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CONNECTION BETWEEN THE ORDER OF FRACTIONAL CALCULUS AND FRACTIONAL DIMENSIONS OF A TYPE OF FRACTAL FUNCTIONS

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Abstract. The linear relationship between fractal dimensions of a type of generalized Weierstrass functions and the order of their fractional calculus has been proved. The graphs and numerical results given here further indicate the corresponding relationship.

Key words: *generalized* Weierstrass *function*, Riemann-Liouville *fractional calculus, fractal dimension, linear, graph*

AMS (2000) subject classification: 41A17, 41A25

1 Introduction

The importance of studying continuous but nowhere differentiable functions was emphasized a long time ago by Perrin Poincaré, Falconer and Mandelbrot(see Refs. [1,3]). It is possible for a continuous function to be sufficiently irregular so that its graph is a fractal curve. Weierstrass function maybe have the most importance which is defined as

$$W(t) = \sum_{j\geq 1} \lambda^{-\alpha j} \sin(\lambda^j t), \qquad 0 < \alpha < 1,\ \lambda > 1. \tag{1.1}$$

Though $W(t)$ is continuous, it is nowhere differentiable, so we appeal to fractional calculus. The Riemann-Liouville transformation is probably very useful fractional calculus which is defined as follows:

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Definition 1[5]**.** *Let f be piecewisely continuous on $(0,\infty)$ and local integrable on $[0,\infty)$. Then for $t>0$, $Re(v)>0$, we call*

$$D^{-v}f(t)=\frac{1}{\Gamma(v)}\int_0^t (t-x)^{v-1}f(x)\mathrm{d}x$$

the Riemann-Liouville *fractional integral of f of order v . For $0<u<1$, we call*

$$D^u f(t)=D[D^{u-1}f(t)]$$

the Riemann-Liouville *fractional derivative of f of order u.*

In [6], Tatom explored the general relationship between the fractional calculus and fractals. But he didn't give out the accurate connection. So in [8], Yao has proved there exist some linear connection between the order of the fractional calculus and the fractal dimensions of the graphs of the Weierstrass function which is defined as (1.1). For both theoretical and practical importance, we consider the generalized Weierstrass function which is defined by:

$$W^*(t)=\sum_{j\geq 1}\lambda^{-\alpha j}f(\lambda^j t), \qquad 0<\alpha<1,\ \lambda>1, \tag{1.2}$$

here

$$f(t)\in S:=\{g(t)\in C^1: g(t)=g(2a+t), g(a+t)=-g(a-t),\ f(t)\not\equiv 0,\ a>0\}. \tag{1.3}$$

The generalized Weierstrass function defined as (1.2) most maybe appear in three contexts: first as repellers for certain functions and such functions can occur as invariant sets in dynamical systems; second as graphs of wavelet functions which are fundamental to wavelet analysis; third as graphs of approximation functions,where they have been used in approximation theory. More details can be found in Refs. [10-13].

Now we consider the Riemann-Liouville fractional calculus of the generalized Weierstrass function. Let

$$D^{-v}f(at)=\frac{1}{\Gamma(v)}\int_0^t (t-x)^{v-1}f(ax)\mathrm{d}x=:G_t(v,a)$$

denote the Riemann-Liouville fractional integral of $f(at)$, and let

$$D^{-v}f'(at)=\frac{1}{\Gamma(v)}\int_0^t (t-x)^{v-1}f'(ax)\mathrm{d}x=:L_t(v,a)$$

denote the Riemann-Liouville fractional integral of $f'(at)$. For $\lambda>1, 0<\alpha,v,u<1$ with $0<\alpha+v<1, 0<u<\alpha$, denote by

$$g^*(t):=D^{-v}(W^*(t))=\sum_{j\geq 1}\lambda^{-\alpha j}G_t(v,\lambda^j) \tag{1.4}$$

the fractional integral of Weierstrass type function $W^*(t)$ of order v. Denote by

$$m^*(t) := D^u(W^*(t)) = \sum_{j\geq 1} \lambda^{(1-\alpha)j} L_t(1-u,\lambda^j) \tag{1.5}$$

the fractional derivative of Weierstrass type function $W^*(t)$ of order u.

We give some symbols and main result of our paper. Let $I=[0,1]$, f be continuous on I and $\Gamma(f,I)$ denote the graph of function $f(t)$ on I. Let

$$\mathrm{OSC}(f,I) = \sup_{t',t''\in I} |f(t') - f(t'')|$$

be the oscillation of f on interval I. Throughout the present paper, more precisely by C we denote a positive constant that may have different values at different occurrences. By $C(v), C_i(v)(i=1,2,\cdots,5)$ we denote positive constants only depending on v.

The main result of our paper is:

Theorem 1. (1) *Let $g^*(t)$ be the fractional integral of generalized Weierstrass function $W^*(t)$, $0<\alpha,v<1$ with $\alpha+v<1$. Then, for sufficiently large $\lambda>1$, it holds*

$$\dim_B \Gamma(g^*,I) = \dim_B \Gamma(W^*,I) - v.$$

(2) *Let $m^*(t)$ be the fractional derivative of generalized Weierstrass function $W^*(t)$, $0<u<\alpha<1$. Then, for sufficiently large λ, it holds that*

$$\dim_B \Gamma(m^*,I) = \dim_B \Gamma(W^*,I) + u.$$

The subsequent discussion will deal with: (1) proof of Theorem 1; (2) graphs and numerical results and (3) conclusions.

2 Proof of Theorem 1

In this section, we give 5 lemmas and the proof of Theorem 1.

2.1 Lemmas

Lemma 1[1,7]**.** *Let f be a continuous function on $I=[0,1]$ and $0\leq s\leq 1$.*

(1) *Suppose*

$$|f(t)-f(u)| \leq C|t-u|^s, \qquad 0\leq t,u\leq 1.$$

Then

$$\overline{\dim}_B \Gamma(f,I) \leq 2-s.$$

(2) *Suppose* $\delta_0 > 0$. *For every* $t \in [0,1]$ *and* $0 < \delta < \delta_0$, *there exists* $u \in [0,1]$ *such that* $|t-u| \le \delta$ *and*

$$|f(t) - f(u)| \ge C\delta^s.$$

Then

$$\underline{\dim}_B \Gamma(f,I) \ge 2 - s.$$

Lemma 2[2]**.** *Let* f *be continuous on* $I = [0,1], 1 < s \le 2$, $\Pi = \{0 = x_0 < x_1 < x_2 < \cdots < x_n = 1\}$ *be a partition of* I, $\delta_i = [x_{i-1}, x_i)$ *and* $|\Pi| = \max\limits_{1\le i\le n} |\delta_i|$. *Then*

$$K^s(\Gamma(f,I)) = \lim_{\delta\to 0^+} \inf_{|\Pi|<\delta} \sum_{\Pi} \mathrm{OSC}(f,\delta_i)|\delta|^{s-1}, \tag{2.1}$$

where $K^s(\Gamma(f,I))$ *denotes the s-dimension K-measure of* $\Gamma(f,I)$. *And*

$$\dim_K \Gamma(f,I) \le \underline{\dim}_B \Gamma(f,I). \tag{2.2}$$

If the values of $\dim_K \Gamma(f,A)$ *are equal for all open intervals* A *of* I. *Then*

$$\dim_K \Gamma(f,I) \le \dim_P \Gamma(f,I). \tag{2.3}$$

From [8] and simple calculation, we have

Lemma 3. *Let* $0 < v < 1, l > 1, I = [0,1]$ *and* $t_j = \dfrac{4aj}{l}(j = 2,3,\cdots)$. *There exists certain* $h \in \left(0, \dfrac{3a}{l}\right)$ *such that*

$$|G_{t_j+h}(v,l) - G_{t_j}(v,l)| \ge C_1(v) l^{-v}. \tag{2.4}$$

We also have

$$|G_t(v,l)| \le C_2(v) l^{-v}, |G_{t+h}(v,l) - G_t(v,l)| \le C_2(v) l^{-v} \tag{2.5}$$

and

$$|G_{t+h}(v,l) - G_t(v,l)| \le C_3(v) h \cdot l^{1-v} \tag{2.6}$$

with $C_1(v), C_2(v)$ *and* $C_3(v)$ *certain positive constants which only depend on* v.

Lemma 4. *Let* $0 < v, \alpha < 1$ *with* $\alpha + v < 1, \lambda > 1$ *and* $I = [0,1]$. *Then*

$$\overline{\dim}_B \Gamma(g^*, I) \le 2 - \alpha - v, \tag{2.7}$$

here $g^*(t)$ *is defined as* (1.4) *of section* 1.

Proof. First we show $g^*(t)$ is continuous on I. Note that $G_t(v,\lambda^j) \le C(v)$, we have

$$|g^*(t)| = \left|\sum_{j\ge 1} \lambda^{-\alpha j} G_t(v,\lambda^j)\right| \le C(v) \sum_{j\ge 1} \lambda^{-\alpha j} < \infty.$$

This shows the continuity of $g^*(t)$. For any given $0 < h < 1$, there exists a non-negative integer N such that $h \in [\lambda^{-(N+1)}, \lambda^{-N})$. Then we have

$$|g^*(t+h) - g^*(t)| \le \left(\sum_{j=1}^{N} + \sum_{j=N+1}^{\infty}\right) \lambda^{-\alpha j} |G_{t+h}(v,\lambda^j) - G_t(v,\lambda^j)| =: I_1 + I_2.$$

By (2.6) of Lemma 3, we have

$$I_1 \le C_3(v) \sum_{j=1}^{N} h\lambda^{(1-\alpha-v)j} \le C_3(v) h^{v+\alpha},$$

and by (2.5) of Lemma 3, we get

$$I_2 \le C_2(v) \sum_{j=N+1}^{\infty} \lambda^{-\alpha j} \lambda^{-vj} \le C_2(v) h^{v+\alpha}.$$

So we have

$$|g^*(t+h) - g^*(t)| \le C(v) h^{v+\alpha}.$$

With Lemma 2(1) we complete the proof of Lemma 4.

Lemma 5. *Let* $0 < v, \alpha < 1, \alpha + v < 1$ *and* $I = [0,1]$. *For sufficiently large* λ, *it holds*

$$\dim_K \Gamma(g^*, I) \ge 2 - \alpha - v, \tag{2.8}$$

here $g^*(t)$ *is defined as* (1.4) *in section* 1.

Proof. Let $\delta < \dfrac{1}{\lambda}$. We consider any partition Π of I. For any given interval δ_i of Π, there exists a positive integer N such that $|\delta_i| \in \left[\dfrac{1}{\lambda^{N-1}}, \dfrac{1}{\lambda^{N-2}}\right)$. Let

$$R = \left\{ t_j : t_j = \frac{4\alpha j}{\lambda^N}, j = 2, 3, \cdots \right\}$$

and $h \in (0, \dfrac{4a}{\lambda^N})$. Because λ is sufficiently large, there exists at least one point t_i of δ_i such that $t_i \in \mathbb{R}$ and $(t_i, t_i + h) \subset \delta_i$.

Due to (2.4) of Lemma 3, there exists certain $h \in (0, \dfrac{a}{\lambda^N})$ such that

$$|G_{t_i+h}(v,\lambda^N) - G_{t_i}(v,\lambda^N)| \ge C_1(v) \lambda^{-vN}.$$

On the other hand, we see that

$$|g^*(t+h)-g^*(t)-\lambda^{-\alpha N}(G_{t+h}(v,\lambda^N)-G_t(v,\lambda^N))|$$
$$\le \left(\sum_{j=1}^{N-1}+\sum_{j=N+1}^{\infty}\right)\lambda^{-aj}|G_{t+h}(v,\lambda^j)-G_t(v,\lambda^j)| =: I_1+I_2.$$

By (2.5) and (2.6) of Lemma 3, we get

$$I_1 \le aC_2(v)\frac{\lambda^{\alpha+v-1}}{1-\lambda^{\alpha+v-1}}\lambda^{-(\alpha+v)N}, \qquad I_2 \le C_3(v)\frac{\lambda^{-\alpha-v}}{1-\lambda^{-\alpha-v}}\lambda^{-(\alpha+v)N}.$$

Here $C_1(v),C_2(v)$ and $C_3(v)$ are the same as that of Lemma 3. Let

$$\lambda^{\alpha+v-1}=s, C_4(v)=\max\{aC_2(v),C_3(v)\}.$$

Hence

$$I_1+I_2 \le C_4(v)\frac{\lambda s^2+1}{(1-s)(\lambda s-1)}\lambda^{-(\alpha+v)N}.$$

For sufficiently large λ and $C_5(v)=\dfrac{2C_4(v)}{C_1(v)}$, we have

$$0<s<\frac{1}{2C_5(v)+1}, \frac{2C_5(v)+2}{\lambda s-1}<1.$$

Then,

$$\frac{\lambda s^2+1}{(1-s)(\lambda s-1)} \le \frac{1}{C_5(v)},$$

so we have

$$|g^*(t+b)-g^*(t)-\lambda^{-\alpha N}(G_{t+h}(v,\lambda^N)-G_t(v,\lambda^N))| \le \frac{1}{2}C_1(v)\lambda^{-(\alpha+v)N}.$$

Based on prior discussion we know that there exists $(t_i,t_i+h)\subset\delta_i$, such that

$$|g^*(t_i+h)-g^*(t_i)| \ge C_1(v)\lambda^{-(\alpha+v)N}-\frac{1}{2}C_1(v)\lambda^{-(\alpha+v)N} > C(v)|\delta_i|^{\alpha+v}.$$

Hence

$$\mathrm{OSC}(g^*,\delta_i) > C|\delta_i|^{\alpha+v}.$$

Combining with (2.2) of Lemma 2, we have

$$K^{2-\alpha-v}(\Gamma(g^*,I)) = \lim_{\delta\to 0^+}\inf_{|\Pi|<\delta}\sum_{\Pi}\mathrm{OSC}(g^*,\delta_i)|\delta_i|^{1-\alpha-v} > C.$$

So we get (2.8).

2.2 Proof of Theorem 1

Proof of Theorem 1. Similar argument to Example 11.3 of [1, pp193-194] and for sufficiently large λ, it holds

$$\dim_B \Gamma(W^*, I) = 2 - \alpha. \tag{2.9}$$

A combination of Lemma 2, 4, 5 and (2.9) leads to the following conclusion:

$$\dim_B \Gamma(g^*, I) = \dim_K \Gamma(g^*, I) = \dim_B \Gamma(W^*, I) - v = \dim_B \Gamma(W^*, I) - v = 2 - \alpha - v.$$

Let $m^*(t)$ be the fractional derivative of generalized Weierstrass function $W^*(t), 0 < u < \alpha < 1$. For sufficiently large λ, similarly we can get

$$\dim_B \Gamma(m^*, I) = \dim_K \Gamma(m^*, I) = \dim_k \Gamma(W^*, I) + u = \dim_B \Gamma(W^*, I) + u = 2 - \alpha + u.$$

Thus we have completed the proof of Theorem 1 in section 1.

3 Graphs and Numerical Results

We give some graphs and numerical results to show the linear connection between the order of the fractional calculus and the fractal dimensions of graphs of the generalized Weierstrass function. Let $\lambda = 2, \alpha = 0.5$, $f(t) = \sin\left(\frac{t}{2}\right) + \cos\left(2t + \frac{\pi}{2}\right)$ and

$$W^*(t) = \sum_{j \geq 1} \lambda^{-\alpha j} f(\lambda^j t). \tag{3.1}$$

Fig.1 shows the graph of $f(t)$ and Fig.2 shows the graph of $W^*(t)$.

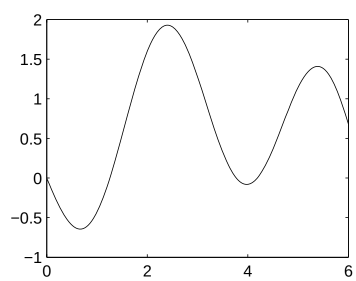

Fig. 1 $f(t)$

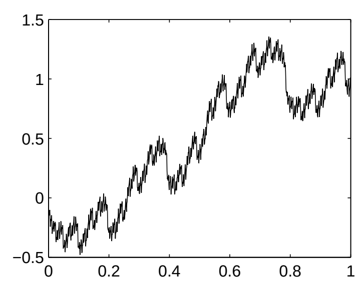

Fig. 2 $W^*(t)$

Fig.3 shows the graph of $g^*(t)$: the fractional integral of $W^*(t)$ of order $1/3$ and Fig.4 shows the graph of $m^*(t)$: the fractional derivative of $W^*(t)$ of order $1/6$,

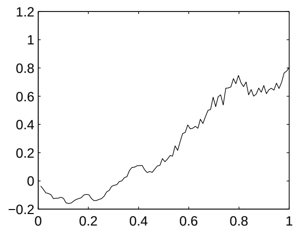

Fig. 3 $g^*(t)$ fractional integral of $W^*(t)$

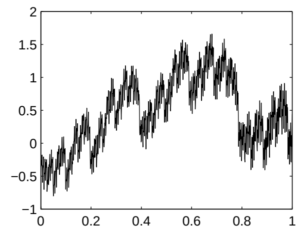

Fig. 4 $m^*(t)$ fractional derivative of $W^*(t)$

Let u and v be 0.1, 0.2, 0.3, 0.4, respectively. Table 1 gives the Box dimension of graphs of $g^*(t)$ and $m^*(t)$.

Table 1

v	$\dim_B\Gamma(g^*,I)$	u	$\dim_B\Gamma(m^*,I)$
0	1.5	0	1.5
0.1	1.3853	0.1	1.5834
0.2	1.2795	0.2	1.6787
0.3	1.1804	0.3	1.7774
0.4	1.0769	0.4	1.8612

Fig.5 shows the connection between v and $\dim_B\Gamma(g^*,I)$ and Fig.6 shows the connection between u and $\dim_B\Gamma(m^*,I)$.

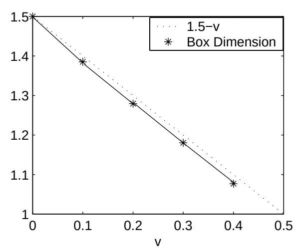

Fig. 5 Connection between v and $\dim_B\Gamma(g^*(I)$

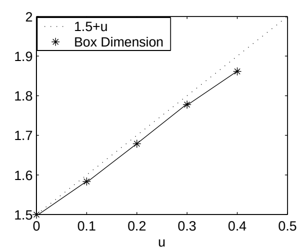

Fig. 6 Connection between u and $\dim_B\Gamma(m^*,I)$

4 Conclusions

To sum up, we have the following conclusions: Let $W^*(t)$ be defined as (1.3), $g^*(t)$ be the fractional integral of $W^*(t)$ and $m^*(t)$ be the fractional derivative of $W^*(t)$. Then, for sufficiently large $\lambda > 1, 0 < \alpha, v < 1, \alpha + v < 1$ and $0 < u < \alpha < 1$, it holds

$$\dim_B\Gamma(g^*,I) = \dim_B\Gamma(W^*,I) - v;$$

$$\dim_B\Gamma(m^*,I) = \dim_B\Gamma(W^*,I) + u.$$

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