigma(v') = 0. \tag{1.1}$$

For suitable functions ϕ and ψ on $[0, \infty)$, we let Γ_ϕ and Λ_ψ be the surfaces of revolution given by $\Gamma_\phi = \{(x, \phi(|x|); x \in \mathbb{R}^m\}$ and $\Lambda_\psi = \{(y, \psi(|y|); y \in \mathbb{R}^n\}$. Define the associated singular integral operator $T_{\phi,\psi}$ (initially for $C_0^\infty(\mathbb{R}^{m+1} \times \mathbb{R}^{n+1})$) by

$$T_{\phi,\psi}(f)(\bar{x}, \bar{y}) = \text{p.v.} \int_{\mathbb{R}^m \times \mathbb{R}^n} K(u, v) f(\bar{x} - \Phi(u), \bar{y} - \Psi(v)) \mathrm{d}u\mathrm{d}v \tag{1.2}$$

and the corresponding maximal truncated singular integral operator $T^*_{\phi,\psi}$ by

$$T^*_{\phi,\psi}(f)(\bar{x}, \bar{y}) = \sup_{\varepsilon_1 > 0, \varepsilon_2 > 0} \left| \int_{|v| \geq \varepsilon_2} \int_{|u| \geq \varepsilon_1} K(u, v) f(\bar{x} - \Phi(u), \bar{y} - \Psi(v)) \mathrm{d}u\mathrm{d}v \right|, \tag{1.3}$$

where $K(u, v) = \Omega(u', v')|u|^{-m}|v|^{-n}$, $\Phi(u) = (u, \phi(|u|))$, $\Psi(v) = (v, \psi(|v|))$, $\bar{x} = (x, x_{m+1}) \in \mathbb{R}^{m+1} = \mathbb{R}^m \times \mathbb{R}$ and $\bar{y} = (y, y_{n+1}) \in \mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$. If $\phi = \psi \equiv 0$, we shall let $T = T_{0,0}$ and $T^* = T^*_{0,0}$.

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L^p boundedness of the operators T and T^* first were established by R. Fefferman and E. M. Stein [1, 2] provided Ω satisfies certain Lipschitz conditions. Subsequently, J. Duoandikoetxea[3] proved T is bounded on L^p for $1<p<\infty$ if $\Omega\in L^q(S^{m-1}\times S^{n-1})$ for $q>1$. Later on, the above results were improved by many authors (see [4-10], among others). In particular, it follows from [7, 8] that T and T^* are bounded on L^p for $1<p<\infty$ if $\Omega\in L\left(\log^+L\right)^2\left(S^{m-1}\times S^{n-1}\right)$, which is nearly optimal in the sense that the exponent 2 in $L\left(\log^+L\right)^2$ can not be replaced by any smaller number.

The other weaker condition on Ω is that for $\alpha>0$,

$$\sup_{\xi'\in S^{m-1},\eta'\in S^{n-1}}\int\int_{S^{m-1}\times S^{n-1}}|\Omega(u',v')|\left(\log\frac{1}{|\xi'\cdot u'|}\log\frac{1}{|\eta'\cdot v'|}\right)^{\alpha}\mathrm{d}\sigma(u')\mathrm{d}\sigma(v')<\infty. \tag{1.4}$$

Y. Ying [10] proved that if Ω satisfies (1.4) for $\alpha>1$, then T is bounded on L^p for $2\alpha/(2\alpha-1)<p<2\alpha$.

It is worth pointing out that the condition (1.4) in one-parameter case was originally defined in Walsh's paper [11] and developed by Grafakos and Stefanov [12] for $\alpha>1$. For the sake of simplicity, we denote for $\alpha>0$,

$$G_\alpha(S^{m-1}\times S^{n-1})=\{\Omega\in L^1(S^{m-1}\times S^{n-1}):\ \ \Omega\quad\text{satisfies}\ \ (1.4)\}.$$

Employing the ideas in [12], ones easily see that $L(\log^+L)^2(S^{m-1}\times S^{n-1})$ and $G_\alpha(S^{m-1}\times S^{n-1})$ for any $\alpha>0$ do not contain each other, and $\bigcup_{q>1}L^q(S^{m-1}\times S^{n-1})$ is a proper subset of $G_\alpha(S^{m-1}\times S^{n-1})$ for any $\alpha>0$, also, of $L(\log^+L)^2(S^{m-1}\times S^{n-1})$.

For the general operators $T_{\phi,\psi}$ and $T^*_{\phi,\psi}$, A. Al-Salman[13] (resp., A. Al-Qassem[14]) showed that $T_{\phi,\psi}$ and $T^*_{\phi,\psi}$ are bounded on $L^p(\mathbb{R}^{m+1}\times\mathbb{R}^{n+1})$ for $1<p<\infty$ if $\phi,\ \psi$ are C^2, convex increasing and $\Omega\in L(\log^+L)^2(S^{m-1}\times S^{n-1})$ (resp., Ω belongs to certain block spaces). The main purpose of this paper is to investigate L^p bondedness of the general operators $T_{\phi,\psi}$ and $T^*_{\phi,\psi}$ when $\Omega\in G_\alpha(S^{m-1}\times S^{n-1})$ for $\alpha>0$. Our main results can be formulated as follows.

Theorem 1.1. *Let Ω be a homogeneous function of degree zero and satisfy (1.1). Suppose that $\phi\in C^1([0,\infty))$, ϕ' is convex and increasing (or ϕ is a polynomial), $\psi\in C^1([0,\infty))$, ψ' is convex and increasing (or ψ is a polynomial). If $\Omega\in G_\alpha(S^{m-1}\times S^{n-1})$ for $\alpha>1$ and one of the following conditions holds, then $T_{\phi,\psi}$ is bounded on $L^p(\mathbb{R}^{m+1}\times\mathbb{R}^{n+1})$ for $2\alpha/(2\alpha-1)<p<2\alpha$.*

(i) $m=n=2$.

(ii) $m\geq 3$, $n=2$ *and* $\phi'(0)=0$.

(iii) $m=2$, $n\geq 3$ *and* $\psi'(0)=0$.

(iv) $m\geq 3$, $n\geq 3$ *and* $\phi'(0)=\psi'(0)=0$.

Moreover, the bounds are independent of the coefficients of $\phi,\ \psi$ when $\phi,\ \psi$ are polynomials.

Theorem 1.2. *Let $\phi,\ \psi$ be given as in Theorem 1. Let Ω is a homogeneous function of degree zero and satisfies (1.1). Suppose that $\Omega\in G_\alpha(S^{m-1}\times S^{n-1})$ for $\alpha>3/2$. Then under the same conditions of Theorem 1, $T^*_{\phi,\psi}$ is bounded on $L^p(\mathbb{R}^{m+1}\times\mathbb{R}^{n+1})$ for $1+1/(2\alpha-2)<p<2\alpha-1$. Moreover, the bounds are independent of the coefficients of $\phi,\ \psi$ when $\phi,\ \psi$ are polynomials.*

This paper is organized as follows. In Section 2 we shall introduce some notations and give some technical lemmas. The proof of Theorem 1 will be given in Section 3. Finally, we shall prove Theorem 2 in Section 4. We remark that some ideas in the proofs of our main results are taken from [3, 6, 15, 16], but our methods and techniques are more delicate and complex than that of [3, 6, 15, 16].

Throughout this paper, we always use the letter C to denote positive constants that may vary at each occurrence but are independent of the essential variables.

2 Main Lemmas

Let ϕ, ψ, Ω be as in Theorem 1 or 2. For $j,k\in\mathbb{Z}$ we denote

$$B_{j,k}=\left\{(u,v)\in\mathbb{R}^m\times\mathbb{R}^n:\ \ 2^j<|u|\le 2^{j+1}, 2^k<|v|\le 2^{k+1}\right\}.$$

Define the measures $\sigma_{j,k}$ and $\lambda_{j,k}$ on $\mathbb{R}^{m+1}\times\mathbb{R}^{n+1}$ by letting their Fourier transforms be

$$\widehat{\sigma}_{j,k}(\overline{\xi},\overline{\eta})=\int\int_{B_{j,k}}\frac{\Omega(u',v')}{|u|^m|v|^n}e^{-i[\xi\cdot u+\xi_{m+1}\phi(|u|)+\eta\cdot v+\eta_{n+1}\psi(|v|)]}\mathrm{d}u\mathrm{d}v \tag{2.1}$$

and

$$\widehat{\lambda}_{j,k}(\overline{\xi},\overline{\eta})=\int\int_{B_{j,k}}\frac{|\Omega(u'v')|}{|u|^m|v|^n}e^{-i[\xi\cdot u+\xi_{m+1}\phi(|u|)+\eta\cdot v+\eta_{n+1}\psi(|v|)]}\mathrm{d}u\mathrm{d}v, \tag{2.2}$$

where $\overline{\xi}=(\xi,\xi_{m+1})\in\mathbb{R}^m\times\mathbb{R}$ and $\overline{\eta}=(\eta,\eta_{n+1})\in\mathbb{R}^n\times\mathbb{R}$. Then we have

$$T_{\phi,\psi}(f)(\overline{x},\overline{y})=\sum_{j,k\in\mathbb{Z}}\sigma_{j,k}*f(\overline{x},\overline{y}). \tag{2.3}$$

It is easy to see that $\|\widehat{\sigma}_{j,k}\|_\infty\le C$, $\|\widehat{\lambda}_{j,k}\|_\infty\le C$ uniformly for $j,k\in\mathbb{Z}$.

Also, we define the maximal operator σ^* by

$$\sigma^*(f)(\overline{x},\overline{y})=\sup_{j,k\in\mathbb{Z}}\left|\lambda_{j,k}*f(\overline{x},\overline{y})\right|.$$

Lemma 2.1. *For $1<p<\infty$, σ^* is bounded on $L^p(\mathbb{R}^{m+1}\times\mathbb{R}^{n+1})$,and the bound is independent of the coefficients of ϕ, ψ when ϕ, ψ are polynomials.*

Proof. By using spherical coordinate, we have

$$\begin{aligned}\sigma^*(f)(\overline{x},\overline{y}) &\le \sup_{j,k\in\mathbb{Z}}\int_{2^j}^{2^{j+1}}\int_{2^k}^{2^{k+1}}\int\int_{S^{m-1}\times S^{n-1}}|\Omega(u',v')|s^{-1}t^{-1}\\ &\qquad\times|f(x-su',x_{m+1}-\phi(s);y-tv',y_{n+1}-\psi(t))|\mathrm{d}\sigma(u')\mathrm{d}\sigma(v')\mathrm{d}s\mathrm{d}t\\ &\le\int\int_{S^{m-1}\times S^{n-1}}|\Omega(u',v')|M_{u',v'}(f)(\overline{x},\overline{y})\mathrm{d}\sigma(u')\mathrm{d}\sigma(v'),\end{aligned}$$

where

$$M_{u',v'}(f)(\overline{x},\overline{y})=\sup_{r,h>0}\frac{1}{rh}\int_0^r\int_0^h|f(x-su',x_{m+1}-\phi(s);y-tv',y_{n+1}-\psi(t))|\mathrm{d}s\mathrm{d}t.$$

By L^p-boundedness results in [17, Corollary 5] (or [18, Proposition 1, p.477]), using iterated integration, it is easy to see that

$$\|M_{u',v'}(f)\|_p \le C\|f\|_p,$$

where C is independent of (u',v') and the coefficients of ϕ, ψ when ϕ, ψ are polynomials. Thus

$$\|\sigma^*(f)\|_p \le \int\int_{S^{m-1}\times S^{n-1}} |\Omega(u',v')|\|M_{u',v'}(f)\|_p \mathrm{d}\sigma(u')\mathrm{d}\sigma(v') \le C\|f\|_p,$$

which completes the proof of Lemma 2.1.

Lemma 2.2. *For arbitrary functions* $\{g_{j,k}\}$, $1 < p_0 < \infty$,

$$\left\|\left(\sum_{j,k\in\mathbb{Z}} \left|\sigma_{j,k} * g_{j,k}\right|^2\right)^{1/2}\right\|_{p_0} \le C\left\|\left(\sum_{j,k\in\mathbb{Z}} |g_{j,k}|^2\right)^{1/2}\right\|_{p_0}$$

where C is independent of the coefficients of ϕ, ψ when ϕ, ψ are polynomials.

Applying Lemma 2.1, the proof of Lemma 2.2 follows by using the similar arguments as the proof of Lemma in [17, p.544]. Here we omit the details.

Lemma 2.3 [17]. *Let $\phi: [0,\infty) \longrightarrow \mathbb{R}$ be a function in C^1 such that ϕ' is convex, increasing and satisfies $\phi'(0) = 0$. Then there exists $C > 0$ such that*

$$\left|\int_1^b e^{i[2^j ar + \rho\phi(2^j r)]}\mathrm{d}r\right| \le C|2^j a|^{-1/2}$$

holds for all $b \ge 1$, a, $\rho \in \mathbb{R}$, and $j \in \mathbb{Z}$.

Lemma 2.4 [19]. *Let $\mu(y) = \sum_{|\beta|\le d} b_\beta y^\beta$ where $b_\beta \in \mathbb{R}$. Then*

$$\left|\int_{[0,1]^n} e^{i\mu(y)}\mathrm{d}y\right| \le C_{d,n}\left(\sum_{0<|\beta|\le d} |b_\beta|\right)^{-1/d}.$$

Moreover, $C_{d,1} \le Cd$ for an absolute constant C.

Lemma 2.5. *Let $j, k \in \mathbb{Z}$, and ϕ, ψ be as in Theorem 1. If $\Omega \in G_\alpha(S^{m-1}\times S^{n-1})$ for $\alpha > 1$ and satisfies (1.1). Then for $\overline{\xi} = (\xi, \xi_{m+1}) \in \mathbb{R}^m \times \mathbb{R}$, $\overline{\eta} = (\eta, \eta_{n+1}) \in \mathbb{R}^n \times \mathbb{R}$, there exists $C > 0$ such that*

(i) $|\widehat{\sigma}_{j,k}(\overline{\xi},\overline{\eta})| \le C|2^j\xi||2^k\eta|$, for all $\xi \in \mathbb{R}^m$, $\eta \in \mathbb{R}^n$;

(ii) $|\widehat{\sigma}_{j,k}(\overline{\xi},\overline{\eta})| \le C|2^j\xi|(\log|2^k\eta|)^{-\alpha}$, for all $\xi \in \mathbb{R}^m$, $|2^k\eta| > 2^\alpha$;

(iii) $|\widehat{\sigma}_{j,k}(\overline{\xi},\overline{\eta})| \le C(\log|2^j\xi|)^{-\alpha}|2^k\eta|$, for all $\eta \in \mathbb{R}^m$, $|2^j\xi| > 2^\alpha$;

(iv) $|\widehat{\sigma}_{j,k}(\overline{\xi},\overline{\eta})| \le C(\log|2^j\xi|)^{-\alpha}(\log|2^k\eta|)^{-\alpha}$, for all $|2^j\xi| > 2^\alpha$, $|2^k\eta| > 2^\alpha$.

Proof. (i) is obvious by (1.1). In what follows, we will prove (ii)–(iv) in the following four cases.

Case 1. ϕ, $\psi \in C^1([0,\infty))$, ϕ', ψ' is convex and increasing.

To prove (ii), we write

$$\begin{aligned}\widehat{\sigma}_{j,k}(\overline{\xi},\overline{\eta}) &= \int\int_{S^{m-1}\times S^{n-1}}\Omega(u',v')\int_1^2 e^{-i[2^js|\xi|\xi'\cdot u'+\phi(2^js)\xi_{m+1}]}\frac{\mathrm{d}s}{s}\\ &\quad\times\int_1^2 e^{-i[2^kt|\eta|\eta'\cdot v'+\psi(2^kt)\eta_{n+1}]}\frac{\mathrm{d}t}{t}\mathrm{d}\sigma(u')\mathrm{d}\sigma(v')\\ &= \int\int_{S^{m-1}\times S^{n-1}}\Omega(u',v')\left[\int_1^2\left(e^{-i[2^js|\xi|\xi'\cdot u'+\phi(2^js)\xi_{m+1}]}-e^{-i\phi(2^js)\xi_{m+1}}\right)\frac{\mathrm{d}s}{s}\right]\\ &\quad\times\left[\int_1^2 e^{i[2^k|\eta|(\eta'\cdot v'+\psi'(0)\eta_{n+1}|\eta|^{-1})+(\psi(2^kt)-\psi'(0)2^kt)\eta_{n+1}]}\frac{\mathrm{d}t}{t}\right]\mathrm{d}\sigma(u')\mathrm{d}\sigma(v').\end{aligned}$$

It is easy to see that the integral in the first brackets is bounded by $C|2^j\xi|$. By Lemma 2.3, the integral in the secondary brackets is bounded by

$$C\left(2^k|\eta|\left|\eta'\cdot v'+\psi'(0)|\eta|^{-1}\eta_{n+1}\right|\right)^{-1/2}.$$

Let $\delta=\min\{|\psi'(0)\eta_{n+1}||\eta|^{-1},2\}\mathrm{sgn}(\psi'(0)\eta_{n+1})$. By combining the preceding inequality with the trivial estimate

$$\left|\int_1^2 e^{-i[2^kt|\eta|\eta'\cdot v'+\psi(2^kt)\eta_{n+1}]}\frac{\mathrm{d}t}{t}\right|\le 1,$$

we have

$$\left|\int_1^2 e^{-i[2^kt|\eta|\eta'\cdot v'+\psi(2^kt)\eta_{n+1}]}\frac{\mathrm{d}t}{t}\right|\le C\min\left\{1,\left(\frac{2^\alpha|\eta'\cdot v'+\delta|^{-1}}{|2^k\eta|}\right)^{1/2}\right\}.$$

Since $t/\log^a t$ is increasing in $(2^a,\ +\infty)$ for any $a>0$, we can deduce that for $\alpha>0$,

$$\left|\int_1^2 e^{-i[2^kt|\eta|\eta'\cdot v'+\psi(2^kt)\eta_{n+1}]}\frac{\mathrm{d}t}{t}\right|\le C\frac{\log^\alpha(2^\alpha|\eta'\cdot v'+\delta|^{-1})}{\log^\alpha|2^k\eta|},\ \text{ if }\ |2^k\eta|>2^\alpha. \tag{2.4}$$

Therefore, when $n\ge 3$, by the additional assumption $\psi'(0)=0$, i.e. $\delta=0$, we get

$$\begin{aligned}|\widehat{\sigma}_{j,k}(\overline{\xi},\overline{\eta})| &\le C|2^j\xi|\left(\log|2^k\eta|\right)^{-\alpha}\int\int_{S^{m-1}\times S^{n-1}}|\Omega(u',v')|\left(\log\frac{2}{|\eta'\cdot v'|}\right)^\alpha\mathrm{d}\sigma(u')\mathrm{d}\sigma(v')\\ &\le C|2^j\xi|\left(\log|2^k\eta|\right)^{-\alpha}, \qquad\qquad \text{if }\ |2^k\eta|>2^\alpha.\end{aligned}$$

When $n=2$, by the similar arguments as those in [15, pp. 167-168], we may assume that $\delta>0$ and set $\delta'=\min\{\delta,1\}$. Let $\theta=\arcsin(\delta')$, and let e_+, e_- denote the vectors obtained by rotating η' by angles θ and $-\theta$, respectively. Then there is a constant $c_0\in(0,1)$ such that

$$|\eta'\cdot v'+\delta|\ge c_0\min\{|e_+\cdot v'|^2,|e_-\cdot v'|^2\}$$

for $v'\in S^1$. Thus

$$|\widehat{\sigma}_{j,k}(\overline{\xi},\overline{\eta})|\le C|2^j\xi|\left(\log|2^k\eta|\right)^{-\alpha}$$

also holds when $n=2$ and $|2^k\eta|>2^\alpha$ (without the additional assumption $\psi'(0)=0$). This completes the proof of Lemma (ii).

Similar to (2.4), we can conclude (iii) (without the additional assumption $\phi'(0)=0$ when $m=2$).

It remains to prove (iv). We write

$$\begin{aligned}\widehat{\sigma}_{j,k}(\overline{\xi},\overline{\eta}) &= \int\int_{S^{m-1}\times S^{n-1}} \Omega(u',v') \int_1^2 e^{-i[2^j s|\xi|\xi'\cdot u'+\phi(2^j s)\xi_{m+1}]}\frac{\mathrm{d}s}{s} \\ &\quad\times \int_1^2 e^{-i[2^k t|\eta|\eta'\cdot v'+\psi(2^k t)\eta_{n+1}]}\frac{\mathrm{d}t}{t}\mathrm{d}\sigma(u')\mathrm{d}\sigma(v') \\ &= \int\int_{S^{m-1}\times S^{n-1}} \Omega(u',v') \left[\int_1^2 e^{-i[2^j s|\xi|(\xi'\cdot u'+\phi'(0)\xi_{m+1}|\xi|^{-1})+(\phi(2^j s)-\phi'(0)2^j s)\xi_{m+1}]}\frac{\mathrm{d}s}{s}\right] \\ &\quad\times\left[\int_1^2 e^{i[2^k|\eta|(\eta'\cdot v'+\psi'(0)\eta_{n+1}|\eta|^{-1})+(\psi(2^k t)-\psi'(0)2^k t)\eta_{n+1}]}\frac{\mathrm{d}t}{t}\right]\mathrm{d}\sigma(u')\mathrm{d}\sigma(v').\end{aligned}$$

Similar to (2.4), we have

$$\left|\int_1^2 e^{-i[2^j s|\xi|\xi'\cdot u'+\psi(2^j s)\xi_{m+1}]}\frac{\mathrm{d}s}{s}\right| \le C\frac{\log^{\alpha}(2^{\alpha}|\xi'\cdot u'+\rho|^{-1})}{\log^{\alpha}|2^j\xi|},\ \text{ if }\ |2^j\xi|>2^{\alpha}, \tag{2.5}$$

where $\rho=\min\{|\phi'(0)\xi_{m+1}|/|\xi|,2\}\mathrm{sgn}(\phi'(0)\xi_{m+1})$. By (2.4)–(2.5) and the similar arguments as those in proving (ii), we get

$$|\sigma_{j,k}(\overline{\xi},\overline{\eta})| \le C(\log|2^j\xi|)^{-\alpha}(\log|2^k\eta|)^{-\alpha}, \quad \text{if }\ |2^j\xi|>2^{\alpha}\ \text{ and }\ |2^k\eta|>2^{\alpha},$$

without the additional assumption $\phi'(0)=0$ or $\psi'(0)=0$ when $m=2$ or $n=2$. (iv) is proved.

Case 2. ϕ, ψ are polynomials. Precisely, $\phi(s)=\sum_{\mu=0}^{d}a_{\mu}s^{\mu}$, $\psi(t)=\sum_{\nu=0}^{l}b_{\nu}t^{\nu}$.

We need only to prove (iv) since the proofs of (ii)-(iii) are similar or even simpler. By spherical coordinate, we write

$$\begin{aligned}\widehat{\sigma}_{j,k}(\overline{\xi},\overline{\eta}) &= \int\int_{S^{m-1}\times S^{n-1}} \Omega(u',v') \int_1^2 e^{-i[2^j(|\xi|\xi'\cdot u'+a_1\xi_{m+1})s+a_0\xi_{m+1}+\sum_{\mu=2}^{d}a_{\mu}\xi_{m+1}2^{\mu j}s^{\mu}]}\frac{\mathrm{d}s}{s} \\ &\quad\times\int_1^2 e^{-i[2^k(|\eta|\eta'\cdot v'+b_1\eta_{n+1})s+b_0\eta_{n+1}+\sum_{\nu=2}^{l}b_{\nu}\eta_{n+1}2^{\nu k}t^{\nu}]}\frac{\mathrm{d}t}{t}\mathrm{d}\sigma(u')\mathrm{d}\sigma(v') \\ &:= \int\int_{S^{m-1}\times S^{n-1}} \Omega(u',v') I_j(\xi,\xi_{m+1},u')I_k(\eta,\eta_{n+1},v')\mathrm{d}\sigma(u')\mathrm{d}\sigma(v').\end{aligned}$$

By Lemma 2.4,

$$|I_j(\xi,\xi_{m+1},u')| \le C|2^j(|\xi|\xi'\cdot u'+a_1\xi_{m+1})|^{-1/d}, \tag{2.6}$$

$$|I_k(\eta,\eta_{n+1},v')| \le C|2^k(|\eta|\eta'\cdot v'+b_1\eta_{n+1})|^{-1/l}. \tag{2.7}$$

Let $\delta_1=\min\{|a_1\xi_{m+1}|/|\xi|,2\}$, $\delta_2=\min\{|b_1\eta_{n+1}|/|\eta|,2\}$. By (2.6), (2.7) and the trivial estimates

$$|I_j(\xi,\xi_{m+1},u')|\le 1, \qquad |I_k(\eta,\eta_{n+1},v')|\le 1,$$

we obtain

$$|I_j(\xi,\xi_{m+1},u')| \le C\left(\frac{\log(2^{\alpha}/|\xi'\cdot u'+\delta_1|)}{\log(2^j|\xi|)}\right)^{\alpha}, \qquad \text{if }\ |2^j\xi|>2^{\alpha}, \tag{2.8}$$

$$|I_k(\eta,\eta_{n+1},v')| \le C\left(\frac{\log(2^{\alpha}/|\eta'\cdot v'+\delta_2|)}{\log(2^k|\eta|)}\right)^{\alpha}, \qquad \text{if } |2^k\eta|>2^{\alpha}. \tag{2.9}$$

By the similar arguments as those in Case 1, we can obtain the desirable estimates.

Case 3. $\phi \in C^1([0,\infty))$, ϕ' is convex, increasing and ψ is a polynomial.

By (2.5) and (2.9), using the similar arguments as those in Case 1, we can get (ii)–(iv). The details are omitted.

Case 4. ϕ is a polynomial, $\psi \in C^1([0,\infty))$, ψ' is convex and increasing:

By (2.8) and (2.4), using the similar arguments as those in Case 1, we can obtain the desirable conclusions.

This completes the proof of Lemma 2.5.

Remark 1. In the proof of Lemma 2.5, we do not use the condition $\phi'(0)=0$ when $m=2$ and the condition $\psi'(0)=0$ when $n=2$. In these case, (1.4) implies that for $\alpha>1$,

$$\sup_{\substack{\xi'\in S^{m-1},\eta'\in S^{n-1},\\ \delta,\rho\in\mathbb{R}}}\int_{S^{m-1}\times S^{n-1}}|\Omega(u',v')|\left(\log\frac{1}{|\xi'\cdot u'+\delta|}\log\frac{1}{|\eta'\cdot v'+\rho|}\right)^{\alpha}\mathrm{d}\sigma(u')\mathrm{d}\sigma(v')<\infty.$$

However, this is no longer true when $m\ge 3$ or $n\ge 3$. For example (see [15,p.168]), let $m=n=3$ and define ω on S^2 by

$$\omega(z_1,z_2,z_3)=\frac{\chi_{(1/2,\sqrt{3}/2)}(z_3)}{(z_3-1/2)[\log(1/(z_3-1/2))]^2},$$

when $z_3\ge 0$, and $\omega(z_1,z_2,z_3)=-\omega(z_1,z_2,-z_3)$ when $z_3<0$. For $u'=(u_1',u_2',u_3'),\ v'=(v_1',v_2',v_3')\in S^2$, let $\Omega(u',v')=\omega(u_1',u_2',u_3')\omega(v_1',v_2',v_3')$. Then for $\alpha>1$

$$\sup_{\xi',\eta'\in S^2}\int\int_{S^2\times S^2}|\Omega(u',v')|\left(\log\frac{1}{|\xi'\cdot u'|}\log\frac{1}{|\eta'\cdot v'|}\right)^{\alpha}\mathrm{d}\sigma(u')\mathrm{d}\sigma(v')<\infty$$

and

$$\int\int_{S^2\times S^2}|\Omega(u',v')|\left(\log\frac{1}{|u_3'-1/2|}\log\frac{1}{|v_3'-1/2|}\right)^{\alpha}\mathrm{d}\sigma(u')\mathrm{d}\sigma(v')=\infty.$$

3 Proof of Theorem 1.1

In this section, we will prove Theorem 1.1. By duality, we may assume $p\in[2,2\alpha)$.

Take two radial Schwartz functions $\psi_1\in\mathscr{S}(\mathbb{R}^m)$, $\psi_2\in\mathscr{S}(\mathbb{R}^n)$ such that

(i) $0\le\psi_i\le 1$, $i=1,2$;

(ii) $\mathrm{supp}(\psi_1)\subset\{x\in\mathbb{R}^m:\ 1/4\le|x|\le 4\}$, $\mathrm{supp}(\psi_2)\subset\{y\in\mathbb{R}^n:\ 1/4\le|y|\le 4\}$;

(iii) $\sum_{d\in\mathbb{Z}}[\psi_1(2^ds)]^2=\sum_{l\in\mathbb{Z}}[\psi_2(2^lt)]^2=1$, for all $s>0$, $t>0$.

For $d,l\in\mathbb{Z}$, define the multiplier operator $S_{d,l}$ in $\mathbb{R}^{m+1}\times\mathbb{R}^{n+1}$ by

$$\widehat{S_{d,l}(f)}(\overline{\xi},\overline{\eta})=\psi_1(2^d|\xi|)\psi_2(2^l|\eta|)\widehat{f}(\overline{\xi},\overline{\eta}),$$

where $\overline{\xi}=(\xi,\xi_{m+1})\in\mathbb{R}^{m+1}=\mathbb{R}^m\times\mathbb{R}$, $\overline{\eta}=(\eta,\eta_{n+1})\in\mathbb{R}^{n+1}=\mathbb{R}^n\times\mathbb{R}$. Then by checking the Fourier transforms, it is easy to see that for any test function f,

$$f(\overline{x},\overline{y})=\sum_{d\in\mathbb{Z}}\sum_{l\in\mathbb{Z}}S^2_{d,l}(f)(\overline{x},\overline{y}).$$

Consequently, by (2.3) we can write

$$T_{\phi,\psi}(f)(\overline{x},\overline{y})=\sum_{j,k\in\mathbb{Z}}\sum_{d,l\in\mathbb{Z}}S_{j-d,k-l}(\sigma_{j,k}*S_{j-d,k-l}(f))(\overline{x},\overline{y}):=\sum_{d,l\in\mathbb{Z}}T_{d,l}(f)(\overline{x},\overline{y}). \tag{3.1}$$

By Plancherel's theorem we have

$$\begin{aligned}\|T_{d,l}(f)\|^2_{L^2(\mathbb{R}^{m+1}\times\mathbb{R}^{n+1})}&=\int\int_{\mathbb{R}^{m+1}\times\mathbb{R}^{n+1}}\Big|\sum_{j,k\in\mathbb{Z}}S_{j-d,k-l}(\sigma_{j,k}*S_{j-d,k-l}(f))(\overline{x},\overline{y})\Big|^2\mathrm{d}\overline{x}\mathrm{d}\overline{y}\\&\le C\int\int_{\mathbb{R}\times\mathbb{R}}\sum_{j,k\in\mathbb{Z}}\int\int_{B_{j-d,k-l}}|\widehat{\sigma}_{j,k}(\overline{\xi},\overline{\eta})|^2|\widehat{f}(\overline{\xi},\overline{\eta})|^2\mathrm{d}\xi\mathrm{d}\eta\mathrm{d}\xi_{m+1}\mathrm{d}\eta_{n+1},\end{aligned}$$

where

$$B_{j-d,k-l}=\Big\{(\xi,\eta)\in\mathbb{R}^m\times\mathbb{R}^n:2^{-j+d-2)}\le|\xi|\le2^{-j+d+2)},2^{-k+l-2)}\le|\eta|\le2^{-k+l+2)}\Big\}.$$

Thus, by Lemma 2.5 (iv) we get that for $d,l>\alpha+20$,

$$\begin{aligned}\|T_{d,l}(f)\|^2_{L^2(\mathbb{R}^{m+1}\times\mathbb{R}^{n+1})}&\le C\int\int_{\mathbb{R}\times\mathbb{R}}\sum_{j,k\in\mathbb{Z}}\int\int_{B_{j-d,k-l}}\big(\log|2^j\xi|\big)^{-2\alpha}\Big(\log|2^k\eta|\Big)^{-2\alpha}\\&\quad\times|\widehat{f}(\overline{\xi},\overline{\eta})|^2\mathrm{d}\xi\mathrm{d}\eta\mathrm{d}\xi_{m+1}\mathrm{d}\eta_{n+1}\\&\le C(dl)^{-2\alpha}\|f\|^2_{L^2(\mathbb{R}^{m+1}\times\mathbb{R}^{n+1})},\end{aligned}$$

which implies

$$\|T_{d,l}(f)\|_{L^2(\mathbb{R}^{m+1}\times\mathbb{R}^{n+1})}\le C(dl)^{-\alpha}\|f\|_{L^2(\mathbb{R}^{m+1}\times\mathbb{R}^{n+1})}. \tag{3.2}$$

On the other hand, by the Littlewood-Paley theory and Lemma 2.2 we obtain that for any $p_0\in(1,\infty)$,

$$\begin{aligned}\|T_{d,l}(f)\|_{L^{p_0}(\mathbb{R}^{m+1}\times\mathbb{R}^{n+1})}&\le C\Bigg\|\Bigg(\sum_{j,k\in\mathbb{Z}}|\sigma_{j,k}*S_{j-d,k-l}(f)|^2\Bigg)^{1/2}\Bigg\|_{L^{p_0}(\mathbb{R}^{m+1}\times\mathbb{R}^{n+1})}\\&\le C\Bigg\|\Bigg(\sum_{j,k\in\mathbb{Z}}|S_{j-d,k-l}(f)|^2\Bigg)^{1/2}\Bigg\|_{L^{p_0}(\mathbb{R}^{m+1}\times\mathbb{R}^{n+1})}\\&\le C\|f\|_{L^{p_0}(\mathbb{R}^{m+1}\times\mathbb{R}^{n+1})}.\end{aligned} \tag{3.3}$$

Noting that $2\le p<2\alpha$, by interpolating between (3.2) and (3.3), we can obtain a $\theta>1$ such that

$$\|T_{d,l}(f)\|_{L^p(\mathbb{R}^{m+1}\times\mathbb{R}^{n+1})}\le C(dl)^{-\theta}\|f\|_{L^p(\mathbb{R}^{m+1}\times\mathbb{R}^{n+1})}\qquad d,l>\alpha+2 \tag{3.4}$$

Similarly, by using Lemma 2.5 (i), we can get $\varepsilon>0$ such that

$$\|T_{d,l}(f)\|_{L^p(\mathbb{R}^{m+1}\times\mathbb{R}^{n+1})}\le C2^{(d+l)\varepsilon}\|f\|_{L^p(\mathbb{R}^{m+1}\times\mathbb{R}^{n+1})},\qquad d,l\le\alpha+2. \tag{3.5}$$

By using Lemma 2.5 (ii) and (iii), it is easy to deduce that

$$\|T_{d,l}(f)\|_{L^p(\mathbb{R}^{m+1}\times\mathbb{R}^{n+1})} \le C2^{d\varepsilon}l^{-\theta}\|f\|_{L^p(\mathbb{R}^{m+1}\times\mathbb{R}^{n+1})}, \qquad d<\alpha+2,\, l\ge\alpha+2, \tag{3.6}$$

and

$$\|T_{d,l}(f)\|_{L^p(\mathbb{R}^{m+1}\times\mathbb{R}^{n+1})} \le Cd^{-\theta}2^{l\varepsilon}\|f\|_{L^p(\mathbb{R}^{m+1}\times\mathbb{R}^{n+1})}, \qquad d\ge\alpha+2,\, l<\alpha+2, \tag{3.7}$$

where $\theta>1$ and $\varepsilon>0$ are the same as that in (3.4) and (3.5), respectively.

Therefore, it follows from (3.1) and (3.4)–(3.7)

$$\begin{aligned}\|T_{\phi,\psi}(f)\|_{L^p(\mathbb{R}^{m+1}\times\mathbb{R}^{n+1})} &\le \sum_{d,l\in\mathbb{Z}}\|T_{d,l}(f)\|_{L^p(\mathbb{R}^{m+1}\times\mathbb{R}^{n+1})}\\ &\le C\left\{\sum_{d,l\le 0}2^{(d+l)\varepsilon}+\sum_{d\le 0,l>0}2^{d\varepsilon}l^{-\theta}+\sum_{d>0,l\le 0}d^{-\theta}2^{l\varepsilon}+\sum_{d,l>0}(dl)^{-\theta}\right\}\\ &\quad\times\|f\|_{L^p(\mathbb{R}^{m+1}\times\mathbb{R}^{n+1})}\\ &\le C\|f\|_{L^p(\mathbb{R}^{m+1}\times\mathbb{R}^{n+1})},\end{aligned}$$

which completes the proof of Theorem 1.1.

4 Proof of Theorem 1.2

In this section, we will prove Theorem 1.2 by employing the techniques developed in [6]. Let us begin by introducing some notations. Let $\mathscr{T}_0^\infty(\mathbb{R}^{m+1}\times\mathbb{R}^{n+1})$ be the subspace of $C_0^\infty(\mathbb{R}^{m+1}\times\mathbb{R}^{n+1})$ generated by functions of the form $f_1\bigotimes f_2$, where $f_1\in C_0^\infty(\mathbb{R}^{m+1})$ and $f_2\in C_0^\infty(\mathbb{R}^{n+1})$. Let $\{\sigma_{j,k}\}_{j,k\in\mathbb{Z}}$ be as before. Define $\tau_{j,k}$ and τ_* by

$$\tau_{j,k}(f)(\bar{x},\bar{y})=\sum_{d=j}^{\infty}\sum_{l=k}^{\infty}\sigma_{d,l}*f(\bar{x},\bar{y})$$

and

$$\tau_*(f)(\bar{x},\bar{y})=\sup_{j,k\in\mathbb{Z}}|\tau_{j,k}(f)(\bar{x},\bar{y})|.$$

To prove Theorem 1.2, we first establish the following two lemmas.

Lemma 4.1. *Let Ω be as in Theorem 1, $\{\sigma_{j,k}\}_{j,k}$ as before. Then for $\alpha>1$ and $2\alpha/(2\alpha-1)<p<2\alpha$,*

$$\|S_1(f)\|_{L^p(\mathbb{R}^{m+1}\times\mathbb{R}^{n+1})}\le C\|f\|_{L^p(\mathbb{R}^{m+1}\times\mathbb{R}^{n+1})}$$

and

$$\|S_2(f)\|_{L^p(\mathbb{R}^{m+1}\times\mathbb{R}^{n+1})}\le C\|f\|_{L^p(\mathbb{R}^{m+1}\times\mathbb{R}^{n+1})},$$

where

$$S_1(f)(\bar{x},\bar{y})=\sup_{d\in\mathbb{Z}}\left|\sum_{l=-\infty}^{\infty}\sigma_{d,l}*f(\bar{x},\bar{y})\right|, \qquad S_2(f)(\bar{x},\bar{y})=\sup_{l\in\mathbb{Z}}\left|\sum_{d=-\infty}^{\infty}\sigma_{d,l}*f(\bar{x},\bar{y})\right|.$$

Proof. First, we consider the case: $f \in \mathscr{T}_0^\infty(\mathbb{R}^{m+1}\times\mathbb{R}^{n+1})$. In this case, it is easy to verify that

$$\begin{aligned}&\int_{2^d\le|u|<2^{d+1}}\lim_{\varepsilon_2\to0}\int_{|v|>\varepsilon_2}\frac{\Omega(u',v')}{|u|^m|v|^n}f(\bar{x}-\Phi(u),\bar{y}-\Psi(v))\mathrm{d}u\mathrm{d}v\\&\quad=\lim_{\varepsilon_2\to0}\int_{|v|>\varepsilon_2}\int_{2^d\le|u|<2^{d+1}}\frac{\Omega(u',v')}{|u|^m|v|^n}f(\bar{x}-\Phi(u),\bar{y}-\Psi(v))\mathrm{d}u\mathrm{d}v,\end{aligned}\tag{4.1}$$

and

$$\begin{aligned}&\int_{2^l\le|v|<2^{l+1}}\lim_{\varepsilon_1\to0}\int_{|u|>\varepsilon_1}\frac{\Omega(u',v')}{|u|^m|v|^n}f(\bar{x}-\Phi(u),\bar{y}-\Psi(v))\mathrm{d}u\mathrm{d}v\\&\quad=\lim_{\varepsilon_1\to0}\int_{|u|>\varepsilon_1}\int_{2^l\le|v|<2^{l+1}}\frac{\Omega(u',v')}{|u|^m|v|^n}f(\bar{x}-\Phi(u),\bar{y}-\Psi(v))\mathrm{d}u\mathrm{d}v.\end{aligned}\tag{4.2}$$

Thus, by (4.1) we have

$$\begin{aligned}S_1(f)(\bar{x},\bar{y})&=\sup_{d\in\mathbb{Z}}\left|\int_{2^d\le|u|<2^{d+1}}\mathrm{p.v.}\int_{\mathbb{R}^{n+1}}\frac{\Omega(u',v')}{|u|^m|v|^n}f(\bar{x}-\Phi(u),\bar{y}-\Psi(v))\mathrm{d}u\mathrm{d}v\right|\\&\le C\int_{S^{m-1}}\sup_{d\in\mathbb{Z}}\left|\int_{2^d}^{2^{d+1}}T_{\psi,u'}(f(x-su',x_{m+1}-\phi(s),\cdot))(\bar{y})\frac{\mathrm{d}s}{s}\right|\mathrm{d}\sigma(u'),\end{aligned}$$

where

$$T_{\psi,u'}(g)(\bar{y})=\mathrm{p.v.}\int_{\mathbb{R}^{n+1}}\frac{\Omega(u',v')}{|v|^n}g(\bar{y}-\Psi(v))\mathrm{d}v.$$

Hence,

$$S_1(f)(\bar{x},\bar{y})\le\int_{S^{m+1}}M_{\phi,u'}(T_{\psi,u'}(f)(\cdot,\bar{y}))(\bar{x})\mathrm{d}\sigma(u'),$$

where

$$M_{\phi,u'}(h)(\bar{x})=\sup_{d\in\mathbb{Z}}\int_{2^d}^{2^{d+1}}|h(x-su',x_{m+1}-\phi(s))|\frac{\mathrm{d}s}{s}.$$

Then by Minkowski's inequality, Theorem 1 in [20] and Theorems 1-2 in [15] together with [17, p.558,Corollary 5.3] and [18, p.477, Proposition 1], we get that for $\alpha>1$ and $2\alpha/(2\alpha-1)<p<2\alpha$,

$$\|S_1(f)\|_{L^p(\mathbb{R}^{m+1}\times\mathbb{R}^{n+1})}\le C\|f\|_{L^p(\mathbb{R}^{m+1}\times\mathbb{R}^{n+1})},\quad f\in\mathscr{T}_0^\infty(\mathbb{R}^{m+1}\times\mathbb{R}^{n+1}).$$

Note that $\mathscr{T}_0^\infty(\mathbb{R}^{m+1}\times\mathbb{R}^{n+1})$ is dense in $L^p(\mathbb{R}^{m+1}\times\mathbb{R}^{n+1})$, we obtain

$$\|S_1(f)\|_{L^p(\mathbb{R}^{m+1}\times\mathbb{R}^{n+1})}\le C\|f\|_{L^p(\mathbb{R}^{m+1}\times\mathbb{R}^{n+1})},\quad f\in L^p(\mathbb{R}^{m+1}\times\mathbb{R}^{n+1}).$$

Similarly, by (4.2) we have

$$\|S_2(f)\|_{L^p(\mathbb{R}^{m+1}\times\mathbb{R}^{n+1})}\le C\|f\|_{L^p(\mathbb{R}^{m+1}\times\mathbb{R}^{n+1})},\quad f\in L^p(\mathbb{R}^{m+1}\times\mathbb{R}^{n+1}),$$

which completes the proof of Lemma 4.1.

Lemma 4.2. *Let ϕ, ψ and Ω be as in Theorem* 2. *Then for $\alpha>3/2$ and $1+1/(2\alpha-2)<p<2\alpha-1$,*

$$\|\tau_*(f)\|_{L^p(\mathbb{R}^{m+1}\times\mathbb{R}^{n+1})}\le C\|f\|_{L^p(\mathbb{R}^{m+1}\times\mathbb{R}^{n+1})}.$$

By Lemma 2.5, Lemma 4.1 and the similar arguments as those in proving Theorem 22 of [6], we can easily prove Lemma 4.2. Here we omit the details.

Proof of Theorem 1.2. For any $\varepsilon_1 > 0$, $\varepsilon_2 > 0$, there exist $j, k \in \mathbb{Z}$ such that $2^j \leq \varepsilon_1 < 2^{j+1}$, $2^k \leq \varepsilon_2 < 2^{k+1}$. Then we have

$$\begin{aligned} T^*_{\phi,\psi}(f)(\bar{x},\bar{y}) &= \sup_{\varepsilon_1,\varepsilon_2>0}\left|\int\int_{|u|>\varepsilon_1,|v|>\varepsilon_2} K(u,v)f(\bar{x}-\Phi(u),\bar{y}-\Psi(v)\mathrm{d}u\mathrm{d}v\right| \\ &\leq \sup_{j,k\in\mathbb{Z}}\int_{2^j\leq|u|<2^{j+1}}\int_{2^k\leq|v|<2^{k+1}}\frac{|\Omega(u',v')|}{|u|^m|v|^n}|f(\bar{x}-\Phi(u),\bar{y}-\Psi(v))|\mathrm{d}u\mathrm{d}v \\ &\quad + \sup_{j,k\in\mathbb{Z}}\int_{2^k\leq|v|<2^{k+1}}\frac{1}{|v|^n}\left|\int_{|u|\geq 2^{j+1}}\frac{\Omega(u',v')}{|u|^m}f(\bar{x}-\Phi(u),\bar{y}-\Psi(v))\mathrm{d}u\right|\mathrm{d}v \\ &\quad + \sup_{j,k\in\mathbb{Z}}\int_{2^j\leq|u|<2^{j+1}}\frac{1}{|u|^m}\left|\int_{|v|\geq 2^{k+1}}\frac{\Omega(u',v')}{|v|^n}f(\bar{x}-\Phi(u),\bar{y}-\Psi(v))\mathrm{d}v\right|\mathrm{d}u \\ &\quad + \sup_{j,k\in\mathbb{Z}}\left|\sum_{d=j}^{\infty}\sum_{l=k}^{\infty}\sigma_{d,l}*f(\bar{x},\bar{y})\right| \\ &:= \sigma^*(|f|)(\bar{x},\bar{y}) + T_1(f)(\bar{x},\bar{y}) + T_2(f)(\bar{x},\bar{y}) + \tau_*(f)(\bar{x},\bar{y}), \end{aligned}$$

where $\Phi(u) = (u, \phi(|u|))$, $\Psi(v) = (v, \psi(|v|))$, $\bar{x} = (x, x_{m+1})$, $\bar{y} = (y, y_{n+1})$.

Thus, by Lemma 2.2 and Lemma 4.2, we need only to estimate $\|T_1(f)\|_p$ and $\|T_2(f)\|_p$. Notice that

$$\begin{aligned} T_1(f)(\bar{x},\bar{y}) &= \sup_{j,k\in\mathbb{Z}}\int_{2^k\leq|v|<2^{k+1}}\frac{1}{|v|^n}\left|\int_{|u|\geq 2^{j+1}}\frac{\Omega(u',v')}{|u|^m}f(\bar{x}-\Phi(u),\bar{y}-\Psi(v))\mathrm{d}u\right|\mathrm{d}v \\ &\leq C\int_{S^{n-1}} M_{v'}(T^*_{v'}(f)(\bar{x},\cdot))(\bar{y})\mathrm{d}\sigma(v'), \end{aligned}$$

where

$$T^*_{\phi,v'}(g)(\bar{x}) = \sup_{j\in\mathbb{Z}}\left|\int_{|u|>2^{j+1}}\frac{\Omega(u',v')}{|u|^m}g(\bar{x}-\Phi(u))\mathrm{d}u\right|,$$

and

$$M_{\psi,v'}(h)(\bar{y}) = \sup_{k\in\mathbb{Z}}\int_{2^k}^{2^{k+1}}|h(y-tv', y_{n+1}-\psi(t))|\frac{\mathrm{d}t}{t}.$$

Then, by Minkowski's inequality, it follows from Theorem 2 in [20] and Theorem 3 in [15] together with [17, p.558, Corollary 5.3] and [18, p.477, Proposition 1] that for $\alpha > 3/2$ and $p \in (1+1/(2\alpha-2), 2\alpha-1)$,

$$\|T_1(f)\|_{L^p(\mathbb{R}^{m+1}\times\mathbb{R}^{n+1})} \leq C\|f\|_{L^p(\mathbb{R}^{m+1}\times\mathbb{R}^{n+1})}.$$

Similarly, for $\alpha > 3/2$ and $1+1/(2\alpha-2) < p < 2\alpha-1$, we have

$$\|T_2(f)\|_{L^p(\mathbb{R}^{m+1}\times\mathbb{R}^{n+1})} \leq C\|f\|_{L^p(\mathbb{R}^{m+1}\times\mathbb{R}^{n+1})}.$$

This completes the proof of Theorem 1.2.

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