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# TIME DECAY FOR SCHRÖDINGER EQUATION WITH ROUGH POTENTIALS

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**Abstract.** We obtain certain time decay and regularity estimates for 3D Schrödinger equation with a potential in the Kato class by using Besov spaces associated with Schrödinger operators.

**Key words:** *functional calculus,* Schrödinger *operator,* Littlewood-Paley *theory*

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## 1 Introduction

The Schrödinger equation $iu_t = -\Delta u$ describes the waves of a free particle in a non-relativistic setting. It is physically important to consider a perturbed dispersive system in the presence of interaction between fields.

Let $H = -\Delta + V$, where $\Delta$ is the Laplacian and $V$ is a real-valued function on $\mathbb{R}^n$. In this note we are concerned with the time decay of Schrödinger equation with a potential

$$\begin{aligned} &iu_t = Hu, \\ &u(x,0) = u_0, \end{aligned}$$

where the solution is given by $u(x,t) = e^{-itH}u_0$. For simple exposition we consider the three dimensional case for $V$ in the Kato class [9, 4]. Recall that $V$ is said to be in the *Kato class* $K_n$, $n \geq 3$ provided

$$\lim_{\delta \to 0+} \sup_{x \in \mathbb{R}^n} \int_{|x-y|<\delta} \frac{|V(y)|}{|x-y|^{n-2}} \mathrm{d}y = 0.$$

Throughout this article we assume that $V = V_+ - V_-$, $V_\pm \geq 0$ so that $V_+ \in K_{n,loc}$ and $V_- \in K_n$, where $V \in K_{n,loc}$ if and only if $V\chi_B \in K_n$ for any characteristic function $\chi_B$ of the balls $B$ centered at 0 in $\mathbb{R}^n$.

We seek to find minimal smoothness condition on the initial data $u_0 = f$ so that $u(x,t)$ has certain global time decay and regularity estimates. The idea is to combine the results of Jensen-Nakamura and Rodnianski-Schlag [4, 7] for short and long time decay by using Besov space method.

In [1, 4, 3, 6, 13] several authors introduced and studied the Besov spaces and Triebel-Lizorkin spaces associated with $H$. Let $\{\varphi_j\}_{j=0}^{\infty} \subset C_0^{\infty}(\mathbb{R})$ be a dyadic system satisfying

(i) $\operatorname{supp}\varphi_0 \subset \{x : |x| \le 1\}$, $\operatorname{supp}\varphi_j \subset \{x : 2^{j-2} \le |x| \le 2^j\},\ j \ge 1$,
(ii) $|\varphi_j^{(k)}(x)| \le c_k 2^{-kj}, \qquad \forall k \ge 0, j \ge 0$,
(iii) $\sum_{j=0}^{\infty} |\varphi_j(x)| = 1, \quad \forall x$.

Let $\alpha \in \mathbb{R}$, $1 \le p \le \infty, 1 \le q \le \infty$. The (inhomogeneous) *Besov space associated with $H$*, denoted by $B_p^{\alpha,q}(H)$, is defined to be the completion of $\mathscr{S}(\mathbb{R}^n)$, the Schwartz class, with respect to the norm

$$\|f\|_{B_p^{\alpha,q}(H)} = \Big(\sum_{j=0}^{\infty} 2^{j\alpha q}\|\varphi_j(H)f\|_{L^p}^q\Big)^{1/q}.$$

Similarly, the (inhomogeneous) Triebel-Lizorkin space associated with $H$, denoted by $F_p^{\alpha,q}(H)$, $\alpha \in \mathbb{R}$, $1 \le p < \infty, 1 \le q \le \infty$ is defined by the norm

$$\|f\|_{F_p^{\alpha,q}(H)} = \Big\|\Big(\sum_{j=0}^{\infty} 2^{j\alpha q}|\varphi_j(H)f|^q\Big)^{1/q}\Big\|_{L^p}.$$

The main result is the following theorem. Let $\|V\|_K$ denote the Kato norm

$$\|V\|_K := \sup_{x\in\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|V(y)|}{|x-y|}\mathrm{d}y.$$

Let $\beta := \beta(p) = n|\frac{1}{p} - \frac{1}{2}|$ be the critical exponent.

**Theorem 1.1.** *Let $1 \le p \le 2$. Suppose $V \in K_n$, $n = 3$ so that $\|V\|_K < 4\pi$ and*

$$\int_{\mathbb{R}^6}\frac{|V(x)|\,|V(y)|}{|x-y|^2}\mathrm{d}x\mathrm{d}y < (4\pi)^2. \tag{1}$$

*The following statements hold: a) If $0 < t \le 1$, then*

$$\|e^{-itH}f\|_{p'} \lesssim \|f\|_{p'} + t^{\beta}\|f\|_{B_{p'}^{\beta,1}(H)}. \tag{2}$$

*b) If in addition, $|\partial_x^{\alpha}V(x)| \le c_\alpha$, $|\alpha| \le 2n$, $n = 3$, then for all $t > 0$*

$$\|e^{-itH}f\|_{L^{p'}} \lesssim \langle t\rangle^{-n(\frac{1}{p}-\frac{1}{2})}\|f\|_{B_p^{2\beta,1}(H)}, \tag{3}$$

*where $p' = p/(p-1)$ is the conjugate of $p$ and $\langle t\rangle = (1+t^2)^{1/2}$.*

*Remark* 1.2. The short time estimate in (2) is an improvement upon[4] since we only demand smoothness order being $\beta$ rather than $2\beta$.

It is well known that if $V$ satisfies (1), then $\sigma(H) = \sigma(H_{ac}) = [0,\infty)$. Note that by Hardy-Littlewood-Sobolev inequality, $V \in L^{3/2}$ implies the finiteness of the L.H.S of (1). Moreover, $V \in L^{3/2+} \cap L^{3/2-}$ implies $\|V\|_K < \infty$ [3, Lemma 4.3]. In particular, if $\|V\|_{L^{3/2+}\cap L^{3/2-}}$ is sufficiently small, then the conditions of Theorem 1.1 (a) are satisfied.

The proof of the main theorem is a careful modification of that of the one dimensional result for a special potential in [6]. For short time we obtain (2) by modifying the proof of [4, Theorem

4.6]. The long time estimates simply follows from the $L^p \to L^{p'}$ estimates for $e^{-itH}$, $1 \le p \le 2$, a result of [7, Theorem 2.6], and the embedding $B_p^{\varepsilon,q}(H) \hookrightarrow L^p$, $\varepsilon > 0$, $1 \le p, q \le \infty$.

Note that from the definitions of $B(H)$ and $F(H)$ spaces we have

$$B_p^{\alpha,\min(p,q)}(H) \hookrightarrow F_p^{\alpha,q}(H) \hookrightarrow B_p^{\alpha,\max(p,q)}(H) \tag{4}$$

for $1 \le p < \infty$, $1 \le q \le \infty$, where $\hookrightarrow$ means continuous embedding.

## 2 Proof of Theorem 1.1

The following lemma is proved in [4, Theorem 2, Remark 2.2].

**Lemma 2.1** [4]**.** *Let* $1 \le p \le \infty$. *Suppose* $V \in K_n$, $n = 3$ *and* $\phi \in C_0^\infty(\mathbb{R})$. *Then there exists a constant* $c > 0$ *independent of* $\theta \in (0,1]$ *so that*

$$\|\phi(\theta H)e^{-it\theta H} f\|_p \le c\langle t\rangle^\beta \|f\|_p .$$

*Remark* 2.2. We can also give a simple proof of this lemma based on the fact that the heat kernel of $H$ satisfies an upper Gaussian bound in short time. The interested reader is referred to [13] and [4, 9].

The long time decay has been studied quite extensively under a variety of conditions on $V$ [5, 7, 8, 11, 12]. The following $L^p \to L^{p'}$ estimates follow via interpolation between the $L^2$ conservation and the $L^1 \to L^\infty$ estimate for $e^{-itH}$ that was proved in [7, Theorem 2.6].

**Lemma 2.3.** *Let* $1 \le p \le 2$. *Suppose* $\|V\|_K < 4\pi$ *and*

$$\int_{\mathbb{R}^6} \frac{|V(x)|\,|V(y)|}{|x-y|^2} \mathrm{d}x\mathrm{d}y < (4\pi)^2.$$

*Then* $\|e^{-itH} f\|_{L^{p'}} \lesssim |t|^{-n(\frac{1}{p}-\frac{1}{2})}\|f\|_{L^p}$.

*Proof of Theorem* 1.1. (a) Let $0 < t \le 1$. Let $\{\varphi_j\}_{j=0}^\infty$ be a smooth dyadic system as given in Section 1. For $f \in \mathscr{S}$ we write

$$e^{-itH} f = \sum_{2^j t \le 1} \varphi_j(H)e^{-itH} f + \sum_{2^j t > 1} \varphi_j(H)e^{-itH} f . \tag{5}$$

According to Lemma 2.1, if $j \ge j_t := [-\log_2 t] + 1$,

$$\|\varphi_j(H)e^{-itH} f\|_{p'} \le c\langle t2^j\rangle^\beta \|\varphi_j(H) f\|_{p'}$$

where we noted that $\varphi_j(H) = \psi_j(H)\varphi_j(H)$, $\psi_j = \psi(2^{-j}x)$ if taking $\psi \in C_0^\infty$ so that $\psi(x) \equiv 1$ on $[-1, -\frac{1}{4}] \cup [\frac{1}{4}, 1]$. It follows that

$$\sum_{2^j t > 1} \|\varphi_j(H)e^{-itH} f\|_{p'} \le c t^\beta \sum_{2^j t > 1} 2^{j\beta} \|\varphi_j(H) f\|_{p'} .$$

For the first term in the R.H.S. of (5), similarly we have by applying Lemma 2.1 again,

$$\|\sum_{2^j t \le 1} \varphi_j(H)e^{-itH} f\|_{p'} \le c\langle t2^{j_t}\rangle^\beta \|\eta(2^{-j}H) f\|_{p'} \le c\|f\|_{p'}$$

where we take $\eta \in C_0^\infty$ with $\eta(x) \equiv 1$ on $[-1,1]$ so that

$$\eta(2^{-j_t}H) \sum_{2^j t \le 1} \varphi_j(H) = \sum_{2^j t \le 1} \varphi_j(H).$$

Therefore we obtain that if $0 < t \le 1$,

$$\|e^{-itH} f\|_{p'} \lesssim \|f\|_{p'} + t^\beta \|f\|_{B^{\beta,1}_{p'}(H)},$$

which proves part (a).

(b) Inequality (3) holds for $t > 1$ in virtue of Lemma 2.3 and the remarks below Theorem 1.1. For $0 < t \le 1$, (3) follows from the Besov embedding $B^{2\beta,1}_p(H) \hookrightarrow B^{\beta,1}_{p'}(H)$, which is valid because of the condition $|\partial_x^\alpha V(x)| \le c_\alpha$, $|\alpha| \le 2n$; cf. e.g. [10, 13].

*Remark* 2.4. It seems from the proof that the smoothness order $2\beta$ in (3) is optimal for the initial data $f$.

*Remark* 2.5. If working a little harder, we can show that

$$\|e^{-itH} f\|_{L^{p'}} \lesssim \langle t\rangle^{-n(\frac{1}{p}-\frac{1}{2})} \|f\|_{B^{2\beta,2}_p(H)}, \tag{6}$$

if assuming the upper Gaussian bound for the gradient of heat kernel of $H$ in short time, in addition to the conditions in Theorem 1.1 (a). The proof of (6) is based on the embedding $B^{0,2}_{p'}(H) \hookrightarrow F^{0,2}_{p'}(H) = L^{p'}$, $p' \ge 2$ which follows from a deeper result by applying the gradient estimates for $e^{-tH}$; see [13] and [2].

**Corollary 2.6.** *Let* $1 \le p \le 2$, $\alpha \in \mathbb{R}$ *and* $\beta = \beta(p)$. *Suppose V satisfies the same conditions as in Theorem* 1.1 *(b). The following estimates hold*

*a) If* $1 \le q \le \infty$, *then*

$$\|e^{-itH} f\|_{B^{\alpha,q}_p(H)} \lesssim \langle t\rangle^{-n(\frac{1}{p}-\frac{1}{2})} \|f\|_{B^{\alpha+2\beta,q}_p(H)}. \tag{7}$$

*b) If* $1 \le q \le p$, *then*

$$\|e^{-itH} f\|_{F^{\alpha,q}_p(H)} \lesssim \langle t\rangle^{-n(\frac{1}{p}-\frac{1}{2})} \|f\|_{B^{\alpha+2\beta,q}_p(H)}. \tag{8}$$

*Proof.* Substituting $\varphi_j(H) f$ for $f$ in (3) we obtain

$$\begin{aligned}\|\varphi_j(H) e^{-itH} f\|_{L^{p'}} &\lesssim \langle t\rangle^{-n(\frac{1}{p}-\frac{1}{2})} \|\varphi_j(H) f\|_{B^{2\beta,1}_p(H)} \\ &\approx \langle t\rangle^{-n(\frac{1}{p}-\frac{1}{2})} 2^{2\beta j} \|\varphi_j(H) f\|_{L^p}\end{aligned}$$

where we used $\|\varphi_j(H) g\|_p \le c\|g\|_p$ by applying Lemma 2.1 with $\theta = 2^{-j}$ and $t = 0$. Now multiplying $2^{j\alpha}$ and taking $\ell^q$ norms in the above inequality gives (7). The estimate in (8) follows from the embedding $B^{\alpha,q}_p(H) \hookrightarrow F^{\alpha,q}_p(H)$ if $q \le p$, according to (4).

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