Analysis in Theory and Applications
Volume 23 Number 4, 2007, 307–314
DOI10.1007/s10496-007-0307-2

AN ENDPOINT ESTIMATE FOR MAXIMAL MULTILINEAR SINGULAR INTEGRAL OPERATORS

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Received Apr. 16, 2007

Abstract. A weak type endpoint estimate for the maximal multilinear singular integral operator

$$T_A^* f(x) = \sup_{\varepsilon>0} \left| \int_{|x-y|>\varepsilon} \frac{\Omega(x-y)}{|x-y|^{n+1}} (A(x) - A(y) - \nabla A(y)(x-y)) f(y) \mathrm{d}y \right|$$

is established, where Ω is homogeneous of degree zero, integrable on the unit sphere and has vanishing moment of order one, and A has derivatives of order one in $\mathrm{BMO}(\mathbb{R}^n)$. A regularity condition on Ω which implies an $L\log L$ type estimate of T_A^* is given.

Key words: *maximal multilinear singular integral operator, bounded mean oscillation, vanishing condition*

AMS (2000) subject classification: 42B20

1 Introduction

We work on $\mathbb{R}^n$, $n \geq 2$. Let Ω be homegeneous of degree zero, integrable on the unit sphere S^{n-1} and satisfy the vanishing condition

$$\int_{S^{n-1}} \Omega(\theta)\theta \mathrm{d}\theta = 0. \tag{1}$$

Let A be a function on $\mathbb{R}^n$ having derivatives of order one in $\mathrm{BMO}(\mathbb{R}^n)$. For $x, y \in \mathbb{R}^n$, set

$$R(A; x, y) = A(x) - A(y) - \nabla A(y)(x-y). \tag{2}$$

Define the multilinear singular integral operator T_A by

$$T_A f(x) = \mathrm{p.v.} \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^{n+1}} R(A; x, y) f(y) \mathrm{d}y \tag{3}$$

and the corresponding maximal operator T_A^* by

$$T_A^* f(x) = \sup_{\varepsilon>0} |T_{A,\varepsilon} f(x)|, \tag{4}$$

where $T_{A,\varepsilon}f(x)$ is the truncated operator defined by

$$T_{A,\varepsilon}f(x)=\int_{|x-y|>\varepsilon}\frac{\Omega(x-y)}{|x-y|^{n+1}}R(A;x,y)f(y)\mathrm{d}y. \tag{5}$$

The operators T_A and T_A* have been considered by many authors. A well known result of Cohen [1] states that if $\Omega\in\mathrm{Lip}_1(S^{n-1})$, then T_A^* is a bounded operator on $L^p(\mathbb{R}^n)$ with the bound $C\|\nabla A\|_{\mathrm{BMO}(\mathbb{R}^n)}$ for any $1<p<\infty$. Hofmann [2] shows that $\Omega\in\bigcup_{q>1}L^q(S^{n-1})$ is a sufficient condition such that T_A is bounded on $L^p(\mathbb{R}^n)$ for any $1<p<\infty$. Recently, Hu [3] gave a regularity condition on Ω which is fairly weaker than that $\Omega\in\mathrm{Lip}_1(S^{n-1})$ and implies the $L^p(\mathbb{R}^n)$ boundedness of T_A^* for any $1<p<\infty$. For $\Omega\in L^1(S^{n-1})$, define the L^1-modulus of continuity of Ω by

$$\omega(\delta)=\sup_{|\rho|\le\delta}\int_{S^{n-1}}|\Omega(\rho x)-\Omega(x)|\mathrm{d}x,$$

where the supremum is taken over all rotations on the unit sphere, and $|\rho|$ denotes the distance of ρ from the identity rotation. Hu [3] shows that if the L^1-modulus of continuity of Ω satisfies

$$\int_0^1\omega(\delta)\log\big(2+\frac{1}{\delta}\big)\frac{1}{\delta}\mathrm{d}\delta<\infty, \tag{6}$$

then T_A^* is bounded on $L^p(\mathbb{R}^n)$ for any p with $1<p<\infty$.

For the endpoint estimates of T_A and T_A^*, Hu and Yang show in [4] that if $\Omega\in\mathrm{Lip}_\beta(S^{n-1})$ for some β with $0<\beta\le1$, then T_A satisfies an $L\log L$ type estimate, that is, there is a positive constant C such that for any bounded function f and $\lambda>0$,

$$\big|\big\{x\in\mathbb{R}^n:|T_Af(x)|>\lambda\big\}\big|\le C\int_{\mathbb{R}^n}\frac{|f(x)|}{\lambda}\log\Big(2+\frac{|f(x)|}{\lambda}\Big)\mathrm{d}x. \tag{7}$$

This estimate is obviously an analogy of the weak type endpoint estimate for commutators of Calderón-Zygmund, which was established by Pérez in [5]. By a Cotlar type inequality and the weak type estimate for the operator T_A, Hu and Li [6] show that if $\Omega\in\mathrm{Lip}_\beta(S^{n-1})$ $(0<\beta\le1)$, then the maximal operator T_A^* also satisfies the estimate (7). The main purpose of this paper is to improve the $L\log L$ type estimate (7) for T_A^* established by Hu and Li. We will show that if Ω satisfies (6), then T_A^* also satisfies the estimate (7). Precisely, we will prove

Theorem 1. *Let Ω be homogeneous of degree zero and satisfy the vanishing condition* (1), *A have derivatives of order one in* $\mathrm{BMO}(\mathbb{R}^n)$. *If Ω satisfies the regularity condition* (6), *then there exists a constant C depending only on n and $\|\nabla A\|_{\mathrm{RBMO}(\mathbb{R}^n)}$, such that for any bounded function f and $\lambda>0$,*

$$\big|\big\{x\in\mathbb{R}^n:|T_A^*f(x)|>\lambda\big\}\big|\le C\int_{\mathbb{R}^n}\frac{|f(x)|}{\lambda}\log\Big(2+\frac{|f(x)|}{\lambda}\Big)\mathrm{d}x.$$

It is obvious that the regularity condition (6) is weaker than that $\Omega\in\mathrm{Lip}_\beta(S^{n-1})$ with $0<\beta\le1$. We point out that in the proof of Theorem 1, we are very much motivated by the work of Grafakos[7].

Throughout this paper, C denotes a constant independent of the main parameters involved but its value may differ from line to line. For a measurable set E, χ_E denote the characteristic function of E. For a cube Q and a locally integrable function f, $m_Q(f)$ denote the mean value of f on Q.

2 Proof of Theorem 1

We begin with some preliminary lemmas.

Lemma 1. *Let b be a function on $\mathbb{R}^n$ with derivatives of order one in $L^q(\mathbb{R}^n)$ for some q, $n<q<\infty$. Then*

$$|b(x)-b(y)| \le C_n|x-y|\Big(\frac{1}{|\widetilde{Q}(x,y)|}\int_{\tilde{Q}(x,y)}|\nabla b(z)|^q \mathrm{d}z\Big)^{1/q},$$

where $\widetilde{Q}(x,y)$ is the cube centered at x and having side length $5\sqrt{n}|x-y|$.

Lemma 2. *There exists a positive constant C such that for any $t_1, t_2 \ge 0$ and $a>0$,*

$$t_1t_2 \le C\big(t_1\log(2+at_1)+a^{-1}\exp t_2\big). \tag{8}$$

Note that the Young functions $\Phi(t)=t\log(2+t)$ and $\Psi(t)=\exp t$ are complementary. It is well known that there is a positive constant C such that for any $t_1, t_2>0$,

$$t_1t_2 \le C(\Phi(t_1)+\Psi(t_2)).$$

The inequality (8) follows from the last inequality immediately.

Proof of Theorem 1. Without loss of generality, we may assume that $\|\nabla A\|_{\mathrm{BMO}(\mathbb{R}^n)}=1$. For each fixed bounded function f and λ, applying the Calderón-Zygmund decomposition to f at the level λ, we can obtain a sequence of cubes $\{Q_j\}$ with disjoint interiors such that

$$\lambda < \frac{1}{|Q_j|}\int_{Q_j}|f(x)|\mathrm{d}x \le 2^n\lambda,$$
$$|f(x)|\le\lambda, \quad \text{a.e.} \quad x\in\mathbb{R}^n\setminus\cup_j Q_j.$$

Let

$$g(x)=f(x)\chi_{\mathbb{R}^n\setminus\cup_j Q_j}(x)+\sum_j m_{Q_j}(f)\chi_{Q_j}(x)$$

and

$$h(x)=f(x)-g(x)=\sum_j(f(x)-m_{Q_j}(f))\chi_{Q_j}(x).$$

Since $\|g\|_\infty \le C\lambda$ and $\|g\|_2 \le C\lambda\|f\|_1$, it follows from $L^2(\mathbb{R}^n)$ boundedness of T_A^* (see [3])

$$|\{x\in\mathbb{R}^n : T_A^*g(x)>\lambda\}| \le C\lambda^{-2}\|T_A^*g\|_2 \le C\lambda^{-1}\|f\|_1.$$

For each fixed j let $Q_j^*=4nQ_j$. Note that

$$\big|\cup_j Q_j^*\big| \le C\sum_j|Q_j| \le C\lambda^{-1}\|f\|_1.$$

Therefore, the proof of Theorem 1 can be reduced to prove that for some positive constant D independent of f and λ,

$$\big|\{x\in\mathbb{R}^n\setminus\cup_j Q_j^* : T_A^*h(x)>D\lambda\}\big| \le C\lambda^{-1}\|f\|_1. \tag{9}$$

For each fixed j, let

$$A_j(y) = A(y) - m_{Q_j}(\nabla A)y.$$

Note that for any $x, y \in \mathbb{R}^n$ and any fixed j,

$$A(x) - A(y) - \nabla A(y)(x-y) = A_j(x) - A_j(y) - \nabla A_j(y)(x-y).$$

For each fixed $\varepsilon > 0$ and k with $1 \le k \le n$, define the operator S_ε^k by

$$S_\varepsilon^k u(x) = \int_{|x-y|>\varepsilon} K(x-y)\frac{x_k - y_k}{|x-y|}u(y)\mathrm{d}y,$$

where $K(x) = \Omega(x)|x|^{-n}$. Write

$$\begin{aligned} T_{A,\varepsilon}h(x) &= \sum_j \int_{|x-y|>\varepsilon} K(x-y)\frac{A_j(x)-A_j(y)}{|x-y|}h_j(y)\mathrm{d}y \\ &\quad + \sum_{k=1}^n \int_{|x-y|>\varepsilon} K(x-y)\frac{x_k-y_k}{|x-y|}\sum_j \left(\frac{\partial^k A(y)}{\partial y_k} - m_{Q_j}(\frac{\partial^k A}{\partial y_k})\right)h_j(y)\mathrm{d}y \\ &= \mathrm{G}^\varepsilon(x) + \sum_{k=1}^n S_\varepsilon^k\left(\sum_j \left(\frac{\partial^k A(y)}{\partial y_k} - m_{Q_j}\left(\frac{\partial^k A}{\partial y_k}\right)\right)h_j\right)(x). \end{aligned}$$

A well known result in the theory of Calderón and Zygmund tells us that the operator $S^{k,*}$ defined by

$$S^{k,*}u(x) = \sup_{\varepsilon>0}\left|S_\varepsilon^k u(x)\right|$$

is bounded from $L^1(\mathbb{R}^n)$ to weak $L^1(\mathbb{R}^n)$. Thus by Lemma 2,

$$\begin{aligned} &\left|\left\{x \in \mathbb{R}^n : \sum_{k=1}^n S^{k,*}\left(\sum_j \left(\frac{\partial^k A(y)}{\partial y_k} - m_{Q_j}(\frac{\partial^k A}{\partial y_k})\right)h_j\right)(x) > \lambda\right\}\right| \\ \le\ & C\sum_j \int_{Q_j} |\nabla A(y) - m_{Q_j}(\nabla A)|\frac{|h_j(y)|}{\lambda}\mathrm{d}y \\ \le\ & C\sum_j \int_{Q_j} \frac{|h_j(x)|}{\lambda}\log\left(2 + \frac{|h_j(x)|}{\lambda}\right)\mathrm{d}x \\ & + C\int_{Q_j} \exp\left(\frac{|\nabla A(x) - m_{Q_j}(\nabla A)|}{B_1}\right)\mathrm{d}x \\ \le\ & C\sum_j \int_{Q_j} \frac{|h_j(x)|}{\lambda}\log\left(2 + \frac{|h_j(x)|}{\lambda}\right)\mathrm{d}x + \sum_j |Q_j| \\ \le\ & C\int_{\mathbb{R}^n} \frac{|f(x)|}{\lambda}\log\left(2 + \frac{|f(x)|}{\lambda}\right)\mathrm{d}x, \end{aligned}$$

where we have invoked the John-Nirenberg inequality, which states that there are two positive constants B_1 and B_2 such that for any $b \in \mathrm{BMO}(\mathbb{R}^n)$ and cube Q,

$$\frac{1}{|Q|}\int_Q \exp\left(\frac{|b(x) - m_Q(b)|}{B_1\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}}\right)\mathrm{d}x \le B_2.$$

We now turn our attention to the term $\sup_{\varepsilon>0}|\mathrm{G}^{\varepsilon}|$. As in [7, p. 175], for each fixed $x\in\mathbb{R}^n\backslash\cup_j Q_j^*$ and $\varepsilon>0$, we define

$$J_1(x,\varepsilon)=\big\{j:\,|x-y|<\varepsilon \text{ for all } y\in Q_j\big\},$$

$$J_2(x,\varepsilon)=\big\{j:\,|x-y|>\varepsilon \text{ for all } y\in Q_j\big\},$$

and

$$J_3(x,\varepsilon)=\big\{j:\,Q_j\cap\{y:\,|y-x|>\varepsilon\}\neq\emptyset,\ Q_j\cap\{y:\,|y-x|<\varepsilon\}\neq\emptyset\big\}.$$

Set

$$h_j(x,\varepsilon)=|Q|^{-1}\int_{Q_j}h_j(y)\chi_{\{y:\,|y-x|>\varepsilon\}}(y)\mathrm{d}y.$$

It is easy to see that $|h_j(x,\varepsilon)|\le 2^n\lambda$. Write

$$\begin{aligned}
|\mathrm{G}^{\varepsilon}(x)| \;\le\;& \sum_{j\in J_2(x,\varepsilon)}\Big|\int_{|x-y|>\varepsilon}K(x-y)\frac{A_j(x)-A_j(y)}{|x-y|}h_j(y)\mathrm{d}y\Big|\\
&+\sum_{j\in J_3(x,\varepsilon)}\Big|\int_{|x-y|>\varepsilon}K(x-y)\frac{A_j(x)-A_j(y)}{|x-y|}h_j(y)\mathrm{d}y\Big|\\
\le\;& \sum_{j\in J_2(x,\varepsilon)}\Big|\int_{\mathbb{R}^n}K(x-y)\frac{A_j(x)-A_j(y)}{|x-y|}h_j(y)\mathrm{d}y\Big|\\
&+\sum_{j\in J_3(x,\varepsilon)}\Big|\int_{Q_j}K(x-y)\frac{A_j(x)-A_j(y)}{|x-y|}\big(h_j(y)\chi_{\{y:\,|y-x|>\varepsilon\}}(y)-h_j(x,\varepsilon)\big)\mathrm{d}y\Big|\\
&+2^n\lambda\sum_{j\in J_3(x,\varepsilon)}\Big|\int_{Q_j}K(x-y)\frac{A_j(x)-A_j(y)}{|x-y|}\mathrm{d}y\Big|\\
=\;& \mathrm{G}_1^{\varepsilon}(x)+\mathrm{G}_2^{\varepsilon}(x)+\mathrm{G}_3^{\varepsilon}(x).
\end{aligned}$$

Choose $x_j\in 3Q_j\backslash 2Q_j$. For each fixed j, define $\mathrm{H}_j(x)$ and $\mathrm{I}_j(x)$ by

$$\begin{aligned}
\mathrm{H}_j(x) \;=\;& \int_{\mathbb{R}^n}\big|K(x-y)-K(x-x_j)\big|\frac{|A_j(x)-A_j(y)|}{|x-y|}|h_j(y)|\mathrm{d}y\\
&+\int_{\mathbb{R}^n}|K(x-x_j)|\frac{|A_j(x_j)-A_j(y)|}{|x-y|}|h_j(y)|\mathrm{d}y\\
&+|K(x-x_j)||A_j(x)-A_j(x_j)|\frac{1}{|x-x_j|^2}\int_{\mathbb{R}^n}|y-x_j||h_j(y)|\mathrm{d}y,
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{I}_j(x) \;=\;& \int_{Q_j}\big|K(x-y)-K(x-x_j)\big|\frac{|A_j(x)-A_j(y)|}{|x-y|}\mathrm{d}y\\
&+\int_{Q_j}|K(x-x_j)|\frac{|A_j(x_j)-A_j(y)|}{|x-y|}\mathrm{d}y\\
&+|Q_j|^{1+1/n}|K(x-x_j)||A_j(x)-A_j(x_j)|\frac{1}{|x-x_j|^2}.
\end{aligned}$$

By the vanishing moment of h_j, a trivial computation shows that for $x \in \mathbb{R}^n \backslash \cup_j Q_j^*$,

$$\mathrm{G}_1^{\varepsilon}(x) \le C \sum_j \mathrm{H}_j(x),$$

and

$$\mathrm{G}_2^{\varepsilon}(x) \le C \sum_j \big(\mathrm{H}_j(x) + \lambda \mathrm{I}_j(x)\big).$$

If we set

$$A^{\varepsilon}(y) = A(y) - m_{B(x,\varepsilon)}(\nabla A)y,$$

an application of Lemma 1 tells us that for any $x, y \in \mathbb{R}^n$ with $\varepsilon/2 \le |x-y| \le 3\varepsilon/2$,

$$|A^{\varepsilon}(x) - A^{\varepsilon}(y) - \nabla A^{\varepsilon}(y)(x-y)| \le C\varepsilon\big(1 + |\nabla A(y) - m_{B(x,\varepsilon)}(\nabla A)|\big).$$

Since for $x \in \mathbb{R}^n \backslash \cup_j Q_j^*$,

$$\bigcup_{j \in J_3(x,\varepsilon)} Q_j \subset B(x, 3\varepsilon/2) \backslash B(x, \varepsilon/2).$$

Applying the inequality (8) with $a = \varepsilon^n$, we then have

$$\begin{aligned}
\mathrm{G}_3^{\varepsilon}(x) \le\ & 2^n \lambda \sum_{j \in J_3(x,\varepsilon)} \int_{Q_j} |K(x-y)| \frac{|A(x) - A(y) - \nabla A(y)(x-y)|}{|x-y|} \mathrm{d}y \\
& + 2^n \lambda \sum_{j \in J_3(x,\varepsilon)} \int_{Q_j} |K(x-y)||\nabla A(y) - m_{Q_j}(\nabla A)| \mathrm{d}y \\
\le\ & 2^n \lambda \int_{B(x,3\varepsilon/2)\backslash B(x,\varepsilon/2)} |K(x-y)| \frac{|A^{\varepsilon}(x) - A^{\varepsilon}(y) - \nabla A^{\varepsilon}(y)(x-y)|}{|x-y|} \mathrm{d}y \\
& + C\lambda \sum_{j \in J_3(x,\varepsilon)} \int_{Q_j} |K(x-y)| \log(2 + \varepsilon^n |K(x-y)|) \mathrm{d}y \\
& + C\lambda \varepsilon^{-n} \sum_{j \in J_3(x,\varepsilon)} \int_{Q_j} \exp\Big(\frac{|\nabla A(y) - m_{Q_j}(\nabla A)|}{C_2}\Big) \mathrm{d}y \\
\le\ & C\lambda \int_{B(x,3\varepsilon/2)\backslash B(x,\varepsilon/2)} |K(x-y)| \log(2 + \varepsilon^n |K(x-y)|) \mathrm{d}y \\
& + C\lambda \varepsilon^{-n} \sum_{j \in J_3(x,\varepsilon)} |Q_j| \\
\le\ & C\lambda.
\end{aligned}$$

We thus obtain that for $x \in \mathbb{R}^n \backslash Q_j^*$,

$$\sup_{\varepsilon > 0} |\mathrm{G}^{\varepsilon}(x)| \le C\lambda + C \sum_j \big(\lambda \mathrm{I}_j(x) + \mathrm{H}_j(x)\big). \tag{10}$$

Now we estimate the integral on $\mathbb{R}^n \backslash \cup_j Q_j^*$ for H_j and I_j. A familiar argument involving Lemma 1 and the John-Nirenberg inequality gives us that for $x \in 2^l Q_j^* \backslash 2^{l-1} Q_j^*$ with $l \ge 1$ and

$y \in Q_j$,

$$\begin{aligned} |A_j(x) - A_j(y)| &\le C|x-y|\Big(\frac{1}{|\widetilde{Q}(x,y)|}\int_{\widetilde{Q}(x,y)} |\nabla A(z) - m_{Q_j}(\nabla A)|^q \mathrm{d}z\Big)^{1/q} \\ &\le Cl|x-y|, \end{aligned}$$

and

$$|A_j(x_j) - A_j(x)| \le Cl|x-x_j|, \qquad |A_j(x_j) - A_j(y)| \le C|x_j - y|,$$

where $n < q < \infty$. Thus for $x \in 2^l Q_j^* \backslash 2^{l-1} Q_j^*$

$$\begin{aligned} \mathrm{H}_j(x) &\le Cl\int_{\mathbb{R}^n} \big|K(x-y) - K(x-x_j)\big||h_j(y)|\mathrm{d}y + Cl\frac{|\Omega(x-x_j)|}{|x-x_j|^{n+1}}|Q_j|^{1/n}\|h_j\|_1 \\ &\le \frac{Cl}{|x-x_j|^n}\int_{\mathbb{R}^n}\big|\Omega(x-y) - \Omega(x-x_j)\big|\mathrm{d}y + Cl\frac{|\Omega(x-x_j)|}{|x-x_j|^{n+1}}|Q_j|^{1/n}\|h_j\|_1. \end{aligned}$$

On the other hand, by the same argument as used in [8], we know that there exist positive constants C_1 and C_2, such that for any positive integer l,

$$\int_{2^l Q_j^* \backslash 2^{l-1} Q^*} \big|\Omega(x-y) - \Omega(x-x_j)\big|\mathrm{d}x \le C_1|2^l Q_j|\int_{C_2 2^{-l-1} < \delta \le C_2 2^{-l}} \omega(\delta)\frac{\mathrm{d}\delta}{\delta}.$$

This via a standard computation gives us

$$\begin{aligned} \sum_j \int_{\mathbb{R}^n\backslash Q_j^*} \mathrm{H}_j(x)\mathrm{d}x &= \sum_j\sum_{l=1}^{\infty}\int_{2^l Q_j^*\backslash 2^{l-1}Q_j^*} \mathrm{H}_j(x)\mathrm{d}x \\ &\le \sum_j\int_{\mathbb{R}^n}|h_j(y)|\sum_{l=1}^{\infty} l|2^l Q_j|\int_{2^l Q_j^*\backslash 2^{l-1}Q_j^*}\big|\Omega(x-y) - \Omega(x-x_j)\big|\mathrm{d}x\mathrm{d}y \\ &\quad + \sum_j \|h_j\|_1\sum_{l=1}^{\infty} l|Q_j|^{1/n}\int_{2^l Q_j^*\backslash 2^{l-1}Q_j^*}\frac{|\Omega(x-x_j)|}{|x-x_j|^{n+1}}\mathrm{d}x \\ &\le C\sum_j\int_{\mathbb{R}^n}|h_j(y)|\mathrm{d}y \le C\|f\|_1. \end{aligned}$$

Similarly, we cam obtain

$$\sum_j\int_{\mathbb{R}^n\backslash Q_j^*}\mathrm{I}_j(x)\mathrm{d}x \le C\sum_j\sum_{l=1}^{\infty}\int_{2^l Q_j^*\backslash 2^{l-1}Q_j^*}\mathrm{I}_j(x)\mathrm{d}x \le C\sum_j|Q_j| \le C\lambda^{-1}\|f\|_1.$$

We can now conclude the proof of Theorem 1. In fact, by the inequality (10) and the argument used in the estimate for H_j and I_j, if we choose D large enough, then

$$\begin{aligned}
&\Big|\big\{x\in\mathbb{R}^n\setminus\cup_j Q_j^*:\sup_{\varepsilon>0}|\mathrm{G}^{\varepsilon}(x)|>(D-1)\lambda\big\}\Big|\\
&\quad\le\Big|\big\{x\in\mathbb{R}^n\setminus\cup_j Q_j^*:\sum_j\mathrm{H}_j(x)>\lambda\big\}\Big|+\Big|\big\{x\in\mathbb{R}^n\setminus\cup_j Q_j^*:\sum_j\mathrm{I}_j(x)>C\big\}\Big|\\
&\quad\le C\lambda^{-1}\sum_j\int_{\mathbb{R}^n\setminus Q_j^*}\mathrm{H}_j(x)\mathrm{d}x\\
&\qquad+C\sum_j\int_{\mathbb{R}^n\setminus Q_j^*}\mathrm{I}_j(x)\mathrm{d}x\\
&\quad\le C\lambda^{-1}\sum_j\int_{Q_j}|f(x)|\mathrm{d}x+\sum_j|Q_j|.
\end{aligned}$$

Combining the estimates for $S^{k,*}\left(\sum_j\big(\frac{\partial^k A}{\partial y_k}-m_{Q_j}\big(\frac{\partial^k A}{\partial y_k}\big)\big)h_j\right)$ and $\sup_{\varepsilon>0}|G^{\varepsilon}|$ we then get the desired result (9).

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