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ELEMENTARY DENSITY BOUNDS FOR SELF-SIMILAR SETS AND APPLICATION[1]

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Abstract. Falconer[1] used the relationship between upper convex density and upper spherical density to obtain elementary density bounds for s-sets at H^s-almost all points of the sets. In this paper, following Falconer[1], we first provide a basic method to estimate the lower bounds of these two classes of set densities for the self-similar s-sets satisfying the open set condition (OSC), and then obtain elementary density bounds for such fractals at all of their points. In addition, we apply the main results to the famous classical fractals and get some new density bounds.

Key words: *self-similar s-set, upper convex density, upper spherical density,*Hausdorff *measure, elementary density bound*

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1 Introduction and Preliminaries

It is well known that Hausdorff measure is the most important notion in the study of fractals and densities play an important role in the study of Hausdorff measure (see Falconer[1, 2], Edgar[3], Freilich[4] and Davies and Samuels[5]). The fractals that people are most interested in are s-sets, especially the self-similar s-sets, and extensively, the s-straight s-sets (see [6–25]). The self-similar s-sets are also the simplest fractals. But up to now, the computation of Hausdorff measure, even for this simplest class of fractals, is still difficult. Especially for the fractals with their their Hausdorff dimensions larger than 1, how to compute Hausdorff measure remains an open problem.

Recently, various authors studied s-sets by virtue of the density properties (see [1–25]). In [1, 2], Falconer studied the local structure of s-sets, including a class of self-similar sets. In his book [1], two classes of densities, i.e., upper convex density and upper spherical density denoted by $\overline{D}_C^s(E,x)$ and $\overline{D}^s(E,x)$ respectively, are introduced in the study of s-sets. These two classes of densities, that reflex the extend to which the s-sets concentrate, have been playing a major role in the developing of geometrical measure theory (see Federer [6]).

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However, since such two classes of densities, like Hausdorff measures, are defined in a sophisticated way, it is usually very difficult to obtain their exact values. Typically, we are only able to give some bounds to cope with such a difficult problem. In [1], Falconer gave elementary density bounds for s-sets as follows.

Proposition 1.1 [1]**.** *Let $E \subset R^n$ be an s-set, then for upper spherical densities, we have*

$$2^{-s} \leq \overline{D}^s(E,x) \leq 1, \tag{1.1}$$

for H^s-almost all $x \in E$.

As in [1], Proposition 1.1 could be easily deduced from the following relationship between upper convex density and upper spherical density together with a classical density property.

Proposition 1.2 [1]**.** *Let $E \subset R^n$ be an s-set, then we have the following relationship between upper spherical densities and upper convex densities:*

$$2^{-s}\overline{D}_C^s(E,x) \leq \overline{D}^s(E,x) \leq \overline{D}_C^s(E,x), \tag{1.2}$$

for all $x \in E$.

Proposition 1.3 [1]**.** *Let $E \subset R^n$ be an s-set, then for upper convex densities, we have*

$$\overline{D}_C^s(E,x) = 1, \tag{1.3}$$

for H^s-almost all $x \in E$.

Recently, many authors investigated the Hausdorff measure of s-straight s-sets, especially the self-similar s-sets, and obtained a number of valuable results (see [8–25]). Since self-similar s-sets are a class of s-sets, a natural problem arises. Would the chain of inequalities (1.1) and the equality (1.3) be also true if E, as the s-set in (1.1) and (1.3), is replaced by the self-similar s-set satisfying the open set condition (OSC)? In this paper, following Falconer [1], we first provide a basic method estimate the lower bounds of upper convex density and upper spherical density for the self-similar sets satisfying OSC. After that, by using this method and some known results on Hausdorff measure of the self-similar s-sets, we obtain the elementary density bounds for such fractals at all of their points. As a result, we show that for some specified self-similar s-sets, (1.1) and (1.3) do hold at all of their points. In addition, we apply the main result to the famous classical fractals such as the Koch curve, the Sierpinski gasket, the Cartesian product of the middle third Cantor set with itself, etc., and obtain some new density bounds.

Some definitions, notations and known results are from the references [1-3].

Let d be the standard distance function on R^n, where R^n is Euclidian n-space. Denote $d(x,y)$ by $|x-y|$, $\forall x,y \in R^n$. If U is a nonempty subset of R^n, we define the diameter of U as $|U| = \sup\{|x-y| : x,y \in U\}$. Let δ be a positive number. If $E \subset \bigcup_i U_i$ and $0 < |U_i| \leq \delta$ for each i, we say that $\{U_i\}$ is a δ-covering of E.

Let $E \subset R^n$ and $s \geq 0$. For $\delta > 0$, define

$$H_\delta^s(E) = \inf\left\{\sum_i |U_i|^s : \bigcup_i U_i \supset E, 0 < |U_i| \leq \delta\right\}. \tag{1.4}$$

Letting $\delta \to 0$, we call the limit

$$H^s(E) = \lim_{\delta \to 0} H^s_\delta(E)$$

the s-dimensional Hausdorff measure of E. Note that the Hausdorff dimension of E is defined as

$$\dim_H E = \inf\{s \geq 0 : H^s(E) = 0\} = \sup\{s \geq 0 : H^s(E) = \infty\}.$$

An H^s-measurable set $E \subset R^n$ with $0 < H^s(E) < \infty$ is termed an s-set. A set $E \subset R^n$ is said to be s-straight (see [15–17]) if

$$H^s_\infty(E) = H^s(E) < \infty,$$

where $H^s_\infty(E)$ is the s-dimensional Hausdorff content, i.e.,

$$H^s_\infty(E) = \inf\left\{\sum_i |U_i|^s : \bigcup_i U_i \supset E\right\}.$$

Let E be an s-set in R^n, and $x \in R^n$. The upper convex density of E at x is defined as (see [1])

$$\overline{D}^s_C(E,x) = \limsup_{\rho \to 0}\left\{\frac{H^s(E \cap U)}{|U|^s} : U \text{ convex in } R^n, x \in U, 0 < |U| \leq \rho\right\}. \tag{1.5}$$

Note that the above supremum may just be taken over all subsets U in R^n with $x \in U$ and $0 < |U| \leq \rho$.

In addition, the upper spherical density of E at x is defined as (see [1])

$$\overline{D}^s(E,x) = \limsup_{\rho \to 0}\left\{\frac{H^s(E \cap B_\rho(x))}{(2\rho)^s}\right\}, \tag{1.6}$$

where $B_\rho(x)$ denotes the closed ball of center x and radius ρ in R^n. Now we review the self-similar s-sets satisfying OSC. Let $D \subset R^n$ be closed. A map $S : D \to D$ is called a contracting similarity, if there is a number r with $0 < r < 1$ such that

$$|S(x) - S(y)| = r|x - y|, \qquad \forall x, y \in D,$$

where r is called the similar ratio. It was proved by Hutchinson (see [7]) that given $m \geq 2$ and contracting similarities $S_i : D \to D$ with similarity ratios $c_i (i = 1, 2, \cdots, m)$ there exists a unique nonempty compact set $E \subset R^n$ satisfying

$$E = \bigcup_{i=1}^m S_i(E).$$

The set E is called the self-similar s-set for the iterated function system (IFS) $\{S_1, ..., S_m\}$, here we assume that there is a bounded nonempty open set V such that

$$\bigcup_{i=1}^m S_i(V) \subset V$$

and

$$S_i(V) \cap S_j(V) = \emptyset, \qquad i \neq j, j = 1, 2, \cdots, m,$$

which is often referred to the open set condition (OSC). In this case, we know that $0 < H^s(E) < \infty$ and so E is an s-set. Furthermore, we call E satisfying the strong separation condition (SSC), if E meets OSC and satisfies

$$S_i(E) \cap S_j(E) = \emptyset, \qquad i \neq j, j = 1, 2, \cdots, m.$$

Denote by J_k the set of all k-sequences $(i_1, \cdots, i_k)$, where $1 \leq i_1, \cdots, i_k \leq m, k \geq 1$ and put $E_{i_1 \cdots i_k} = S_{i_1} \circ \cdots \circ S_{i_k}(E)$, which is referred to k-contracting-copy of E. Obviously, $\forall (i_1, \cdots, i_k) \in J_k$, we have

$$|E_{i_1 \cdots i_k}|^s = |S_{i_1} \circ \cdots \circ S_{i_k}(E)|^s = c_{i_1} \cdots c_{i_k} |E|^s.$$

It is not hard to see that for each $k \geq 1$,

$$E = \bigcup_{J_k} E_{i_1 \cdots i_k} = \bigcup_{J_k} S_{i_1} \circ \cdots \circ S_{i_k}(E). \tag{1.7}$$

2 A Basic Method to Estimate the Lower Bounds of Densities for Self-Similar Sets

In this section, we will provide a basic method to estimate the lower bounds of upper convex density and upper spherical density for the self-similar sets satisfying OSC.

Theorem 2.1. *Let $E \subset R^n$ be a self-similar s-set satisfying OSC. Suppose that there exist a finite collection $\{U_j\}_{j=1}^p$ (where p is some positive integer) of subsets in R^n with $E = \bigcup\limits_{j=1}^p U_j$ and a real number $d > 0$ such that*

$$\frac{H^s(U_j)}{|U_j|^s} \geq d \quad \text{for all} \quad j = 1, 2, \cdots, p.$$

Then the upper convex densities of E at all of its points have the following lower bound estimates:

$$\overline{D}_C^s(E, x) \geq d \quad \text{for all} \quad x \in E.$$

Proof. Suppose that $\{U_j\}_{j=1}^p$ is as in the assumptions of Theorem 2.1. Then for any integer $k \geq 1$, by (1.7) we have

$$E = \bigcup_{J_k} S_{i_1} \circ \cdots \circ S_{i_k}\Big(\bigcup_{j=1}^p U_j\Big) = \bigcup_{J_k} \bigcup_{j=1}^p S_{i_1} \circ \cdots \circ S_{i_k}(U_j).$$

Thus, for any point $x \in E$ and any integer $k \geq 1$, there exist $(i_1, \cdots, i_k) \in J_k$ and a positive integer j_k with $1 \leq j_k \leq p$ such that

$$x \in S_{i_1} \circ \cdots \circ S_{i_k}(U_{j_k}).$$

Now setting

$$V_k = S_{i_1} \circ \cdots \circ S_{i_k}(U_{j_k}),$$

we then get a sequence $\{V_k\}_{k=1}^\infty$ with $x \in V_k$ and

$$\frac{H^s(V_k)}{|V_k|^s} = \frac{H^s(S_{i_1} \circ \cdots \circ S_{i_k}(U_{j_k}))}{|S_{i_1} \circ \cdots \circ S_{i_k}(U_{j_k})|^s} = \frac{(c_{i_1} \cdots c_{i_k})^s H^s(U_{j_k})}{(c_{i_1} \cdots c_{i_k})^s |U_{j_k}|^s} = \frac{H^s(U_{j_k})}{|U_{j_k}|^s} \geq d$$

for all $k \geq 1$. Moreover, we also see $|V_k| \to 0$ as $k \to \infty$ since

$$0 \leq |V_k| = |S_{i_1} \circ \cdots \circ S_{i_k}(U_{j_k})| = c_{i_1} \cdots c_{i_k} |U_{j_k}| \leq c^k |E| \to 0, \qquad k \to \infty,$$

where $c = \max\{c_{i_1}, \cdots, c_{i_k}\} < 1$. Thus, by the definition of upper convex density, we have

$$\overline{D}_C^s(E,x) \geq \limsup_{n\to\infty} \frac{H^s(V_k)}{|V_k|^s} \geq d,$$

which is the desired result.

As an application of Theorem 2.1, a basic method for investigating the lower bounds of upper spherical densities is obtained as follows.

Theorem 2.2. *Let $E \subset R^n$ be a self-similar s-set satisfying OSC. Suppose that there exist a finite collection $\{U_j\}_{j=1}^p$ (where p is some positive integer) of subsets in R^n with $E = \bigcup_{j=1}^p U_j$ and a real number $d > 0$ such that*

$$\frac{H^s(U_j)}{|U_j|^s} \geq d \quad \textit{for all} \quad j = 1, 2, \cdots, p.$$

Then the upper spherical densities of E at all of its points have the following lower bound estimates:

$$\overline{D}^s(E,x) \geq 2^{-s} d \quad \textit{for all} \quad x \in E.$$

Proof. Theorem 2.2 is immediately deduced from Theorem 2.1 and (1.2).

3 Elementary Density Bounds for Self-Similar s-Sets

In this section, we will use the basic method offered in the last section and some known results on Hausdorff measure of the self-similar s-sets to obtain elementary density bounds for the self-similar s-sets satisfying OSC at all of their points. As consequences, we will point out under what conditions the elementary density bounds for s-sets at H^s-almost all points of the sets mentioned by Falconer in [1] remain true for the self-similar s-sets at all points of the sets.

Proposition 3.1. *Let $E \subset R^n$ be an s-straight s-set, then*

$$\overline{D}_C^s(E,x) \leq 1 \quad \textit{for all} \quad x \in E. \tag{3.1}$$

Proof. By the definition of upper convex density and the equivalent definition of s-straight s-set (see Delaware [17]), we can easily see that the conclusion of Proposition 3.1 holds.

Corollary 3.1. *Let $E \subset R^n$ be a self-similar s-set satisfying OSC, then*

$$\overline{D}^s(E,x) \leq 1 \quad \textit{for all} \quad x \in E. \tag{3.2}$$

Proof. Since any self-similar s-set must be an s-straight s-set (see [10, 25]), Corollary 3.1 is immediately deduced from Proposition 3.1.

Theorem 3.1. *Let $E \subset R^n$ be a self-similar s-set satisfying OSC. Then the upper convex densities of E at all of its points have the following density bounds:*

$$\frac{H^s(E)}{|E|^s} \leq \overline{D}_C^s(E,x) \leq 1 \quad \textit{for all} \quad x \in E. \tag{3.3}$$

Proof. By (1.7), for any given integer $k \geq 1$, we have

$$E = \bigcup_{J_k} V_{j_1\cdots j_k}, \tag{3.4}$$

where $V_{j_1\cdots j_k} = S_{i_1} \circ \cdots \circ S_{i_k}(E)$. Considering

$$\frac{H^s(V_{j_1\cdots j_k})}{|V_{j_1\cdots j_k}|^s} = \frac{(c_{i_1}\cdots c_{i_k})^s H^s(E)}{(c_{i_1}\cdots c_{i_k})^s |E|^s} = \frac{H^s(E)}{|E|^s}, \tag{3.5}$$

we conclude from Theorem 2.1 that

$$\overline{D}_C^s(E,x) \geq \frac{H^s(E)}{|E|^s} \quad \text{for all} \quad x \in E, \tag{3.6}$$

which together with (3.2) implies that (3.3) is true.

Theorem 3.2. *Let $E \subset R^n$ be a self-similar s-set satisfying OSC. Then the upper spherical densities of E at all of its points have the following density bounds:*

$$2^{-s}\frac{H^s(E)}{|E|^s} \leq \overline{D}^s(E,x) \leq 1 \quad \textit{for all} \quad x \in E. \tag{3.7}$$

Proof. By Theorem 3.1, (3.3) is true. This together with (3.2) and (1.2) leads that (3.7) holds.

Corollary 3.2. *Let $E \subset R^n$ be a self-similar s-set satisfying OSC. Suppose that $H^s(E) = |E|^s$, then*

$$\overline{D}_C^s(E,x) = 1 \quad \textit{for all} \quad x \in E. \tag{3.8}$$

Proof. We can use Theorem 3.1 to give the proof here. The fact that $H^s(E) = |E|^s$ together with (3.3) implies that (3.8) holds.

Similarly, the following corollary, which can be easily deduced from Theorem 3.2, deals with the bounds of upper spherical densities of the self-similar s-set E at all of its points. We will omit its proof.

Corollary 3.3. *Let $E \subset R^n$ be a self-similar s-set satisfying OSC. Suppose that $H^s(E) = |E|^s$, then the upper spherical densities of E at all of its points have the following density bounds:*

$$2^{-s} \leq \overline{D}^s(E,x) \leq 1 \quad \textit{for all} \quad x \in E. \tag{3.9}$$

4 Applications

In this section, we use some of the main results to discuss a number of famous classical fractals.

Example 4.1. Let E be any one of the following fractals and $s = \dim_H E$:
(i) the famous middle third Cantor set (see [2, Figure 0.1]);
(ii) the Sierpinski carpet as in [10] or [18] (see [18, Figure 1]);
(iii) the Cantor dust in [2] (see [2, Figure 0.4]).
Then we conclude that

$$\overline{D}_C^s(E,x) = 1$$

and

$$2^{-s} \leq \overline{D}^s(E,x) \leq 1$$

for all $x \in E$.

Proof. In case (i), (ii)and (iii), we have $H^s(E) = |E|^s$. In fact, in (i), we have

$$H^s(E) = 1 = 1^{\log_3 2} = |E|^s.$$

In (ii), we know

$$H^s(E) = \sqrt{2} = (\sqrt{2})^1 = |E|^s.$$

In (iii), we can get

$$H^s(E) = \frac{\sqrt{10}}{3} = (\frac{\sqrt{10}}{3})^1 = |E|^s.$$

Therefore, it follows from Corollary 3.2 and 3.3 that the conclusions of Example 4.1 hold.

Example 4.2. Let $s = \log_2 3$ and denote by S the Sierpinski gasket (see [2, Figure 0.3] or [18, Figure 2]). Then the density bounds of S at all points in S are as follows:

$$0.5 \leq \overline{D}_C^s(S,x) \leq 1 \quad \text{for all} \quad x \in E \tag{4.1}$$

and

$$2^{-(1+s)} \leq \overline{D}^s(S,x) \leq 1 \quad \text{for all} \quad x \in E. \tag{4.2}$$

Proof. From [11], we see $H^s(S) \geq 0.5$. It follows from Theorem (3.1) and (3.2) that (4.1) and (4.2) hold.

Example 4.3. Let $s = \log_3 4$ and denote by K the Koch curve (see [2, Figure 0.2(a)] or [18, Figure 3]). Then the density bounds of K at all points in K are as follows:

$$0.5 \leq \overline{D}_C^s(K,x) \leq 1 \quad \text{for all} \quad x \in E \tag{4.3}$$

and

$$2^{-(1+s)} \leq \overline{D}^s(K,x) \leq 1 \quad \text{for all} \quad x \in E. \tag{4.4}$$

Proof. From [12], we see $H^s(K) \geq 0.5$. It follows from Theorem (3.1) and (3.2) that (4.3) and (4.4) hold.

Example 4.4. Let $s = \log_3 4$ and denote by $C \times C$ the Cartesian product of the middle third Cantor set with itself (see [18, Figure 4]). Then we have the density bounds of $C \times C$ at all points in $C \times C$ as follows:

$$0.9557 < \overline{D}_C^s(C \times C, x) \le 1 \quad \text{for all} \quad x \in E \tag{4.5}$$

and

$$0.3985 < \overline{D}^s(C \times C, x) \le 1 \quad \text{for all} \quad x \in E. \tag{4.6}$$

Proof. From [13], we see $H^s(C \times C) > 1.48$. It follows from Theorem (3.1) and (3.2) that

$$1 \ge \overline{D}_C^s(C \times C, x) \ge \frac{1.48}{(\sqrt{2})^s} > 0.9557 \quad \text{for all} \quad x \in E$$

and

$$1 \ge \overline{D}^s(C \times C, x) > 0.9557 \cdot \frac{1}{2^s} \ge 0.3985 \text{ for all } x \in E,$$

which are the desired results.

Remark 4.1. Example 4.2–4.4 show the new density bounds for the Sierpinski gasket, the Koch curve and the Cartesian product of the middle third Cantor set with itself, respectively.

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