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AN APPLICATION OF BERNSTEIN-DURRMEYER OPERATORS*

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Abstract. In the present paper, we find that the Bernstein-Durrmeyer operators, besides their better applications in approximation theory and some other fields, are good tools in constructing translation network. With the help of the de la Vallée properties of the Bernstein-Durrmeyer operators a sequence of translation network operators is constructed and its degree of approximation is dealt.

Key words: *Bernstein-Durrmeyer operator, translation network, approximation*
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1 Introduction

Let $W_{\alpha,\beta}(x)=x^{\alpha}(1-x)^{\beta},\alpha,\beta>-1,$ be a given Jacobi-weight function on the interval [0,1], and let $L^p_{W_{\alpha,\beta}}[0,1],1\le p\le+\infty$, denote the space of Lebesgue-measurable functions on [0,1] for which the norm

$$\|f\|_{W_{\alpha,\beta}}=\left(\int_0^1\Big|f(u)\Big|^pW_{\alpha,\beta}(u)\mathrm{d}u\right)^{\frac{1}{p}}$$

is finite. Let $\mathbf{N}_0$ be the set of non-negative integers. Define

$$V_n^{(\alpha,\beta)}(f,x)=\sum_{k=0}^{n}a_{k,n}^{(\alpha,\beta)}(f)p_{n,k}(x),\quad x\in[0,1],\quad f\in L^1_{W_{\alpha,\beta}}[0,1],\quad \forall n\in\mathbf{N}_0,$$

where

$$a_{k,n}^{(\alpha,\beta)}(f)=\frac{\displaystyle\int_0^1 f(u)p_{n,k}(u)W_{\alpha,\beta}(u)\mathrm{d}u}{\displaystyle\int_0^1 p_{n,k}(u)W_{\alpha,\beta}(u)\mathrm{d}u}.$$

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$V_n^{(\alpha,\beta)}(f,x)$ is the $n-$th Bernstein-Durrmeyer polynomial of f with Jacobi-weight $W_{\alpha,\beta}(x)$, $p_{n,k}(x)$ are the Bernstein basis polynomials, i.e.,

$$p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \quad k = 0,1,2,\cdots,n, \quad n \in \mathbf{N}_0.$$

We list some interesting properties of $V_n^{(\alpha,\beta)}$(see [1-2]).

(1) $\forall n \in \mathbf{N}_0, V_n^{(\alpha,\beta)} : L^p_{W_{\alpha,\beta}}[0,1] \to P_n$($P_n$ is the set of all algebraic polynomials of order $\leq n$),is a positive, linear contraction.

The sequence $\{V_n^{(\alpha,\beta)}\}_{n=0}^{\infty}$ forms an approximate identity on $L^p_{W_{\alpha,\beta}}[0,1], 1 \leq p < +\infty$, and $\forall k,n \in \mathbf{N}_0, V_n^{(\alpha,\beta)}(P_k) \subset P_k$.

Denoted by $Q_k^{(\alpha,\beta)}$ the Jacobi polynomial of degree k on the interval [0,1] normalized by $Q_k^{(\alpha,\beta)}(0) = 1$. The sequence $\{Q_k^{(\alpha,\beta)}(x)\}_{k=0}^{+\infty}$ forms an orthogonal system on [0,1] with respect to the inner product $\langle\cdot,\cdot\rangle_{W_{\alpha,\beta}}$(see G.Szegö [3,Chapter IV]).

Let

$$h_k^{(\alpha,\beta)} = \left(\int_0^1 Q_k^{(\alpha,\beta)}(x)^2 W_{\alpha,\beta}(x) \mathrm{d}x \right)^{-1}.$$

For a given function $f \in L^1_{W_{\alpha,\beta}}[0,1]$ its Jacobi series is defined as

$$\sum_{k=0}^{\infty} a_k^{(\alpha,\beta)}(f) h_k^{(\alpha,\beta)} Q_k^{(\alpha,\beta)}(x), \qquad x \in [0,1],$$

where

$$a_k^{(\alpha,\beta)}(f) = \int_0^1 f(u) Q_k^{(\alpha,\beta)}(u) W_{\alpha,\beta}(u) \mathrm{d}u.$$

(2) $\forall n \in \mathbf{N}_0$, the Jacobi polynomials $\{Q_k^{(\alpha,\beta)}(x)\}_{k=0}^{n}$ are the eigenfunctions of $V_n^{(\alpha,\beta)}$ and

$$V_n^{(\alpha,\beta)}(Q_k^{(\alpha,\beta)},x) = \lambda_{k,n}^{(\alpha,\beta)} Q_k^{(\alpha,\beta)}(x), \qquad x \in [0,1], \quad k = 0,1,\cdots,n,$$

where

$$\lambda_{k,n}^{(\alpha,\beta)} = \frac{n!}{(n-k)!} \frac{\Gamma(n+\alpha+\beta+2)}{\Gamma(n+k+\alpha+\beta+2)}.$$

In particular,

$$V_n^{(\alpha,\beta)}(f,x) = \sum_{k=0}^{n} \lambda_{k,n}^{(\alpha,\beta)} a_k^{(\alpha,\beta)}(f) h_k^{(\alpha,\beta)} Q_k^{(\alpha,\beta)}(x), \qquad x \in [0,1].$$

Denote by $x_k^{(n)}(\alpha,\beta), 1 \leq k \leq n$, the zeros of $Q_n^{(\alpha,\beta)}(x)$, and its order is arranged as

$$1 > x_1^{(n)}(\alpha,\beta) > x_2^{(n)}(\alpha,\beta) > \cdots > x_n^{(n)}(\alpha,\beta) > 0.$$

Let $\phi \in L^p_{W_{\alpha,\beta}}[0,1], 1 \leq p < +\infty$. Then, in [4] a kind of translation operators $S_t^{(\alpha,\beta)}(\phi,x)$ are defined as

$$S_t(\phi,x) := S_t^{(\alpha,\beta)}(\phi,x) = \sum_{k=0}^{\infty} a_k^{(\alpha,\beta)}(\phi) h_k^{(\alpha,\beta)} Q_k^{(\alpha,\beta)}(x) Q_k^{(\alpha,\beta)}(t), \quad x,t \in [0,1].$$

Set

$$\triangle_{\phi}^{(\alpha,\beta)}(x) := \{S_x(\phi,t): \ t\in[0,1]\}\cup\{1\}, \quad x\in[0,1], \phi\in L^p_{W_{\alpha,\beta}}[0,1],$$

then $\triangle_{\phi}^{(\alpha,\beta)}(x)$ is a kind of translation networks which is a new kind of function class formed by $S_t(\phi,\, x)$.

In the present paper, we shall show that, besides the interesting properties (1)-(2), $V_n^{(\alpha,\beta)}(f,x)$ are also good tools in constructing translation network $\triangle_{\phi}^{(\alpha,\beta)}(x)$. In fact, a kind of translation operators of the type $\triangle_{\phi}^{(\alpha,\beta)}(x)$ is constructed with the help of the property (2) and its degree of approximation in the weighted L^p space is investigated.

For $L^p_{W_{\alpha,\beta}}[0,1]$, $1\le p<+\infty$, we define the subspace

$$D^p_{W_{\alpha,\beta}}[0,1] = \{f\in L^p_{W_{\alpha,\beta}}[0,1]: f, f' \quad \text{are local absolutely continuous on} \quad [0,1],$$

$[W_{1+\alpha,1+\beta}f'](x)\to 0$, as $x\to 0,1$, and $[W_{1+\alpha,1+\beta}f']'/W_{\alpha,\beta}\in L^p_{W_{\alpha,\beta}}[0,1]\}$.

Furthermore, we define the linear operator U on $L^p_{W_{\alpha,\beta}}[0,1]$, via $Uf=[W_{1+\alpha,1+\beta}f']'/W_{\alpha,\beta}$. U is a closed operator for all f in its dense domain $D^p_{W_{\alpha,\beta}}[0,1]$.

For $f\in D^p_{W_{\alpha,\beta}}[0,1]$ we equip with the seminorms $\|U(f)\|_{p,W_{\alpha,\beta}}$ and define the Peetre K-modulus $K(f,t;L^p_{W_{\alpha,\beta}},D^p_{W_{\alpha,\beta}})$, $1\le p<+\infty$, by

$$K(f,t;L^p_{W_{\alpha,\beta}},D^p_{W_{\alpha,\beta}}) = \inf_{g\in D^p_{W_{\alpha,\beta}}}(\|f-g\|_{p,W_{\alpha,\beta}}+t\|U(g)\|_{p,W_{\alpha,\beta}}), \quad t>0.$$

Let $N,n\in\mathbf{N}_0$ with $N\ge n$. Let $f\in L^1_{W_{\alpha,\beta}}[0,1]$, and $\phi\in L^1_{W_{\alpha,\beta}}[0,1]$ satisfies $a_k^{(\alpha,\beta)}(\phi)\neq 0$, $k=1,2,\cdots$. Define

$$\begin{aligned} V_{n,N,\phi}^{(\alpha,\beta)}(f,x) &= \lambda_{0,n}^{(\alpha,\beta)}a_0^{(\alpha,\beta)}(f)h_0^{(\alpha,\beta)}Q_0^{(\alpha,\beta)}(x) \\ &\quad+\sum_{l=1}^{N+n}\Big(\sum_{k=1}^{n}\frac{a_k^{(\alpha,\beta)}(V_n^{(\alpha,\beta)}(f))h_k^{(\alpha,\beta)}}{\lambda_{k,n}^{(\alpha,\beta)}a_k^{(\alpha,\beta)}(\phi)}Q_k^{(\alpha,\beta)}(x_l^{(N+n)}(\alpha,\beta))\Big) \\ &\quad\times S_x(\phi,x_l^{(N+n)}(\alpha,\beta))\,\lambda_l^{(N+n)}(\alpha,\beta), \qquad x\in[0,1], \end{aligned}$$

then, $V_{n,N,\phi}^{(\alpha,\beta)}(f,x)\in\triangle_{\phi}^{(\alpha,\beta)}(x)$.

Theorem 1. *Let $N,n\in\mathbf{N}_0$ with $N\ge n\ge 1$, and $\alpha,\beta>-1$. If $f\in L^p_{W_{\alpha,\beta}}[0,1]$, and $\phi\in L^p_{W_{\alpha,\beta}}[0,1]$ satisfies $a_k^{(\alpha,\beta)}(\phi)\neq 0, k=1,2,\cdots$. Then, for $1\le p<+\infty$ there holds*

$$\begin{aligned} \|V_{n,N,\phi}^{(\alpha,\beta)}(f)-f\|_{p,W_{\alpha,\beta}} &\le C\Big[K(f,\frac{1}{n+1};L^p_{W_{\alpha,\beta}},D^p_{W_{\alpha,\beta}}) \\ &\quad+\frac{n^{\sigma}\|f\|_{p,W_{\alpha,\beta}}\,K(\phi,\frac{1}{N+1};L^p_{W_{\alpha,\beta}},D^p_{W_{\alpha,\beta}})}{\phi_n}\Big], \end{aligned}$$

where $\phi_n=\min_{0\le k\le n}|\lambda_{k,n}^{(\alpha,\beta)}a_k^{(\alpha,\beta)}(\phi)|$, $\sigma=\dfrac{1}{\min(p,2)}$.

2 Some Lemmas

Lemma 1. *Let $\alpha,\beta>-1$, and let $f\in L^p_{W_{\alpha,\beta}}[0,1], 1\le p<+\infty$. Then there exists a positive constant depending only on α,β, and p such that for all $n\in\mathbf{N}_0$*

$$\|V_n^{(\alpha,\beta)}(f)-f\|_{p,W_{\alpha,\beta}}\le \text{const}\, K\left(f,\frac{1}{n+1};L^p_{W_{\alpha,\beta}},\, D^p_{W_{\alpha,\beta}}\right),\quad 1\le p<+\infty.$$

Proof. See [1,2].

Lemma 2. *We have the following Gauss integral formula and Marcinkiewicz-Zygmund inequality*

$$\int_0^1 p(x)W_{\alpha,\beta}(x)\mathrm{d}x=\sum_{k=1}^{n}p(x_k^{(n)}(\alpha,\beta))\lambda_k^{(n)}(\alpha,\beta),\qquad p\in P_n,$$

where $\lambda_k^{(n)}(\alpha,\beta)$ are the Cotes numbers about $Q_n^{(\alpha,\beta)}(x)$, and moreover, there exists a constant C depending on p,α,β, such that for $1\le p<+\infty$,

$$\sum_{k=1}^{n}\left|p(x_k^{(n)}(\alpha,\beta))\right|^p\lambda_k^{(n)}(\alpha,\beta)\le C\int_0^1\left|p(u)\right|^p W_{\alpha,\beta}(u)\,\mathrm{d}u,\qquad p\in P_n.$$

Proof. See [3,5].

Lemma 3. *Let $N,n\in\mathbf{N}_0$ with $N\ge n\ge 1$. If $f\in L^p_{W_{\alpha,\beta}}[0,1]$, and $\phi\in L^p_{W_{\alpha,\beta}}[0,1]$ satisfies $a_k^{(\alpha,\beta)}(\phi)\ne 0,\ k=1,2,\cdots$. Then*

$$\begin{aligned}V_n^{(\alpha,\beta)}(f,x) &= \lambda_{0,n}^{(\alpha,\beta)}a_0^{(\alpha,\beta)}(f)h_0^{(\alpha,\beta)}Q_0^{(\alpha,\beta)}(x)\\ &\quad+\sum_{l=1}^{N+n}\left[\sum_{k=1}^{n}\frac{a_k^{(\alpha,\beta)}(V_n^{(\alpha,\beta)}(f))h_k^{(\alpha,\beta)}}{\lambda_k^{(\alpha,\beta)}a_k^{(\alpha,\beta)}(\phi)}\right]Q_k^{(\alpha,\beta)}(x_l^{(N+n)}(\alpha,\beta))\\ &\quad\times S_x(V_N^{(\alpha,\beta)}(\phi),x_l^{(N+n)}(\alpha,\beta))\ \lambda_l^{(N+n)}(\alpha,\beta),\qquad x\in[0,1].\end{aligned}$$

Proof. Obviously,

$$\int_0^1 S_t(\phi,x)Q_k^{(\alpha,\beta)}(x)W_{\alpha,\beta}(x)\mathrm{d}x=a_k^{(\alpha,\beta)}(\phi)Q_k^{(\alpha,\beta)}(t),\qquad k=1,2,\cdots,$$

which implies for $k=1,2,\cdots,N$,

$$\begin{aligned}\int_0^1 S_t(V_N^{(\alpha,\beta)}(\phi),x)Q_k^{(\alpha,\beta)}(x)W_{\alpha,\beta}(x)\mathrm{d}x &= a_k^{(\alpha,\beta)}(V_N^{(\alpha,\beta)}(\phi))\ Q_k^{(\alpha,\beta)}(t)\\ &= \lambda_k^{(\alpha,\beta)}a_k^{(\alpha,\beta)}(\phi)Q_k^{(\alpha,\beta)}(t),\end{aligned}$$

or

$$Q_k^{(\alpha,\beta)}(t)=\frac{1}{\lambda_k^{(\alpha,\beta)}a_k^{(\alpha,\beta)}(\phi)}\int_0^1 S_t(V_N^{(\alpha,\beta)}(\phi),x)Q_k^{(\alpha,\beta)}(x)W_{\alpha,\beta}(x)\mathrm{d}x.$$

Hence

$$\begin{aligned} V_n^{(\alpha,\beta)}(f,x) &= \lambda_{0,n}^{(\alpha,\beta)} a_0^{(\alpha,\beta)}(f) h_0^{(\alpha,\beta)} Q_0^{(\alpha,\beta)}(x) \\ &\quad + \sum_{k=1}^{n} \frac{a_k^{(\alpha,\beta)}(V_n^{(\alpha,\beta)}(f))\, h_k^{(\alpha,\beta)}}{\lambda_k^{(\alpha,\beta)} a_k^{(\alpha,\beta)}(\phi)} \\ &\quad \times \int_0^1 S_x(V_N^{(\alpha,\beta)}(\phi),t)\, Q_k^{(\alpha,\beta)}(t) W_{\alpha,\beta}(t)\mathrm{d}t, \qquad x \in [0,1]. \end{aligned}$$

Since $Q_k^{(\alpha,\beta)}(\cdot) \in P_k, S_t(V_N^{(\alpha,\beta)}(\phi),\cdot) \in P_N$, we know $S_t(V_N^{(\alpha,\beta)}(\phi),\cdot) Q_k^{(\alpha,\beta)}(\cdot) \in P_{N+k} \subset P_{N+n}$. It follows by Lemma 2

$$\begin{aligned} \int_0^1 S_x(V_N^{(\alpha,\beta)}(\phi),u) Q_k^{(\alpha,\beta)}(u) W_{\alpha,\beta}(u)\mathrm{d}u &= \sum_{l=1}^{N+n} S_x(V_N^{(\alpha,\beta)}(\phi),\, x_l^{(N+n)}(\alpha,\beta)) \\ &\quad \times Q_k^{(\alpha,\beta)}(x_l^{(N+n)}(\alpha,\beta)) \lambda_l^{(N+n)}(\alpha,\beta). \end{aligned}$$

Lemma 3 is thus proved.

Lemma 4. *Let $\phi \in L^p_{W_{\alpha,\beta}}[0,1], 1 \le p < +\infty$. Then*

$$S_t(\phi,x) = S_x(\phi,t), \qquad x,\ t \in [0,1],$$

and there exists a constant $C > 0$ independent of t and ϕ such that

$$\|S_t(\phi)\|_{p,W_{\alpha,\beta}} \le C\|\phi\|_{p,W_{\alpha,\beta}}.$$

Proof. See [4].

3 The Proof of Theorem 1

Obviously,

$$\begin{aligned} |V_{n,N,\phi}^{(\alpha,\beta)}(f,x) - f(x)| &\le |V_{n,N,\phi}^{(\alpha,\beta)}(f,x) - V_n^{(\alpha,\beta)}(f,x)| + |V_n^{(\alpha,\beta)}(f,x) - f(x)| \\ &\le \sum_{l=1}^{N+n} \sum_{k=1}^{n} \left| \frac{a_k^{(\alpha,\beta)}(V_n^{(\alpha,\beta)}(f)) h_k^{(\alpha,\beta)}}{\lambda_k^{(\alpha,\beta)} a_k^{(\alpha,\beta)}(\phi)} \right| |Q_k^{(\alpha,\beta)}(x_l^{(N+n)}(\alpha,\beta))| \\ &\quad \times S_x(|\phi - V_N^{(\alpha,\beta)}(\phi)|, x_l^{(N+n)}(\alpha,\beta)) \lambda_l^{(N+n)}(\alpha,\beta) + |V_n^{(\alpha,\beta)}(f,x) - f(x)| \\ &= \sum_{l=1}^{N+n} \sum_{k=1}^{n} \left| \frac{a_k^{(\alpha,\beta)}(V_n^{(\alpha,\beta)}(f)) h_k^{(\alpha,\beta)}}{\lambda_k^{(\alpha,\beta)} a_k^{(\alpha,\beta)}(\phi)} \right| |Q_k^{(\alpha,\beta)}(x_l^{(N+n)}(\alpha,\beta))| \\ &\quad \times S_{x_l^{(N+n)}(\alpha,\beta)}(|\phi - V_N^{(\alpha,\beta)}(\phi)|, x) \lambda_l^{(N+n)}(\alpha,\beta) + |V_n^{(\alpha,\beta)}(f,x) - f(x)|, \quad x \in [0,1]. \end{aligned}$$

Therefore

$$\begin{aligned} \|V_{n,N,\phi}^{(\alpha,\beta)}(f) - V_n^{(\alpha,\beta)}(f)\|_{p,W_{\alpha,\beta}} &\le \sum_{l=1}^{N+n} \Big[\sum_{k=1}^{n} \left| \frac{a_k^{(\alpha,\beta)}(V_n^{(\alpha,\beta)}(f)) h_k^{(\alpha,\beta)}}{\lambda_k^{(\alpha,\beta)} a_k^{(\alpha,\beta)}(\phi)} \right| |Q_k^{(\alpha,\beta)}(x_l^{(N+n)}(\alpha,\beta))| \Big] \\ \times \Big(\int_0^1 \Big| S_{x_l^{(N+n)}(\alpha,\beta)}(\Big|\phi - V_N^{(\alpha,\beta)}(\phi)\Big|, x) \Big|^p W_{\alpha,\beta}(x)\, dx \Big)^{\frac{1}{p}} &\lambda_l^{(N+n)}(\alpha,\beta) + \|V_n^{(\alpha,\beta)}(f) - f\|_{W_{\alpha,\beta}}. \end{aligned}$$

By Lemma 4 we have

$$\begin{aligned}
&\sum_{l=1}^{N+n}\Big[\sum_{k=1}^{n}\Big|\frac{a_k^{(\alpha,\beta)}(V_n^{(\alpha,\beta)}(f))h_k^{(\alpha,\beta)}}{\lambda_k^{(\alpha,\beta)}a_k^{(\alpha,\beta)}(\phi)}\Big||Q_k^{(\alpha,\beta)}(x_l^{(N+n)}(\alpha,\beta))|\Big]\\
&\times\Big(\int_0^1\Big|S_{x_l^{(N+n)}(\alpha,\beta)}(|\phi-V_N^{(\alpha,\beta)}(\phi)|,x)\Big|^pW_{\alpha,\beta}(x)\mathrm{d}x\Big)^{\frac1p}\lambda_l^{(N+n)}(\alpha,\beta)\\
\le\ &\|\phi-V_N^{(\alpha,\beta)}(\phi)\|_{p,W_{\alpha,\beta}}\sum_{l=1}^{N+n}\sum_{k=1}^{n}\Big|\frac{a_k^{(\alpha,\beta)}(V_n^{(\alpha,\beta)}(f))h_k^{(\alpha,\beta)}}{\lambda_k^{(\alpha,\beta)}a_k^{(\alpha,\beta)}(\phi)}\Big|\\
&\times|Q_k^{(\alpha,\beta)}(x_l^{(N+n)}(\alpha,\beta))|\ \lambda_l^{(N+n)}(\alpha,\beta)\\
\le\ &\|\phi-V_N^{(\alpha,\beta)}(\phi)\|_{p,W_{\alpha,\beta}}\sum_{k=1}^{n}\Big|\frac{a_k^{(\alpha,\beta)}(V_n^{(\alpha,\beta)}(f))h_k^{(\alpha,\beta)}}{\lambda_k^{(\alpha,\beta)}a_k^{(\alpha,\beta)}(\phi)}\Big|\\
&\times\int_0^1|Q_k^{(\alpha,\beta)}(u)|W_{\alpha,\beta}(u)\mathrm{d}u\\
\le\ &\frac{\|\phi-V_N^{(\alpha,\beta)}(\phi)\|_{p,W_{\alpha,\beta}}}{\phi_n}\sum_{k=1}^{n}\Big|a_k^{(\alpha,\beta)}(V_n^{(\alpha,\beta)}(f))h_k^{(\alpha,\beta)}\Big|\\
&\times\Big(\int_0^1\Big|Q_k^{(\alpha,\beta)}(u)\Big|^2W_{\alpha,\beta}(u)\,du\Big)^{\frac12}\Big(\int_0^1W_{\alpha,\beta}(u)\mathrm{d}u\Big)^{\frac12}\\
\le\ &\frac{C\|\phi-V_N^{(\alpha,\beta)}(\phi)\|_{p,W_{\alpha,\beta}}}{\phi_n}\sum_{k=1}^{n}\Big|a_k^{(\alpha,\beta)}(V_n^{(\alpha,\beta)}(f))\Big|\,(h_k^{(\alpha,\beta)})^{\frac12}.
\end{aligned}$$

Since $\{(h_k^{(\alpha,\beta)})^{\frac12}Q_k^{(\alpha,\beta)}(x)\}_{k=0}^{+\infty}$ forms an orthonormal basis, we have by the Parseval formula

$$\Big(\sum_{k=1}^{n}\Big|a_k^{(\alpha,\beta)}(V_n^{(\alpha,\beta)}(f))\Big|^2h_k^{(\alpha,\beta)}\Big)^{\frac12}=\Big(\int_0^1\Big|V_n^{(\alpha,\beta)}(f,x)\Big|^2W_{\alpha,\beta}(x)\mathrm{d}x\Big)^{\frac12}.$$

Therefore, by the Nikolskii inequality (see [6])

$$\|p\|_{p,W_{\alpha,\beta}}\le Cn^{(\frac{1}{p'}-\frac1p)_+}\|p\|_{p',W_{\alpha,\beta}},\qquad 1\le p,\ \ p'\le+\infty,\quad p\in P_n,$$

(here $(a)_+=\max(a,0)$) we have

$$\begin{aligned}
&\sum_{l=1}^{N+n}\sum_{k=1}^{n}\Big|\frac{a_k^{(\alpha,\beta)}(V_n^{(\alpha,\beta)}(f))h_k^{(\alpha,\beta)}}{\lambda_k^{(\alpha,\beta)}a_k^{(\alpha,\beta)}(\phi)}\Big||Q_k^{(\alpha,\beta)}(x_l^{(N+n)}(\alpha,\beta))|\\
&\quad\times\Big(\int_0^1\Big|S_{x_l^{(N+n)}(\alpha,\beta)}(|\phi-V_N^{(\alpha,\beta)}(\phi)|,x)\Big|^pW_{\alpha,\beta}(x)\mathrm{d}x\Big)^{\frac1p}\lambda_l^{(N+n)}(\alpha,\beta)
\end{aligned}$$

$$\begin{aligned}
&\le \frac{C\|\phi - V_N^{(\alpha,\beta)}(\phi)\|_{p,W_{\alpha,\beta}}}{\phi_n}\sum_{k=0}^{n}\left|a_k^{(\alpha,\beta)}(V_n^{(\alpha,\beta)}(f))\right|(h_k^{(\alpha,\beta)})^{\frac{1}{2}}\\
&\le \frac{Cn^{\frac{1}{2}}\|\phi - V_N^{(\alpha,\beta)}(\phi)\|_{p,W_{\alpha,\beta}}}{\phi_n}\Big(\int_0^1\left|V_n^{(\alpha,\beta)}(f,u)\right|^2 W_{\alpha,\beta}(u)\mathrm{d}u\Big)^{\frac{1}{2}}\\
&\le \frac{Cn^{\sigma}\|\phi - V_N^{(\alpha,\beta)}(\phi)\|_{p,W_{\alpha,\beta}}}{\phi_n}\times\|V_n^{(\alpha,\beta)}(f)\|_{p,W_{\alpha,\beta}}\\
&\le \frac{Cn^{\sigma}\|\phi - V_N^{(\alpha,\beta)}(\phi)\|_{p,W_{\alpha,\beta}}}{\phi_n}\times\|f\|_{p,W_{\alpha,\beta}}\\
&\le \frac{Cn^{\sigma}\|f\|_{W_{\alpha,\beta}}K(\phi,\frac{1}{N+1},L^p_{W_{\alpha,\beta}},D^p_{W_{\alpha,\beta}})}{\phi_n}.
\end{aligned}$$

4 The Applications

Obviously, for $\alpha = \beta = 0$ we obtain the Durrmeyer's operators

$$M_n(f,x) = (n+1)\sum_{k=0}^{n} p_{n,k}(x)\int_0^1 f(u)p_{n,k}(u)\mathrm{d}u, \quad x\in[0,1],\ f\in L^1([0,1]),$$

$Q_m^{(0,0)}(x)$ reduce to the Legendre polynomials of degree m, i.e.,

$$Q_m(x) = \begin{cases} \dfrac{\sqrt{2m+1}}{m!}\dfrac{\mathrm{d}^m}{\mathrm{d}x^m}\big[x(1-x)\big]^m, & m\ge 1,\\ 1, & m=0.\end{cases}$$

$\|f\|_{p,W_{\alpha,\beta}}$ reduces to $\|f\|_p = \left(\int_0^1\left|f(x)\right|^p\mathrm{d}x\right)^{\frac{1}{p}}$, $\lambda_k^{(\alpha,\beta)}$ reduce to

$$\lambda_k = \frac{n!}{(n-k)!}\frac{(n+1)!}{(n+k+1)!},$$

and $h_k^{(\alpha,\beta)}$ reduces to $\dfrac{2k+1}{4}, k=0,1,\cdots,n.$

W. Chen, Z. Ditzian and K. Ivanov showed(see [7])

$$\|M_n(f)-f\|_p \sim K(f,n^{-1})_p, \qquad f\in L^p[0,1],$$

where

$$K_2(f,n^{-1}) = \inf_{g\in C^2[0,1]}\left(\|f-g\|_p + n^{-1}\|P(D)g\|_p\right), \qquad 1\le p<+\infty,$$

and

$$P(D) = \frac{\mathrm{d}}{\mathrm{d}x}x(1-x)\frac{\mathrm{d}}{\mathrm{d}x}.$$

Let $x_l^{(n)}$, $l=1,2,\cdots,n$, be the zeros of $Q_n(x)$. Then, the Cotes numbers $\lambda_l^{(n)}$, $l=1,2,\cdots,n$, are

$$\lambda_l^{(n)}=\frac{2\sqrt{x_l^{(n)}(1-x_l^{(n)})}}{n},\qquad l=1,2,\cdots,n.$$

The translation operators $S_t^{(\alpha,\beta)}(\phi,x)$ for a given t therefore reduces to (see [8])

$$\begin{aligned}S_t(\phi,x) &= \int_0^1\phi\Big[(2x-1)(2t-1)+4(2u-1)\sqrt{x(1-x)t(1-t)}\Big]\\ &\quad\times[u(1-u)]^{-\frac12}\Big[\int_0^1[y(1-y)]^{-\frac12}dy\Big]^{-1}\mathrm{d}u,\qquad x,t\in[0,1].\end{aligned}$$

Set $a_k(f)=\int_0^1 f(u)Q_k(u)\mathrm{d}u$, then $V_{n,N,\phi}^{(\alpha,\beta)}(f)$ reduce to

$$\begin{aligned}V_{n,N,\phi}(f,x) &= a_0(f)+\sum_{l=1}^{N+n}\Big(\sum_{k=1}^{n}\frac{a_k(M_n(f))}{a_k(\phi)}\times\frac{(n-k)!(n+k+1)!(2k+1)}{4n!(n+1)!}\\ &\quad\times Q_k(x_l^{(N+n)})\Big)\int_0^1\phi[(2x-1)(2x_l^{(N+n)}-1)\\ &\quad+4(2u-1)\sqrt{x(1-x)x_l^{(N+n)}(1-x_l^{(N+n)})}][u(1-u)]^{-\frac12}\\ &\quad\times\Big[\int_0^1[y(1-y)]^{-\frac12}\mathrm{d}y\Big]^{-1}\mathrm{d}u\times\frac{\sqrt{x_l^{(N+n)}(1-x_l^{(N+n)})}}{n},\qquad x\in[0,1].\end{aligned}$$

In this case, Theorem 1 reduces to the following Theorem 2.

Theorem 2. *Let $N,n\in\mathbf{N}_0$ with $N\ge n\ge 1$. If $f\in L^p[0,1]$, and $\phi\in L^p[0,1]$ satisfies $a_k(\phi)\ne 0,k=1,2,\cdots$. Then, for $1\le p<+\infty$ there holds*

$$\|V_{n,N,\phi}(f)-f\|_p \le C\Big[K(f,n^{-1})_p+\frac{n^{\sigma}\|f\|_pK(\phi,N^{-1})_p}{\phi_n}\Big].$$

If $\alpha=\beta=-\frac{1}{2}$, then $V_n^{(\alpha,\beta)}(f,x)$ reduce to

$$V_n^{(-\frac12)}(f,x)=\sum_{k=0}^{n}C_{n,k}^{-1}\int_0^1\frac{p_{n,k}(u)f(u)}{\sqrt{u(1-u)}}\mathrm{d}u\,p_{n,k}(x),\qquad x\in[0,1],$$

where

$$C_{n,k}^{-1} = \int_0^1\frac{p_{n,k}(u)}{\sqrt{u(1-u)}}\mathrm{d}u,$$

$Q_k^{(\alpha,\beta)}(x)$ therefore reduce to $T_k(x)=\cos k\arccos(2x-1), x\in[0,1], \lambda_{k,n}^{(\alpha,\beta)}$ reduce to

$$\lambda_{k,n}^{(-\frac12)}=\frac{(n!)^2}{(n-k)!(n+k)!},$$

and $h_k^{(\alpha,\beta)}$ reduce to $\frac{4}{\pi}$. The Cotes number $\lambda_k^{(n)}(\alpha,\beta)$ reduce to $\frac{1}{n}$.

The corresponding translation operators thus reduce to(see [8])

$$\begin{aligned} S_t^{(-\frac{1}{2})}(\phi,x) &= \frac{1}{2}\Big(\phi\Big[(2x-1)(2t-1)+4\sqrt{x(1-x)t(1-t)}\Big] \\ &\quad +\phi\Big[(2x-1)(2t-1)-4\sqrt{x(1-x)t(1-t)}\Big]\Big), \qquad x,t\in[0,1]. \end{aligned}$$

Set $c_k(f)=\int_0^1 \frac{f(x)T_k(x)}{\sqrt{x(1-x)}}\mathrm{d}x$, then $V_{n,\phi}^{(\alpha,\beta)}(f,x)$ reduce to

$$\begin{aligned} V_{n,N,\phi}^{(-\frac{1}{2})}(f,x) &= \frac{2c_0(f)}{\pi}+\frac{1}{n}\sum_{l=1}^{N+n}\Big(\sum_{k=1}^{n}\frac{c_k(V_n^{(-\frac{1}{2})}(f))}{c_k(\phi)} \\ &\quad \times\frac{2(n!)^2}{\pi(n-k)!(n+k)!}\times\cos k\arccos(2z_l^{(N+n)}-1)\Big) \\ &\quad \times\Big(\phi\Big[(2x-1)(2z_l^{(N+n)}-1) \\ &\quad +4\sqrt{x(1-x)z_l^{(N+n)}(1-z_l^{(N+n)})}\Big] \\ &\quad +\phi\Big[(2x-1)(2z_l^{(N+n)}-1) \\ &\quad -4\sqrt{x(1-x)z_l^{(N+n)}(1-z_l^{(N+n)})}\Big]\Big), \qquad x\in[0,1], \end{aligned}$$

where $z_k^{(n)}=\frac{1}{2}\left(\cos\frac{(2k-1)\pi}{N+n}+1\right), k=1,2,\cdots,n$, are the zeros of $T_n(x)$.

In this case, Theorem 1 reduces to the following Theorem 3.

Theorem 3. *Let $N,n\in\mathbf{N}_0$ with $N\ge n\ge 1$. If $f\in L^p_{W_{-\frac{1}{2}}}[0,1]$, and $\phi\in L^p_{W_{-\frac{1}{2}}}[0,1]$ satisfies $c_k(\phi)\neq 0, k=1,2,\cdots$. Then for $1\le p<+\infty$ there holds*

$$\begin{aligned} \|V_{n,N,\phi}^{(-\frac{1}{2})}(f)-f\|_{p,W_{-\frac{1}{2}}} &\le C\Big[K(f,\frac{1}{n+1};L^p_{W_{-\frac{1}{2}}},D^p_{W_{-\frac{1}{2}}}) \\ &\quad +\frac{n^\sigma\|f\|_{p,W_{-\frac{1}{2}}}K(\phi,\frac{1}{N+1};L^p_{W_{-\frac{1}{2}}},D^p_{W_{-\frac{1}{2}}})}{\phi_n}\Big], \end{aligned}$$

where $W_{-\frac{1}{2}}(x)=(1-x^2)^{-\frac{1}{2}}$.

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