Analysis in Theory and Applications
Volume 23 Number 4 (2007), 363–374
DOI10.1007/s10496-007-0363-7

SCORE SETS IN ORIENTED 3-PARTITE GRAPHS

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Received Dec. 26, 2006, Revised Oct. 6, 2007

Abstract. Let $D(U,V,W)$ be an oriented 3-partite graph with $|U|=p, |V|=q$ and $|W|=r$. For any vertex x in $D(U,V,W)$, let d_x^+ and d_x^- be the outdegree and indegree of x respectively. Define a_{u_i} (or simply a_i) $= q+r+d_{u_i}^+ - d_{u_i}^-, b_{v_j}$ (or simply b_j) $= p+r+d^+v_j - d_{v_j}^-$ and c_{w_k} (or simply c_k) $= p+q+d_{w_k}^+ - d_{w_k}^-$ as the scores of u_i in U, v_j in V and w_k in W respectively. The set A of distinct scores of the vertices of $D(U,V,W)$ is called its score set. In this paper, we prove that if a_1 is a non-negative integer, $a_i (2 \le i \le n-1)$ are even positive integers and a_n is any positive integer, then for $n \ge 3$, there exists an oriented 3-partite graph with the score set $A = \left\{a_1, \sum_{i=1}^{2} a_i, \cdots, \sum_{i=1}^{n} a_i\right\}$, except when $A = \{0,2,3\}$. Some more results for score sets in oriented 3-partite graphs are obtained.

Key words: *oriented graph, oriented 3-partite graph, tournament score set*

AMS (2000) subject classification: 05C20

1 Introduction

An oriented graph is a digraph with no symmetric pairs of directed arcs and without loops. Let D be an oriented graph with the vertex set $V = \{v_1, v_2, \cdots, v_p\}$, and let d_v^+ and d_v^- denote the outdegree and indegree of the vertex v respectively. Avery [1] defined a_{v_i} (or simply a_i) $= p-1+d_{v_i}^+ - d_{v_i}^-$, the score of v_i, so $0 \le a_{v_i} \le 2p-2$. The sequence $[a_1, a_2, \cdots, a_p]$ in non-decreasing order is called the score sequence of D.

Avery obtained the following criterion for score sequences in oriented graphs.

Theorem 1.1 [1]**.** *A non-decreasing sequence of non-negative integers* $[a_1, a_2, \cdots, a_p]$ *is the score sequence of an oriented graph if and only if*

$$\sum_{i=1}^{k} a_i \ge k(k-1), \quad \textit{for} \quad 1 \le k \le p,$$

with equality when $k = p$.

The set A of distinct scores of the vertices of an oriented graph D is called its score set. Pirzada and Naikoo[4] obtained the following results.

Theorem 1.2 [4]**.** *Let* $A = \{a, ad, ad^2, \cdots, ad^n\}$*, where* a *and* d *are positive integers with* $d > 1$*. Then there exists an oriented graph with the score set A, except for* $a=1, d=2, n>0$ *and for* $a=1, d=3, n>0$*.*

Theorem 1.3 [4]**.** *If* $a_1, a_2, \cdots, a_n$ *are non-negative integers with* $a_1 < a_2 < \cdots < a_n$*. Then there exists an oriented graph with the score set* $A = \{a_1', a_2', \cdots, a_n'\}$*, where*

$$a_i' = \begin{cases} a_{i-1} + a_i + 1, & \textit{for} \quad i > 1, \\ a_i, & \textit{for} \quad i = 1. \end{cases}$$

Various results regarding score sets in complete oriented graphs (tournaments) can be found in [2, 5, 8, 9, 10].

An oriented bipartite graph is the result of assigning a direction to each edge of a simple bipartite graph. Suppose $U = \{u_1, u_2, \cdots, u_p\}$ and $V = \{v_1, v_2, \cdots, v_q\}$ be the parts of an oriented bipartite graph $D(U,V)$. For any vertex x in $D(U,V)$, let d_x^+ and d_x^- be the outdegree and indegree of x respectively. Define a_{u_i} (or simply a_i) $= q + d_{u_i}^+ - d_{u_i}^-$ and b_{v_j} (or simply b_j) $= p + d_{v_j}^+ - d_{v_j}^-$ as the scores of u_i in U and v_j in V respectively. Clearly, $0 \le a_{u_i} \le 2q$ and $0 \le b_{v_j} \le 2p$. The sequences $[a_1, a_2, \cdots, a_p]$ and $[b_1, b_2, \cdots, b_q]$ in non-decreasing order are called the score sequences of $D(U,V)$.

The following result due to Pirzada, Merajuddin and Yin[6] is the bipartite version of Theorem 1.1.

Theorem 1.4 [6]**.** *Two non-decreasing sequences* $[a_1, a_2, \cdots, a_p]$ *and* $[b_1, b_2, \cdots, b_q]$ *of non-negative integers are the score sequences of some oriented bipartite graph if and only if*

$$\sum_{i=1}^{l} a_i + \sum_{j=1}^{m} b_j \ge 2lm, \quad 1 \le l \le p, \quad 1 \le m \le q,$$

with equality when $l = p$ *and* $m = q$*.*

The set A of distinct scores of the vertices of an oriented bipartite graph $D(U,V)$ is called its score set. In [7], Pirzada, Naikoo and Chishti proved that every set A of positive integers is the score set of an oriented bipartite graph when $|A| = 1,2,3$ or when A is a geometric or arithmetic progression.

An oriented 3-partite graph is the result of assigning a direction to each edge of a simple 3-partite graph. Suppose $U = \{u_1, u_2, \cdots, u_p\}, V = \{v_1, v_2, \cdots, v_q\}$ and $W = \{w_1, w_2, \ldots, w_r\}$ be the parts of an oriented 3-partite graph $D(U,V,W)$. For any vertex x in $D(U,V,W)$, let d_x^+ and d_x^- be the outdegree and indegree of x respectively. Define a_{u_i} (or simply a_i) $= q + r + d_{u_i}^+ - d_{u_i}^-, b_{v_j}$ (or simply b_j) $= p + r + d_{v_j}^+ - d_{v_j}^-$ and c_{w_k} (or simply c_k) $= p + q + d_{w_k}^+ - d_{w_k}^-$ as the scores of u_i in U, v_j in V and w_k in W respectively. Clearly, $0 \le a_{u_i} \le 2(q+r), 0 \le b_{v_j} \le 2(p+r)$ and $0 \le c_{w_k} \le 2(p+q)$. The sequences $[a_1, a_2, \cdots, a_p], [b_1, b_2, \cdots, b_q]$ and $[c_1, c_2, \cdots, c_r]$ in non-decreasing order are called the score sequences of $D(U,V,W)$.

The next result is the 3-partite version of Theorem 1.1 given by Pirzada, and Merajuddin[3].

Theorem 1.5 [3]**.** *Three non-decreasing sequences* $[a_1, a_2, \cdots, a_p]$*,* $[b_1, b_2, \cdots, b_q]$ *and* $[c_1, c_2, \cdots, c_r]$ *of non-negative integers are the score sequences of some oriented 3-partite graph if and only if*

$$\sum_{i=1}^{l} a_i + \sum_{j=1}^{m} b_j + \sum_{k=1}^{n} c_k \ge 2(lm + mn + nl), \quad 1 \le l \le p, 1 \le m \le q, \quad 1 \le n \le r,$$

with equality when $l = p, m = q$ *and* $n = r$*.*

The set A of distinct scores of the vertices of an oriented 3-partite graph $D(U,V,W)$ is called its score set.

For any nonempty vertex sets X and $Y, X \rightarrow Y$ means that each vertex of X dominates every vertex of Y. Also for any two vertices x and y, $x \rightarrow y$ means that there is an arc from x to y, and $x \uparrow y$ or $y \uparrow x$ means that neither $x \rightarrow y$ nor $y \rightarrow x$.

2 Results

We give the following results.

Theorem 2.1. *Every singleton set of positive integer, except* $\{1\}$, *is the score set of some oriented 3-partite graph.*

Proof. Let $A = \{a\}$, where $a > 1$ is a positive integer.

There are the following three cases:

(i) $a = 2r$ where $r \geq 1$, (ii) $a = 4r-1$ where $r \geq 1$, (iii) $a = 4r-3$ where $r \geq 2$.

Now we give the proofs.

(i) $a = 2r$ where $r \geq 1$.

Consider an oriented 3-partite graph $D(U,V,W)$ with $|U| = |V| = |W| = r$, and $U \rightarrow V; V \rightarrow W$, and $W \rightarrow U$. Then the scores of the vertices of $D(U,V,W)$ are

$$a_u = a_v = a_w = |V| + |W| + r - r = r + r = 2r = a$$

for all $u \in U, v \in V, w \in W$.

Therefore, the score set of $D(U,V,W)$ is $A = \{a\}$.

(ii) $a = 4r-1$ where $r \geq 1$.

Consider an oriented 3-partite graph $D(U,V,W)$ with $U = \{u_1, u_2, \cdots, u_{6r^2-2r}\}, V = \{v_1, v_2, \cdots, v_r\}$ and $W = \{v_{r+1}, v_{r+2}, \cdots, v_{2r}\}$ in which

$$u_{i+j} \rightarrow v_1, v_2, \cdots, v_{i-1}, v_{i+1}, \cdots, v_{2r} \quad \text{for all} \quad i, j,$$

where $1 \leq i \leq 2r, j \in \{0, 2r, 4r, \cdots, 6r^2 - 4r\} = S$ so that $|S| = 3r-1$. Then the scores of the vertices of $D(U,V,W)$ are

$$\begin{aligned} a_{u_{i+j}} &= |V| + |W| + \sum_{\substack{g=1 \\ g \neq i}}^{2r} |v_g| - 0 = r + r_g + \sum_{\substack{g=1 \\ g \neq i}}^{2r} 1 \\ &= 2r + (2r-1) = 4r - 1 = a \end{aligned}$$

for all $u_{i+j} \in U$, where $1 \leq i \leq 2r, j \in S$ and

$$\begin{aligned} a_{v_g} &= |U| + |W| + 0 - \sum_{\substack{i=1 \\ i \neq g}}^{2r} \sum_{j \in S} |u_{i+j}| \\ &= 6r^2 - 2r + r - \sum_{\substack{i=1 \\ i \neq g}}^{2r} \sum_{j \in S} 1 = 6r^2 - r - \sum_{\substack{i=1 \\ i \neq g}}^{2r} (3r-1) \\ &= 6r^2 - r - (3r-1) \sum_{\substack{i=1 \\ i \neq g}}^{2r} 1 = 6r^2 - r - (3r-1)(2r-1) \\ &= 4r - 1 = a \end{aligned}$$

for all $v_g \in V \cup W$, where $1 \leq g \leq 2r$.

Therefore, the score set of $D(U,V,W)$ is $A = \{a\}$.

(iii) $a = 4r-3$ where $r \geq 2$.

Consider an oriented 3-partite graph $D(U,V,W)$ with $U = \{u_1, u_2, \cdots, u_{2r^2-2r}\}$, $V = \{v_1, v_2, \cdots, v_r\}$ and $W = \{v_{r+1}, v_{r+2}, \cdots, v_{2r}\}$ in which

$$u_{i+j} \rightarrow v_1, v_2, \cdots, v_{i-1}, v_{i+3}, \cdots, v_{2r}, \qquad i, \quad j,$$

where $1 \leq i \leq 2r, j \in \{0, 2r, 4r, \cdots, 2r^2-4r\} = T$ so that $|T| = r-1$. Then the scores of the vertices of $D(U,V,W)$ are

$$\begin{aligned} a_{u_{i+j}} &= |V| + |W| + \sum_{\substack{g=1 \\ g \neq i, i+1, i+2}}^{2r} |v_g| - 0 \\ &= r + r + \left(\sum_{g=1}^{2r} |v_g|\right) - (|v_i| + v_{i+1}| + |v_{i+2}|) = 2r + \left(\sum_{g=1}^{2r} 1\right) - 3 \\ &= 2r + 2r - 3a \end{aligned}$$

for all $u_{i+j} \in U$, where $1 \leq i \leq 2r, j \in T$. Note that v_{2r+1} and v_{2r+2} are treated v_1 and v_2 respectively, and

$$\begin{aligned} a_{v_g} &= |U| + |W| + 0 - \sum_{\substack{i=1 \\ i \neq g-2, g-1, g}}^{2r} \sum_{j \in T} |u_{i+j}| \\ &= 2r^2 - 2r + r - \sum_{\substack{i=1 \\ i \neq g-2, g-1, g}}^{2r} \sum_{j \in T} 1 = 2r^2 - r - \sum_{\substack{i=1 \\ i \neq g-2, g-1, g}}^{2r} (r-1) \\ &= 2r^2 - r - (r-1) \sum_{\substack{i=1 \\ i \neq g-2, g-1, g}}^{2r} 1 \\ &= 2r^2 - r - (r-1)(\sum_{i=1}^{2r} 1 - (|u_{g-2}| + |u_{g-1}| + |v_g|)) \\ &= 2r^2 - r - (r-1)(2r-3) = 4r - 3 = a \end{aligned}$$

for all $v_g \in V \cup W$, where $1 \leq g \leq 2r$, and note that u_0 and u_{-1} are treated as u_{2r^2-2r} and u_{2r^2-2r-1} respectively. Therefore, the score set of $D(U,V,W)$ is $A = \{a\}$.

Theorem 2.2. *Let $A = \{a_1, a_2\}$, where $a_1 \geq 0$ is an even integer and a_2 is any positive integer such that $a_1 < a_2$. Then, there exists an oriented 3-partite graph with the score set A except for $a_1 = 0, a_2 = 1, 2$.*

Proof. First assume that $a_1 = 0$ and $a_2 > 2$ so that $a_2 - 2 > 0$. Consider an oriented 3-partite graph $D(U,V,W)$ with $|U| = 1, |V| = |W| = a_2 - 2$, and $V \rightarrow U$ and $W \rightarrow U$. Then, the scores of the vertices of $D(U,V,W)$ are

$$\begin{aligned} a_u &= |V| + |W| + 0 - (a_2 - 2 + a_2 - 2) = a_2 - 2 + a_2 - 2 - a_2 + 2 - a_2 + 2 \\ &= 0 = a_1, \quad u \in U, \\ a_v &= |U| + |W| + 1 - 0 = 1 + a_2 - 2 + 1 = a_2, \qquad v \in V, \end{aligned}$$

and

$$a_w = |U| + |V| + 1 - 0 = 1 + a_2 - 2 + 1 = a_2, \qquad w \in W.$$

Therefore, the score set of $D(U,V,W)$ is $A = \{a_1, a_2\}$.

Now, assume $a_1 = 2r$ where $r \geq 1$. Since $a_1 < a_2$, then $a_2 - a_1 > 0$. Construct an oriented 3-partite graph $D(U,V,W)$ as follows.

Let

$$U = X_1,\ V = Y_1 \cup Y_2,\ W = Z_1,\ Y_1 \cap Y_2 = \varphi,\ |X_1| = |Y_1| = |Z_1| = r,\ |Y_2| = a_2 - a_1.$$

Let $X_1 \to Y_1; Y_1 \to Z_1$, and $Z_1 \to X_1$, so that we get the oriented 3-partite graph $D(U,V,W)$ with

$$|U| = |X_1| = r, |V| = |Y_1| + |Y_2| = r + a_2 - a_1, |W| = |Z_1| = r,$$

and the scores of vertices

$$\begin{aligned} a_{x_1} &= |V| + |W| + r - r = r + a_2 - a_1 + r = 2r + a_2 - a_1 \\ &= a_1 + a_2 - a_1 = a_2, \quad x_1 \in X_1, \\ a_{y_1} &= |U| + |W| + r - r = r + r = 2r = a_1, \quad y_1 \in Y_1, \\ a_{y_2} &= |U| + |W| + 0 - 0 = r + r + 2r = a_1, \quad y_2 \in Y_2, \end{aligned}$$

and

$$\begin{aligned} a_{z_1} &= |U| + |V| + r - r = r + r + a_2 - a_1 = 2r + a_2 - a_1 \\ &= a_1 + a_2 - a_1 = a_2, \quad z_1 \in Z_1. \end{aligned}$$

Therefore, the score set of $D(U,V,W)$ is $A = \{a_1, a_2\}$.

The following result shows that every set of three non-negative integers in arithmetic progression, except $\{0, 1, 2\}$, is a score set of some oriented 3-partite graph.

Theorem 2.3. *Let $A = \{a, a+d, a+2d\}$, where a and d are non-negative integers with $d > 0$. Then there exists an oriented 3-partite graph with the score set A, except for $a = 0, d = 1$.*

Proof. First assume that $a = 0$ and $d = 2$. Consider an oriented 3-partite graph $D(U,V,W)$ with $|U| = |V| = |W| = 1$, and $V \to U$ and $W \to U, V$. Then the scores of the vertices of $D(U,V,W)$ are

$$\begin{aligned} a_u &= |V| + |W| + 0 - 2 + 1 + 1 - 2 = 0 = a,\ \ u \in U, \\ a_v &= |U| + |W| + 1 - 1 + 1 + 1 = 2 = a + d,\ \ v \in V, \end{aligned}$$

and

$$a_w = |U| + |V| + 2 - 0 = 1 + 1 + 2 = 4 = a + 2d,\ \ w \in W.$$

Therefore, the score set of $D(U,V,W)$ is $A = \{a, a+d, a+2d\}$.

Now, assume that $a = 0$ and $d > 2$ so that $d - 2 > 0$. Consider an oriented 3-partite graph $D(U,V,W)$ with $|U| = 1, |V| = d-2, |W| = 2d-2$, and $V \to U$ and $W \to U, V$. Then the scores of the vertices of $D(U,V,W)$ are

$$\begin{aligned} a_u &= |V| + |W| + 0 - (d - 2 + 2d - 2) = d - 2 + 2d - 2 - d + 2 - 2d + 2 \\ &= 0 = a, \quad u \in U, \\ a_v &= |U| + |W| + 1 - 0 = 1 + 2d - 2 + 1 = 2d = a + 2d, \quad v \in V, \end{aligned}$$

and

$$a_w = |U| + |V| + 1 - 0 = 1 + d - 2 + 1 = d = a + d, \quad w \in W.$$

Therefore, the score set of $D(U,V,W)$ is $A = \{a, a+d, a+2d\}$.

Finally, assume that $a > 0$. Consider an oriented 3-partite graph $D(U,V,W)$ with $|U| = d, |V| = |W| = a$, and $W \rightarrow U$. Then the scores of the vertices of $D(U,V,W)$ are

$$\begin{aligned} a_u &= |V| + |W| + 0 - a = a + a - a = a, \quad u \in U, \\ a_v &= |U| + |W| + 0 - 0 = d + a = a + d, \quad v \in V, \end{aligned}$$

and

$$a_w = |U| + |V| + d - 0 = d + a + d = a + 2d, \quad w \in W.$$

Therefore, the score set of $D(U,V,W)$ is $A = \{a, a+d, a+2d\}$.

The next result shows that every set of four non-negative integers in arithmetic progression, except $\{0,1,2,3\}$, is a score set of some oriented 3-partite graph.

Theorem 2.4. *Let $A = \{a, a+d, a+2d, a+3d\}$, where a and d are non-negative integers with $d > 0$. Then there exists an oriented 3-partite graph with the score set A, except for $a = 0, d = 1$.*

Proof. First assume that $a = 0$ and $d = 2$. Construct an oriented 3-partite graph $D(U,V,W)$ as follows.

Let $U = X_1, V = Y_1 \cup Y_2, W = Z_1, Y_1 \cap Y_2 = \emptyset, |X_1| = |Y_1| = |Y_2| = 1, |Z_1| = 2$. Let $Y_1 \rightarrow X_1, Z_1; Y_2 \rightarrow X_1$, and $Z_1 \rightarrow X_1, Y_2$, so that we get the oriented 3-partite graph $D(U,V,W)$ with

$$|U| = |X_1| = 1, |V| = |Y_1| + |Y_2| = 1 + 1 = 2, |W| = |Z_1| = 2,$$

and the scores of vertices

$$\begin{aligned} a_{x_1} &= |V| + |W| + 0 - 4 = 2 + 2 - 4 = 0 = a, \quad x_1 \in X_1, \\ a_{y_1} &= |U| + |W| + 3 - 0 = 1 + 2 + 3 = 6 = a + 3d, \quad y_1 \in Y_1, \\ a_{y_2} &= |U| + |W| + 1 - 2 = 1 + 2 - 1 = 2 = a + d, \quad y_2 \in Y_2, \end{aligned}$$

and

$$a_{z_1} = |U| + |V| + 2 - 1 = 1 + 2 + 1 = 4 = a + 2d, \quad z_1 \in Z_1.$$

Therefore, the score set of $D(U,V,W)$ is $A = \{a, a+d, a+2d, a+3d\}$.

Now, assume that $a = 0$ and $d > 2$ so that $d - 2 > 0$. Construct an oriented 3-partite graph $D(U,V,W)$ as follows.

Let

$$\begin{aligned} &U = X_1,\ V = Y_1 \cup Y_2,\ W = Z_1 \cup Z_2,\ Y_1 \cap Y_2 = \emptyset,\ Z_1 \cap Z_2 = \emptyset,\ |X_1| = 1, \\ &|Y_1| = |Z_1| = d - 2,\ |Y_2| = d,\ |Z_2| = 2d. \end{aligned}$$

Let $Y_1 \rightarrow X_1; Y_2 \rightarrow X_1; Z_1 \rightarrow X_1$, and $Z_2 \rightarrow X_1, Y_2$, so that we get the oriented 3-partite graph $D(U,V,W)$ with

$$\begin{aligned} &|U| = |X_1| = 1,\ |V| = |Y_1| + |Y_2| = d - 2 + d = 2d - 2, \\ &|W| = |Z_1| + |Z_2| = d - 2 + 2d = 3d - 2, \end{aligned}$$

and the scores of vertices

$$\begin{aligned} a_{x_1} &= |V|+|W|+0-(d-2+d+2d+d-2) \\ &= 2d-2+3d-2-5d+4=0=a, \quad x_1\in X_1, \\ a_{y_1} &= |U|+|W|+1-0=1+3d-2+1=3d=a+3d, \qquad y_1\in Y_1, \\ a_{y_2} &= |U|+|W|+1-2d=1+3d-2+1-2d=d=a+d, \qquad y_2\in Y_2, \\ a_{z_1} &= |U|+|V|+1-0=1+2d-2+1=2d=a+2d, \qquad z_1\in Z_1, \end{aligned}$$

and

$$a_{z_2}=|U|+|V|+(1+d)-0=1+2d-2+1+d=3d=a+3d, \qquad z_2\in Z_2.$$

Therefore, the score set of $D(U,V,W)$ is $A=\{a,a+d,a+2d,a+3d\}$.

Finally, assume that $a>0$. Construct an oriented 3-partite graph $D(U,V,W)$ as follows. Let

$$U=X_1,\ V=Y_1\cup Y_2,\ W=Z_1,\ Y_1\cap Y_2=\emptyset,\ |X_1|=|Y_1|=d,\ |Y_2|=|Z_1|=a.$$

Let $Y_1\to X_1$, and $Z_1\to X_1$, so that we get the oriented 3-partite graph $D(U,V,W)$ with

$$|U|=|X_1|=d,\ |V|=|Y_1|+|Y_2|=d+a=a+d,\ |W|=|Z_1|=a,$$

and the scores of vertices

$$\begin{aligned} a_{x_1} &= |V|+|W|+0-(d+a)=a+d+a-d-a=a, \quad x_1\in X_1, \\ a_{y_1} &= |U|+|W|+d-0=d+a+d=a+2d, \quad y_1\in Y_1, \\ a_{y_2} &= |U|+|W|+0-0=d+a=a+d, \quad y_2\in Y_2, \end{aligned}$$

and

$$a_{z_1}=|U|+|V|+d-0=d+a+d+d=a+3d, \quad z_1\in Z_1.$$

Therefore, the score set of $D(U,V,W)$ is $A=\{a,a+d,a+2d,a+3d\}$.

Finally, we have the following main result.

Theorem 2.5. *Let a_1 be a non-negative integer, a_i $(2\le i\le n-1)$ be even positive integers and a_n be any positive integer. Then for $n\ge 3$, there exists an oriented 3-partite graph with the score set $A=\left\{a_1,\sum_{i=1}^{2}a_i,\cdots,\sum_{i=1}^{n}a_i\right\}$, except when $A=\{0,2,3\}$.*

Proof. For $2\le i\le n-1$, let $a_i=2r_i$ where $r_i\ge 1$.

First assume that $a_1=0$ and $n=3$. For $a_2=2, a_3=2$, consider an oriented 3-partite graph $D(U,V,W)$ with $|U|=|V|=|W|=1$, and $V\to U$ and $W\to U,V$. Then, the scores of the vertices of $D(U,V,W)$ are

$$\begin{aligned} a_u &= |V|+|W|+0-2=1+1-2=0=a_1, \quad u\in U, \\ a_v &= |U|+|W|+1-1=1+1=2=a_1+a_2, \quad v\in V, \end{aligned}$$

and

$$a_w=|U|+|V|+2-0=1+1+2=4=a_1+a_2+a_3, \quad w\in W.$$

Therefore, the score set of $D(U,V,W)$ is $A=\{a_1,a_1+a_2,a_1+a_2+a_3\}$.

For $a_2 \geq 2, a_3 > 2$, construct an oriented 3-partite graph $D(U,V,W)$ as follows.
Let

$$U = X_1,\ V = Y_1 \cup Y_2,\ W = Z_1,\ Y_1 \cap Y_2 = \emptyset,\ |X_1| = r_2,\ |Y_1| = 1,\ |Y_2| = a_3 - 2,\ |Z_1| = a_3.$$

Let $Y_1 \to X_1$; $Y_2 \to X_1$, and $Z_1 \to X_1,\ Y_1$, so that we get the oriented 3-partite graph $D(U,V,W)$ with

$$|U| = |X_1| = r_2,\ |V| = |Y_1| + |Y_2| = 1 + a_3 - 2 = a_3 - 1,\ |W| = |Z_1| = a_3,$$

and the scores of vertices

$$\begin{aligned} a_{x_1} &= |V| + |W| + 0 - (|Y_1| + |Y_2| + |Z_1|) = a_3 - 1 + a_3 - (1 + a_3 - 2 + a_3) = 0 = a_1, \quad x_1 \in X_1, \\ a_{y_1} &= |U| + |W| + |X_1| - |Z_1| = r_2 + a_3 + r_2 - a_3 = 2r_2 = a_1 + a_2, \qquad y_1 \in Y_1, \\ a_{y_2} &= |U| + |W| + |X_1| - 0 = r_2 + a_3 + r_2 = 2r_2 + a_3 = a_1 + a_2 + a_3, \qquad y_2 \in Y_2, \end{aligned}$$

and

$$a_{z_1} = |U| + |V| + (|X_1| + |Y_1|) - 0 = r_2 + a_3 - 1 + r_2 + 1 = 2r_2 + a_3 = a_1 + a_2 + a_3, \qquad z_1 \in Z_1.$$

Therefore, the score set of $D(U,V,W)$ is $A = \{a_1, a_1 + a_2, a_1 + a_2 + a_3\}$.

Now, let $a_1 = 0$ and $n \geq 4$. Construct an oriented 3-partite graph $D(U,V,W)$ as follows.
Let

$$\begin{aligned} U &= X \cup X_1 \cup X_2 \cup \cdots \cup X_{n-3}, \\ V &= Y, \\ W &= Z \cup Z_1 \cup Z_2 \cup \cdots \cup Z_{n-3}, \end{aligned}$$

with $X \cap X_i = \emptyset, X_i \cap X_j = \emptyset, Z \cap Z_i = \emptyset, Z_i \cap Z_j = \emptyset (i \neq j), |X| = |Z| = r_2, |Y| = r_3, |X_i| = |Z_i| = r_{i+3}$ for all i, where $1 \leq i \leq n-4, |X_{n-3}| = |Z_{n-3}| = a_n$.

Let $X_i \to Y, Z, Z_1, Z_2, \cdots, Z_i$ for all i, where $1 \leq i \leq n-4; X_{n-3} \to Y, Z, Z_1, Z_2, \cdots, Z_{n-4}; Y \to X; Z \to Y, X$, and $Z_i \to Y, X, X_1, X_2, \cdots, X_{i-1}$ for all i, where $1 \leq i \leq n-3$, so that we get the oriented 3-partite graph $D(U,V,W)$ with

$$|U| = |X| + \sum_{i=1}^{n-3} |X_i| = |Z| + \sum_{i=1}^{n-3} |Z_i| = |W| = r_2 + \sum_{i=1}^{n-4} r_{i+2} + a_n, |V| = |Y| = r_3,$$

and the scores of vertices

$$\begin{aligned} a_x &= |V| + |W| + 0 - \left(|Y| + |Z| + \sum_{i=1}^{n-4} |Z_i| + |Z_{n-3}|\right) \\ &= r_3 + r_2 + \sum_{i=1}^{n-4} r_{i+3} + a_n - \left(r_3 + r_2 + \sum_{i=1}^{n-4} r_{i+3} + a_n\right) \\ &= 0 = a_1, \quad x \in X, \qquad 1 \leq i \leq n-4, \end{aligned}$$

$$\begin{aligned}
a_{x_i} &= |V|+|W|+\left(|Y|+|Z|+\sum_{j=1}^{i}|Z_j|\right)-\left(\sum_{j=i+1}^{n-4}|_j|+|Z_{n-3}|\right)\\
&= r_3+r_2+\sum_{i=1}^{n-4}r_{i+3}+a_n+r_3+r_2+\sum_{j=1}^{i}r_{j+3}-\left(\sum_{i=1}^{n-4}r_{i+3}+a_n\right)\\
&= 2r_2+2r_3+(r_4+r_3+\cdots+r_{i+3}+r_{i+4}+\cdots+r_{n-1})\\
&\quad +(r_4+r_5+\cdots+r_{i+3})-(r_{i+4}+r_{i+5}+\cdots+r_{n-1})\\
&= 2r_2+2r_3+2r_4+2r_5+\cdots+2r_{i+3}\\
&= a_1+a_2+a_3+\cdots+a_{i+3},\quad x_i\in X_i,\\
a_{x_{n-3}} &= |V|+|W|+\left(|Y|+|Z|+\sum_{i=1}^{n-4}|Z_i|\right)-0\\
&= r_3+r_2+\sum_{i=1}^{n-4}r_{i+3}+a_n+r_3+r_2+\sum_{i=1}^{n-4}r_{i+3}\\
&= 2r_2+2r_3+2\sum_{i=1}^{n-4}r_{i+3}+a_n\\
&= a_1+a_2+a_3+\cdots+a_{n-1}+a_n,\quad x_{n-3}\in X_{n-3},\\
a_y &= |U|+|W|+|X|-\left(\sum_{i=1}^{n-4}|X_i|+|X_{n-3}|+|Z|+\sum_{i=1}^{n-4}|Z_i|+|Z_{n-3}|\right)\\
&= r_2+\sum_{i=1}^{n-4}r_{i+3}+a_n+r_2+\sum_{i=1}^{n-4}r_{i+3}+a_n+r_2\\
&\quad -\left(\sum_{i=1}^{n-4}r_{i+3}+a_n+r_2+\sum_{i=1}^{n-4}r_{i+3}+a_n\right)\\
&= 2r_2=a_1+a_2,\quad y\in Y,\\
a_z &= |U|+|V|+(|Y|+|X|)-\left(\sum_{i=1}^{n-4}|X_i|+|X_{n-3}|\right)\\
&= r_2+\sum_{i=1}^{n-4}r_{i+3}+a_n+r_3+r_3+r_2-\left(\sum_{i=1}^{n-4}r_{i+3}+a_n\right)\\
&= 2r_2+2r_3=a_1+a_2+a_3,\quad z\in Z,
\end{aligned}$$

for $1\le i\le n-4$

$$\begin{aligned}
a_{z_i} &= |U|+|V|+\left(|Y|+|X|+\sum_{j=2}^{i}|X_{j-1}|\right)-\left(\sum_{j=1}^{n-4}|X_j|+|X_{n-3}|\right)\\
&= r_2+\sum_{j=1}^{n-4}r_{i+3}+a_n+r_3+r_3+r_2+\sum_{j=2}^{i}r_{j+2}-\left(\sum_{j=1}^{n-4}r_{j+3}+a_n\right)\\
&= 2r_2+2r_3+(r_4+r_5+\cdots+r_{i+2}+r_{i+3}+\cdots+r_{n-1})\\
&\quad +(r_4+r_5+\cdots+r_{i+2})-(r_{i+3}+r_{i+4}+\cdots+r_{n-1})
\end{aligned}$$

$$\begin{aligned}
&= 2r_2+2r_3+2r_4+2r_5+\cdots+2r_{i+2}\\
&= a_1+a_2+a_3+\cdots+a_{i+2}, \quad z_i\in Z_i,
\end{aligned}$$

and

$$\begin{aligned}
a_{z_{n-3}} &= |U|+|V|+\left(|Y|+|X|+\sum_{j=1}^{n-4}|X_i|\right)-0\\
&= r_2+\sum_{i=1}^{n-4}r_{i+3}+a_n+r_3+r_3+r_2+\sum_{j=2}^{i}r_{j+2}-\left(\sum_{j=1}^{n-4}r_{j+3}+a_n\right)\\
&= 2r_2+2r_3+2r_4+\cdots+2r_{n-1}+a_n\\
&= a_1+a_2+a_3+\cdots+a_{n-1}+a_n, \quad z_{n-3}\in Z_{n-3}.
\end{aligned}$$

Therefore, the score set of $D(U,V,W)$ is $A=\left\{a_1,\sum_{i=1}^{2}a_i,\cdots,\sum_{i=1}^{n}a_i\right\}$.

Now, assume that $a_1>0$. Construct an oriented 3-partite graph $D(U,V,W)$ as follows. Let

$$\begin{aligned}
U &= X,\\
V &= Y\cup Y_1\cup Y_2\cup\cdots\cup Y_{n-3}\cup Y_{n-2},\\
W &= Z\cup Z_1\cup Z_2\cup\cdots\cup Z_{n-3},
\end{aligned}$$

with $Y\cap Y_i=\emptyset, Y_i\cap Y_j=\emptyset, Z\cap Z_i=\emptyset, Z_i\cap Z_j=\emptyset (i\neq j), |X|=a_1, |Y|=|Z|=r_2, |Y_i|=|Z_i|=r_{i+2}$ for all i, where $1\le i\le n-3, |Y_{n-2}|=a_1+a_n$.

Let $Y_1\to Z,Z_1;Y_i\to Z,Z_1,Z_2,\cdots,Z_{i-1}$ for all i, where $2\le i\le n-2; Z\to Y; Z_1\to Y$, and $Z_i\to Y,Y_1,Y_2,\cdots,Y_i$ for all i, where $2\le i\le n-3$, so that we get the oriented 3-partite graph $D(U,V,W)$ with

$$|U|=|X|=a_1, |V|=|Y|+\sum_{i=1}^{n-2}|Y_i|=r_2+\sum_{i=1}^{n-3}r_{i+2}a_1+a_n,$$

$$|W|=|Z|+\sum_{i=1}^{n-3}|Z_i|=r_2+\sum_{i=1}^{n-3}r_{i+2},$$

and the scores of vertices

$$\begin{aligned}
a_x &= |V|+|W|+0-0=r_2+\sum_{i=1}^{n-3}r_{i+2}+a_1+a_n+r_2+\sum_{i=1}^{n-3}r_{i+2}\\
&= a_1+2r_2+2r_3+\cdots+2r_{n-1}+a_n\\
&= a_1+a_2+a_3+\cdots+a_{n-1}+a_n, \qquad x\in X,\\
a_y &= |U|+|W|+0-\left(|Z|+\sum_{i=1}^{n-3}|Z_i|\right)\\
&= a_1+r_2+\sum_{i=1}^{n-3}r_{i+2}-\left(r_2+\sum_{i=1}^{n-3}r_{i+2}\right)=a_1, \qquad y\in Y,
\end{aligned}$$

$$\begin{aligned} a_{y_1} &= a_{y_2} = |U| + |W| + (|Z| + |Z_1|) - \sum_{i=2}^{n-3} |Z_i| \\ &= a_1 + r_2 + \sum_{i=1}^{n-3} r_{i+2} + r_2 + r_3 - \sum_{i=2}^{n-3} r_{i+2} = a_1 + 2r_2 + 2r_3 \\ &= a_1 + a_2 + a_3, \quad y_1 \in Y_1, y_2 \in Y_2, \end{aligned}$$

for $3 \le i \le n-3$

$$\begin{aligned} a_{y_1} &= |U| + |W| + \left(|Z| + \sum_{j=2}^{i} |Z_{j-1}| \right) - \sum_{j=i}^{n-3} |Z_j| \\ &= a_1 + r_2 + \sum_{i=1}^{n-3} r_{i+2} + r_2 + \sum_{j=2}^{i} r_{j+1} - \sum_{j=i}^{n-3} r_{j+2} \\ &= a_1 + 2r_2 + (r_3 + r_4 + \cdots + r_{i+1} + r_{i+2} + \cdots + r_{n-1}) \\ &\quad + (r_3 + r_4 + \cdots + r_{i+1}) - (r_{i+2} + r_{i+3} + \cdots + r_{n-1}) \\ &= a_1 + 2r_2 + 2r_3 + 2r_4 + \cdots + 2r_{i+1} \\ &= a_1 + a_2 + a_3 + \cdots + a_{i+1}, \quad y_i \in Y_i, \\ a_{y_{n-2}} &= |U| + |W| + \left(|Z| + \sum_{j=2}^{n-2} |Z_{j-1}| \right) - 0 \\ &= a_1 + r_2 + \sum_{i=1}^{n-3} r_{i+2} + r_2 + \sum_{j=2}^{n-2} r_{j+1} \\ &= a_1 + 2r_2 + 2r_3 + 2r_4 + \cdots + 2r_{n-1} \\ &= a_1 + a_2 + a_3 + \cdots + a_{n-1}, \quad y_{n-2} \in Y_{n-2}, \\ a_z &= a_{z_1} = |U| + |V| + |Y| - \sum_{i=1}^{n-2} |Y_i| \\ &= a_1 + r_2 + \sum_{i=1}^{n-3} r_{i+2} + a_1 + a_n + r_2 - \left(\sum_{i=1}^{n-3} r_{i+2} + a_1 + a_n \right) \\ &= a_1 + 2r_2 = a_1 + a_2, \quad z \in Z, z_1 \in Z_1, \end{aligned}$$

and for $2 \le i \le n-3$

$$\begin{aligned} a_{z_i} &= |U| + |V| + |Y| + \sum_{j=1}^{i} |Y_j| - \sum_{j=i+1}^{n-2} |Y_j| \\ &= a_1 + r_2 + \sum_{i=1}^{n-3} r_{i+2} + a_1 + a_n + r_2 \\ &\quad + \sum_{j=1}^{i} r_{j+2} - \left(\sum_{j=i+1}^{n-2} r_{j+2} + a_1 + a_n \right) \\ &= a_1 + 2r_2 + (r_3 + r_4 + \cdots + r_{i+2} + r_{i+3} + \cdots + r_{n-1}) \\ &\quad + (r_3 + r_4 + \cdots + r_{i+2}) - (r_{i+3} + r_{i+4} + \cdots + r_{n-1}) \\ &= a_1 + 2r_2 + 2r_3 + 2r_4 + \cdots + 2r_{i+2} \\ &= a_1 + a_2 + a_3 + \cdots + a_{i+2}, \quad z_i \in Z_i. \end{aligned}$$

Therefore, the score set of $D(U,V,W)$ is $A = \left\{ a_1, \sum_{i=1}^{2} a_i, \cdots, \sum_{i=1}^{n} a_i \right\}$.

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