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RESEARCH

The Conjecture of a General Law of the Wall for Classical Turbulence Models, Implying a Structural Limitation

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Abstract

We call classical a transport model in which each governing equation comprises a production term proportional to velocity gradients and terms such as diffusion and dissipation which are built from the internal quantities of the model and are local. They may depend on the wall-normal coordinate y. We consider the layer along a wall in which the total shear stress is uniform, and y is much smaller than the thickness of the full wall layer. We only use channels and boundary layers, but have seen no evidence that pipe flow is different. The conjectured General Law of the Wall (GLW), in contrast with the classical law for mean velocity U only, states that every quantity Q in the model (e.g., dissipation, stresses) is the product of four quantities: powers of the friction velocity u_τ and y which satisfy dimensional analysis; a constant C characteristic of the model; and a function f of the wall distance y in wall units, which closely approaches 1 outside the viscous and buffer layers. This is independent of any flow Reynolds number such as the friction Reynolds number in a channel, once it is large enough, and it rigidly constrains the y-dependence of Q outside the wall region: in particular, all the stresses are on plateaus. In the widely accepted velocity law of the wall, the shear rate dU/dy satisfies such a law with C the inverse of the Karman constant κ. We cannot prove the GLW property as a theorem, but we provide extensive arguments to the effect that any Classical equation set allows it, and many numerical results support it. A Structural Limitation any Classical Model would suffer from then arises because the results of experiments (not shown here) and Direct Numerical Simulations contradict the GLW, already for some of the Reynolds stresses in simple flows and all the way to the wall (the conflict between the GLW as predicted by Classical turbulence theory, on which the models are based, and measurements was discussed by Townsend as early as Townsend in J Fluid Mech 11:97–120, 1961). This implies that no modification of a model that remains within the classical type can make it agree closely with this key body of results. This has been tolerated for decades, but the GLW is stated here more precisely than it has been implicitly in the literature, it extends all the way to the wall, and it has theoretical interest. It creates a danger for the developing "data-driven" efforts based on Machine Learning in turbulence modelling, which generally involve all six Reynolds stresses and possibly other quantities such as budget terms.

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1 Introduction

The creation of turbulence modelling approximations for the Reynolds-Averaged Navier–Stokes (RANS) equations combines physical intuition, theory, and mathematical planning. It is a century old, remembering Prandtl's mixing-length theory, and its evolution from algebraic to transport models has been driven more by the growth of boundary-layer and then Navier–Stokes CFD codes than by new physical thinking. The boundary between Turbulence Theory and Turbulence Modelling is faint; Theory may state that "there exists a constant" while Modelling must propose a value for the constant.

The community has the sense of a stagnation which would have started in the 1990's, from the observation that models up to two equations have not evolved in depth, and Reynolds-Stress models are more available and better understood, but do not achieve the radical improvement in accuracy which was expected beginning in the 1970's. All the models have a large number of terms and constants, and the intellectual effort needed to master and truly improve them is considerable. Partly for this reason, in the 2020's there are many calls to apply Machine Learning (ML). Another reason is the availability of large DNS datasets for non-trivial flows, for which the power of computers and ML is expected to permit the assessment and improvement of models grounded on cases much more varied than the small sets of Thin Shear Layers used in the 1990's (modern experimental measurement systems also provide rich databases). This ML work would, in particular, involve all six Reynolds stresses instead of only the shear stress in simple shear flows, and the question of whether this would be successful is a strong motivation for the present study. We now proceed with formulating our conjecture.

Unusually in turbulence, the result presented here is analytical and fits in a single concrete equation, already described in words in the abstract:

$$Q = f_Q\left(y^+\right) C_Q u_\tau^{\alpha_Q} y^{\beta_Q} \tag{1}$$

where Q is any quantity in the model, α_Q and β_Q ensure dimensional analysis, C_Q is a constant of the model, and f_Q is a nondimensional function of the model which reaches 1 outside the buffer layer. As usual $u_\tau \equiv \sqrt{\nu dU/dy_{wall}}$ is the friction velocity and $y^+ \equiv y u_\tau / \nu$ the wall distance in wall units. Equation (1) applies in the Constant-Stress Layer (CSL).

This GLW is a conjecture: we do not have a theorem, but we have detailed arguments in favour of it, and very supportive numerical results for four models. Unfortunately, it leads to a negative result, a limitation. The motivation however is to pre-empt research efforts, especially in Machine Learning, which would have no chance of full success and much potential for overfitting. There is also the remote possibility of inspiring the creation of a model that is not "classical" and is free of the Structural Limitation associated with the GLW.

Comments are in order regarding the word "conjecture." No flow is accessible to DNS or experiment that does not have an external scale, such as channel width or boundary-layer thickness, and we can base a Reynolds number Re on such a length and u_τ. At a given y^+, all results depend on Re. A more precise statement of the GLW is that RANS-model predictions rapidly approach (1) as $Re \to \infty$ (the deviation plausibly scaling like Re^{-1})

whereas the true behaviours approach it far more slowly, or fail to approach it altogether, instead tending to infinity.

Some of the facts claimed here are known to the competent experts in modelling (such as the property $k^+ = 1/\sqrt{c_\mu}$ of the k-ϵ model in the logarithmic layer) but there are very few such experts, and in addition we are introducing three new elements. First, the GLW would apply to any quantity Q, and not only to k. Second, it would apply all the way to the wall, instead of only outside the viscous and buffer layers, thanks to the f functions. Third, we are proposing an explicit definition of Classical Turbulence Models, which will be examined, and claiming that it applies to all the known transport models and also to any that are likely to be proposed in the near future, barring a breakthrough in physics (Durbin and Pettersson Reif 2011, Rumsey 2022). We begin with this key part of the work, then cover the CSL, formulate the GLW and apply it to the k-ϵ model, verify it numerically on four different models, and then examine whether DNS results satisfy the GLW or disprove it, thus creating a conflict.

2 Classical Turbulence Models

The framework by which modelled turbulence evolves due to Production, Diffusion and Source Terms is essentially universal (we emphasize that we are considering transport models, as opposed to the algebraic boundary-layer models which preceded them in CFD). It emulates the exact transport equations for the Reynolds Stresses, as described by Hanjalić and Launder (2023) and others, which can make the form of the models appear self-evident.

However, writing equations for one-point moments of the turbulence or for artificial quantities such as an eddy viscosity does not constitute a rigorous step, and the ansatz is somewhat driven by the mundane requirements of turbulent CFD. An obvious physical objection is that pressure effects are not local, yet the terms that approximate them are local. This local/non-local fact has sometimes been called the Fundamental Paradox of Turbulence Modelling and alone could lead to the conclusion that modelling is hopeless (Spalart 2015). Another prominent debate in modelling practice is that some useful models include in the equation at a field point its distance y to the wall. This is of course not emulating any term in the stress-transport equation; it does however have an empirical physical motivation, precisely based on the non-local influence of the wall. It will be seen that the GLW is not affected by the presence of y. Models involving an Elliptic Relaxation have a similar motivation but a very different construction (Durbin and Pettersson Reif 2011).

In this paper, the letter Q can designate any quantity related to the model, from the eddy viscosity to the pressure term for the stress $< v'w' >$, or any others and including derived quantities such as k^2/ϵ and functions of nondimensional combinations. Although we have found no contradictions yet, for any Q the GLW could be readily falsified by theory, or in numerical solutions of the model equations, or in real facts obtained from experiments or simulations. Such a falsification would be an exciting consequence of the present paper.

We examine the terms type by type. In models with k as the first variable, the Production normally is the true production of Turbulent Kinetic Energy; however, some versions of the SST model, for instance, use an analogous quantity built from the vorticity instead of the strain; this has better behaviour outside the turbulent vortical region (Rumsey 2022). We also note how k and ϵ are well-defined and measurable quantities, but the equation which provides the eddy viscosity, namely $\nu_t = c_\mu k^2/\epsilon$, simply comes from dimensional analysis and the consideration that the turbulent Reynolds number $k^2/\nu\epsilon$ is large. In any case, in a Classical Model the production P_k is a combination of some of the Q quantities and the first derivatives of the velocity field, that is the tensor $\partial U_i/\partial x_j$. This is inspired by the stress-transport equations. The production for other quantities is built on the k production and dimensional analysis, for instance $P_\epsilon = c_{\epsilon 1}(\epsilon/k)P_k$.

There is a wide variety of local source terms besides production, usually dominated by dissipation or destruction. Some are directly provided, for instance ϵ in the k equation, but other destruction terms are fully empirical such as the one for ϵ or the one in the SA model (Rumsey 2022). This latter term is "semi-local" since it contains y, but otherwise involves only local Q quantities.

The diffusion terms are not purely local since they involve first and second derivatives of the Qs, and thus depend on a neighbourhood of the field point; this is discussed in detail below. The GLW argument also applies to the viscous diffusion terms, because the molecular viscosity itself satisfies the GLW: indeed we have $\nu = (1/y^+)u_\tau y$ which is of the form of (1). The form of the modelled diffusion terms is quite universal. Exceptions include the c_{b2} term in the SA model, and the cross-diffusion terms in other models, but they satisfy the GLW.

3 The Constant-Stress Layer (CSL) and the Logarithmic Layer

The setting is wall-bounded flows. We suspect that Classical Models can also conflict with measurements or DNS in free shear flows; for instance, in a time-developing mixing layer the velocity difference and the time would play the role of the friction velocity and wall distance here. The model would have a simplistic behaviour at maturity. However, fully clear findings from simulations are made difficult by the issue of initial, inflow, and boundary conditions, and we leave this for future work. Channel flow is free of initial or inflow conditions, and the boundary condition in y is straightforward. There is consensus over the size of the periodic DNS domains in x and z, except maybe in Couette flow. This makes channel flow a solid foundation.

The CSL is an idealization, but the Reynolds numbers now reached in DNS are high enough to approach it closely. The definition of the CSL is of course that the total shear stress is $\tau^+ \equiv \tau/(\rho u_\tau^2) = 1$, but the condition $y \ll h$ where h is the half-width of the tunnel is also required. In Poiseuille flow, $\tau^+ = 1 - y/h$ so the two conditions are equivalent at a fixed value of the flow Reynolds number $Re_\tau \equiv hu_\tau/\nu$, but this is not the case in Couette flow.

In view of the fact that $\tau^+ = 1 - y^+/Re_\tau$, a Reynolds-number dependence scaling with $1/Re_\tau$ would appear normal; it has been called Inverse Reynolds-number Dependence, or IRD (Spalart and Abe 2021). Therefore, the interest here is in Reynolds-number effects which do not follow the IRD, for instance a growth that is logarithmic in Re_τ and therefore unbounded (Smits 2022). Mild asymptotic behaviour such as the IRD apply only to some quantities, and then may depend on whether the behaviour is studied at fixed y^+ or at fixed y/h.

The GLW applies all the way to the wall, but there is great interest in the layer that is free of viscous effects, as expressed by the condition $y^+ \gg 1$. This is when the f functions of the GLW are essentially 1. This property is the foundation of the classical argument in favour of the log law for velocity, and we will also call this the "log layer." In reality, y^+

needs to be well above 100 for this to apply. The full requirement is then $1 \ll y^+ \ll Re_\tau$ which of course demands $Re_\tau \gg 1$.

We can outline this layer with $1 \ll y^+ \ll Re_\tau$ in the latest channel DNS of Hoyas et al. (2022). It has $Re_\tau = 10^4$ and the peak Reynolds shear stress is $- < u'v' >^+ \approx 0.97$ near $y^+ = 156 = 0.0156 Re_\tau$. This is quite consistent with the CSL conditions, and another way to judge the limiting process is to compare results at Reynolds number 5000 and 10,000, for instance. In any case, from here on, h is not involved, being considered essentially infinite, and the total shear stress is set to $\tau^+ = 1$.

4 General Law of the Wall and Mathematical Arguments in its Favour

Understanding the GLW requires extensive notation, and careful discussions of how nontrivial combinations and derivatives of the Q quantities at the core of the model also satisfy the GLW, which is the key to expressing the governing equations in terms of the C_Q constants.

The velocity Law of the Wall enjoys considerable support, and we both use it as the prototype for the GLW and as a foundation for the mathematical arguments. The consensus is that the velocity satisfies $U^+ = f\left(y^+\right)$, *f()* meaning "a function of only," and that for large y^+ we have $U^+ = \frac{\log(y^+)}{\kappa} + C$ where κ and C are (nondimensional) constants of Nature. We switch to the derivative of U, which is a local quantity whereas U itself is a difference relative to the wall velocity. This gives

$$\frac{dU}{dy} = f_{U_y}\left(y^+\right)\frac{u_\tau}{\kappa y} \tag{2}$$

which has the form of the GLW already given in Eq. (1), with $C_{U_y} = 1/\kappa$, and $\alpha_{U_y} = 1$ and $\beta_{U_y} = -1$ as per dimensional analysis. Channel DNS results from different groups now agree very closely on the function f_{U_y} up to $y^+ \approx 100$; beyond 100, disagreements remain and prevent us from determining the exact value of κ, but undulations in the results suggest that the DNS results are not fully converged, especially in terms of the length of the time average. We treat (2) as fully established, with a narrow range of possible values for κ.

As mentioned, we cannot present a theorem proving the GLW. We are making the "soft" argument that it gives the "natural" behaviour of Classical models. However, such a GLW theorem would contain a proof of existence and uniqueness for the RANS equations, which has not been found. The theorem could also be different for each RANS model. It could also be conditional: in particular, establishing the presence of hysteresis for CFD solutions over an aerofoil at high angle of attack would not be a surprise. The situation is that in order to build confidence in the predictability of models the community has studied the consistency of RANS solutions between CFD solvers and under wide variations of grid resolution, as exemplified by the Turbulence Modeling Resource (Rumsey 2022).

Any Classical Model consists in a choice of Q variables such as k and ϵ, constitutive relations such as the $\nu_t = c_\mu k^2/\epsilon$ eddy-viscosity equation, and transport equations. Our approach is to demonstrate that all these building blocks are compatible with the GLW. Then, it is partly a matter of intuition to decide whether the GLW indeed applies, guided by numerical tests with different solvers, different models, and a range of Reynolds numbers. Therefore, we posit that all Q quantities satisfy the GLW, that (2) applies and satisfies the GLW, and consider each transport equation, which in channel flow is written $DQ/Dt = 0$, that is, the RHS of the equation represented by the sum of e.g., Production, Dissipation, and Diffusion is equal to zero. As shown shortly, in Classical Models all these terms satisfy the GLW and so their sum also does, with the constant denoted by $C_{DQ/Dt}$; the $f_{DQ/Dt}$ function can also be calculated. Hence, finding the solution will amount to finding which set of C_Q values will give $C_{DQ/Dt} = 0$ for each Q. Then, the function $f_{DQ/Dt} = 0$ exists but is irrelevant, a fact that was not obvious beforehand. The example of k-ϵ will be helpful.

First, it is easy to see that the product of two quantities which each satisfy the GLW also satisfies the GLW, by simply taking the product of the C constants and the f functions. This is key for the Production term, which is the product of dU/dy which is given by (2) and of quantities from the model itself. This motivates the earlier emphasis on Production as it applies to Classical Models.

Now consider non-dimensional functions, denoted by $g(Q)$ where Q is nondimensional and therefore has $\alpha = \beta = 0$ and is merely $f_Q(y^+)C_Q$. Examples are f_μ in the k-ϵ model and f_{v1} in the SA model. They tend to 1 when their argument, such as Re_t or χ, tends to infinity along with y^+; some models may have functions which tend to values different from 1, but that is absorbed by a C_Q constant. The needed function is then $g(Q) = g(f(y^+))$ and therefore also has the properties of a GLW quantity. This covers nonlinear eddy-viscosity models, including Explicit Algebraic Reynolds-Stress Models (EARSM).

Terms which depend on y are not an issue, since y is a GLW quantity ($\alpha = 0$, $\beta = 1$, $C = 1$, and $f = 1$).

Finally, the Diffusion terms contain y derivatives, which require more attention. Assume Q satisfies the GLW; then

$$\frac{dQ}{dy} = \frac{d}{dy}\left(f_Q(y^+)C_Q u_\tau^{\alpha_Q} y^{\beta_Q}\right) = \left(\beta_Q f_Q + y^+ f'_Q\right)C_Q u_\tau^{\alpha_Q} y^{\beta_Q - 1} \tag{3}$$

which also satisfies the GLW since the quantity in parentheses is a function of y^+.

A minor constraint on these functions is the following: $y^+ f'_Q$ must tend to a constant (often 0) as $y^+ \to \infty$ for the GLW to apply, as seen in (3). Recall that mathematically the condition $\lim_{y^+\to\infty} f_Q = 1$ is not sufficient to ensure that $y^+ f'_Q$ tends to a constant or even remains bounded (M. Strelets, personal communication, 2022). In fact, even the derivative f' itself could be unbounded (think of $\sin(x^3)/x$ as $x \to \infty$). However, only a very narrow class of functions has such a behaviour, and the functions encountered in modelling are not oscillatory or exhibiting steep steps, which are two easy ways to create counterexamples. Many are decaying exponentials (see f_μ) or rational fractions. The constraint that $y^+ f'$ tends to a constant is not actually complete. If it tended to a non-zero value A, then f itself would diverge like $A log(y^+)$; therefore, the precise condition is $\lim_{y^+\to\infty} y^+ f'(y^+) = 0$.

Algebra between the various equations will produce deviations of f'_Q from 1 that are inverse powers of y^+, and these satisfy the constraint which was just described. The "specified functions" f_μ in the k-ϵ model and f_{v1} in the SA model also satisfy it, often being based on exponentials of $-y^+$. In the case of f_{v1}, there is no exponential but we have $f'_{v1} = 3\kappa c_{v1}^3 (\kappa y^+)^2 / \left[(\kappa y^+)^3 + c_{v1}^3\right]^2 = O(1/y^{+4})$. A complete survey of the rather wide collection of models found in the TMR (Rumsey 2022) found 24 "specified functions," all of which satisfy the constraint on $y^+ f'$.

The von-Karman length scale $L_{VK} \equiv |U_y|/|U_{yy}|$ used in a few models also satisfies the GLW. A simple examination of the Elliptic Blending equations fails to reveal any reason why their solution would not, even though it introduces a non-local effect (Rumsey 2022, Durbin and Pettersson Reif 2011).

Some quantities satisfy the GLW trivially, for instance $-<u'v'>^+ = 1 - \frac{dU^+}{dy^+}$ as well as the TKE production P. Note that the function $f_{<u'v'>}$ approaches 1 rather slowly, since it takes place only as fast as $1/\kappa y^+$. However, the product $y^+ f'_{<u'v'>}$ tends to 0 like $1/\kappa y^+$, as needed.

To summarize, we have shown that if dU/dy and all the Q quantities satisfy the GLW, then the RHS of the transport equations for all Qs satisfy the GLW. So does the RHS of the mean momentum equation, namely $\tau^+ - 1 = 0$. Proving that a complex quantity such as DQ/Dt is zero for all y appears very difficult until we observe that, for a GLW quantity, it is only a matter of proving that $C_{DQ/Dt} = 0$ (we are following the convention that the letter C with a subscript such as DQ/Dt is the constant in (1) for that particular quantity); in other words, once the transport equation is satisfied in the CSL/log layer, the $f_{DQ/Dt}$ function (same convention for f functions as for C constants) is irrelevant and $DQ/Dt = 0$ is satisfied down to the wall.

The task is reduced to $N+1$ scalar equations, where N is the number of quantities Q; the extra unknown being κ. The equations for the C_Q constants are nonlinear, but algebraic, because all the nondimensional functions equal 1. This does not guarantee existence nor uniqueness (think of $C_Q^2 + 1 = 0$ or $C_Q^2 - 1 = 0$). As of today, we consider that the GLW is very plausible; we have not encountered failures for the common models, and we will now present examples of success, after discussing a familiar example.

4.1 Example: The *k*-ϵ Model

Our purpose here is to fully present the concrete equations for a relatively simple case, keeping in mind background concerns over existence and uniqueness, and the novel feature of having laws that extend to the wall. We nominally consider the Chien model, but as discussed, only the log layer requires attention, so that this would apply to any k-ϵ model, if with slightly different constants (Rumsey 2022). Capital C's will be used as in the GLW with as subscript the quantity of interest which may be a combination or derivative of the basic Q quantities of the model, while lower-case c's are the constants of the model, e.g. c_μ. In the log layer the fundamental equation of k-ϵ gives the eddy viscosity: $\nu_t = \frac{c_\mu k^2}{\epsilon}$ since $f_\mu = 1$ (here f_μ is internal to the model). The three unknowns κ, C_k and C_ϵ must satisfy the momentum equation and the two k-ϵ transport equations (Rumsey 2022), which via simple algebra and building on the dimensional analysis of (1) become

$$\tau - \tau_{wall} = u_\tau^2 \left(\frac{c_\mu C_k^2}{\kappa C_\epsilon} - 1 \right) = 0,$$

$$\frac{Dk}{Dt} = \frac{u_\tau^3}{y} \left(\frac{1}{\kappa} - C_\epsilon \right) = 0, \quad \frac{D\epsilon}{Dt} = \frac{u_\tau^4}{y^2} \left(\frac{c_{\epsilon 1}}{\kappa} \frac{C_\epsilon}{C_k} - c_{\epsilon 2} \frac{C_\epsilon^2}{C_k} + \frac{2\kappa}{\sigma_\epsilon} \right) = 0. \tag{4}$$

The first two combined give $c_\mu C_k^2 = 1$, which is the well-known property $k^+ = 1/\sqrt{c_\mu}$, and $C_{dU/dy} = C_\epsilon = 1/\kappa$, which is also the accepted behavior for the logarithmic velocity dependence and the match between production and dissipation. The third equation sets

$k = 0.444$. In this respect, other of k-ϵ versions differ slightly. In fact, even with the same κ, the f functions will alter the eddy viscosity in the buffer layer, and therefore alter the log-law intercept C (with the traditional notation), but not dU/dy.

To sum up, the GLW equations in the CSL for k-ϵ indeed result in existence and uniqueness for the κ, C_k and C_ϵ constants, and directly confirm the well-known behavior of this model. Furthermore, within the GLW, the equations in (4) simply get multiplied by f functions, for instance $f_{Dk/Dt}$, and remain equal to 0 through the viscous region to the wall. This makes the GLW expressed in (1) a stronger statement than those in the literature.

5 Numerical Verifications

For the first two cases, Reynolds-Stress models are chosen, as they are closer to first principles than eddy-viscosity models, and their complexity could have opened the door to unexpected behaviour. The first one was verified extensively by DLR and NASA and is called the SSG/LRR-ω model after Speziale, Sarkar & Gatski, and Launder, Reece & Rodi, complemented with the second equation of the Wilcox k-ω model (near the wall, the LRR model prevails over the SSG model) (Rumsey 2022). Reynolds-Stress models have a reputation of difficult convergence in non-trivial geometries, but not of multiple solutions.

We begin in the boundary layer, and must note that the facts are not quite as simple here as they are in channel flow. The quantities DQ/Dt and du_τ/dx are not exactly zero, so that the GLW formally vanishes. However, the established thinking has been that these derivatives are small enough to justify reasoning very similar to that in the channel, the more so the closer to the wall ($\frac{y}{\delta} \ll 1$ just like $\frac{y}{h} \ll 1$). This includes for the velocity U both a Law of the Wall (a universal one) and a Law of the Wake, and for the stresses a near-wall behaviour very similar to that in the channel.

The results in Fig. 1 fully conform to the GLW: as the flow Reynolds number increases the turbulence intensities each approach a plateau in the log layer (i.e., $\beta = 0$), and the distributions in the buffer layer are identical between flow Reynolds numbers, i.e., each f is a function of y^+ only. The plateau levels could be predicted analytically like they were for the k-ϵ model, but only by solving a 6-by-6 system of nonlinear equations.

The second model, in Fig. 2, is very different in its formulation from the first one and has a near-wall peak of the streamwise fluctuations, yet the GLW behaviour again

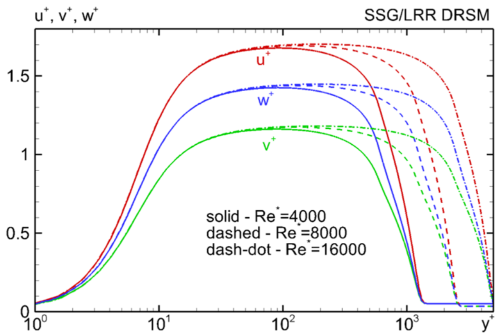


Fig. 1 Root-mean square fluctuations in a Reynolds-Stress Model in a boundary layer at three Reynolds numbers. Courtesy of B. Eisfeld and C. Rumsey

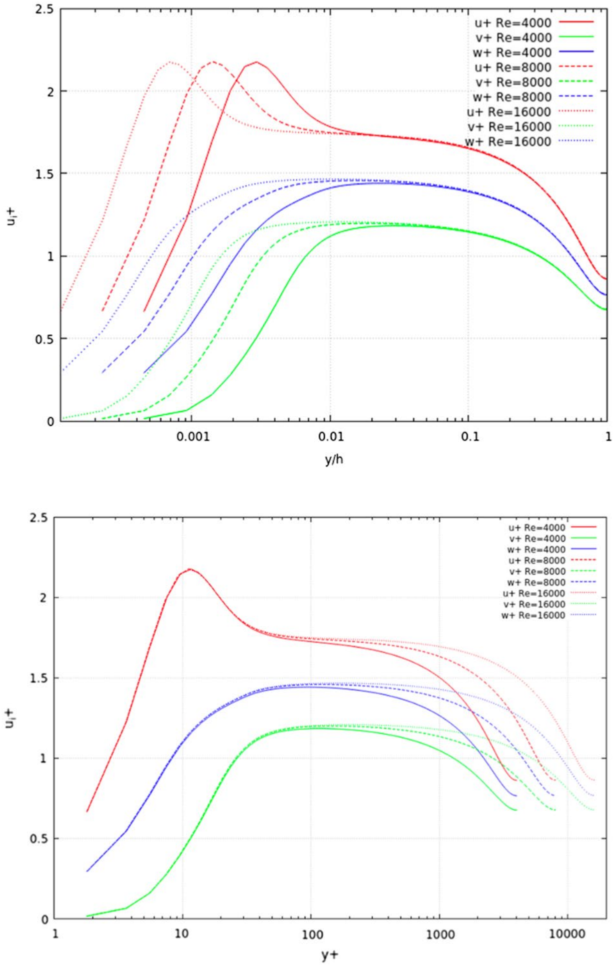


Fig. 2 Root-mean-square fluctuations in a Reynolds-Stress Model in a channel at three Reynolds numbers (Batten et al. 1999). **a** versus y^+; **b** versus y/h. Courtesy of P. Batten

is unmistakeable in Fig. 2a (Fig. 2b illustrates behavior in the outer layer, which may be the subject of further work). This supports our contention that the structure of the Classical equation sets leads to the GLW.

6 Confrontation of the GLW with Nature

Essentially, here, we are seeking evidence of the Structural Limitation mentioned in the abstract. Recall that the claim is as follows: the very form of the Classical Models, independent of features such as the number of equations, nonlinearity, y-dependence, or terms inherited from the exact Reynolds-Stress transport equations, impose a specific property we call the GLW and expressed in (1). We then gather results in real turbulence which strongly deviate from the GLW; this puts predicting these results out of reach of any such model.

Below, we present channel and then boundary-layer results, using DNS which has a strong advantage over experiments in the viscous sublayer. Databases provided by Lee, Moser, Hoyas, Jimenez and Oberlack among others are extremely helpful. However, we first attend to experimental findings which established the failure of the GLW many years before DNS confirmed it.

The conflict between the GLW and Nature was known to Townsend (1961) who cites "numerous observations that turbulent intensities in constant-stress layers vary considerably between different flows of the same stress." He was not focusing on RANS models, but on the natural extension by theory of the velocity Law of the Wall to the stresses, which is fully consistent with the GLW. Bradshaw in 1967 confirmed Townsend's concept of "inactive" motion, which is dominated by wavelengths much larger than y and is reflected in $\overline{u'^2}$ and $\overline{w'^2}$. The line of thought also leads to specific predictions including spectra with k^{-1} dependence (k here is the wavenumber) and a logarithmic decrease of $\overline{u'^2}^+$ with y, which disobey the GLW, are convincing, and have much support in experiments and now in DNS. However, we know of no attempt to make a RANS model or other one-point description of turbulence capture Inactive Motion in the CSL.

An illustration of the expectations in the turbulence community is the work of Kalitzin et al. in 2005. In their development of a Wall Model, they use the words "It is based on the assumption of wall layer universality, applied to the entire model." This is a clear expression of the GLW.

DeGraaf and Eaton, in 2000, presented detailed experiments in a flat-plate boundary layer at Reynolds numbers far outside the reach of DNS, and they still do not obey the GLW. The peak of $\overline{u'^2}^+$ clearly depends on R_θ, in their Fig. 7; a trend towards a plateau at the highest Reynolds number cannot be ruled out. On the other hand, the asymptotic behaviour at the wall fails to display such an R_θ dependence (their Fig. 10); see DNS results below.

We turn to the k-ϵ model. In Fig. 3 the shear stresses strongly approach the GLW both in DNS and RANS, which is a trivial consequence of the shear rate *dU/dy* satisfying it in the velocity LW. For the turbulent kinetic energy (TKE) on the other hand, the model satisfies the GLW near the wall, but the present Reynolds numbers do not allow a true plateau to form in the CSL. The DNS does not give a conclusive trend about a plateau, but it strikingly violates the GLW in the buffer layer, near $y^+ = 20$: the peak values are very different at the two Reynolds numbers. The dissipation is instructive. In the CSL/log layer, both models agree with the GLW result, namely $\epsilon^+ = 1/\kappa y^+$, which is related to the same fact for Production, itself deriving from the agreement on the gradient dU^+/dy^+. However, in the viscous layer the DNS results strongly depend on Re_τ. The k-ϵ model results are roughly ½ of the DNS results, which is consistent with the same phenomenon seen for the TKE in the middle frame, since the dissipation is in balance with the viscous diffusion (Rumsey 2022). An unfortunate consequence of these results is that making the modeled k match the DNS result (by means we do not have) would actually degrade the eddy viscosity and mean velocity, unless a matching correction were made to the c_μ quantity.

The Spalart–Allmaras model is exercised in Fig. 4. The quantity $\nu_t/[u_\tau y(1\text{-}y/h)]$ is the most revealing. Its log-layer value is κ. The GLW f function for it is equivalent to the f_{v1} function internal to the model, when $Re_\tau \to \infty$. First, observe that the agreement with experiment in the viscous region especially near $y^+ = 70$ is imperfect; in other words, a more accurate f_{v1} function exists relative to the SA92 model; however its practical effect

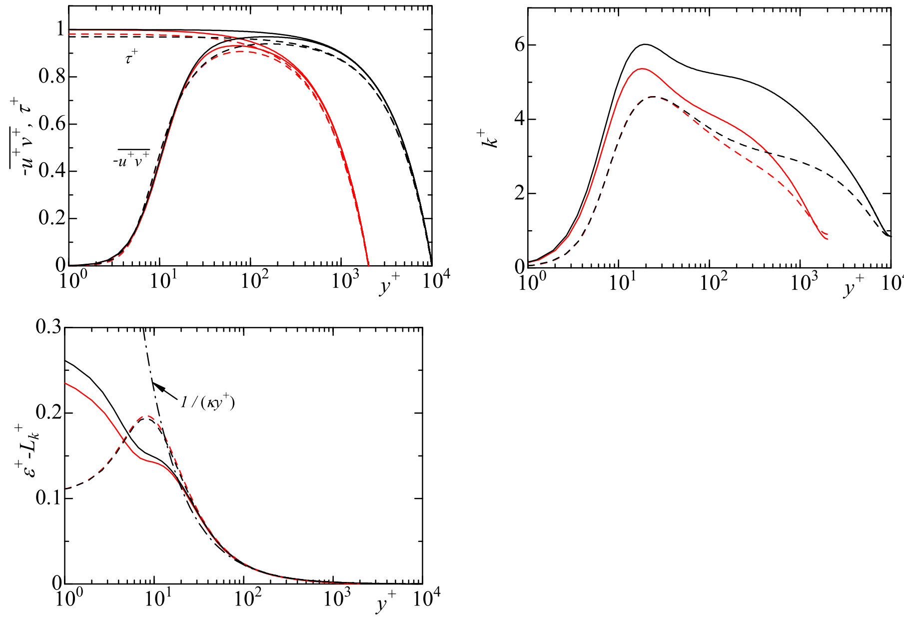
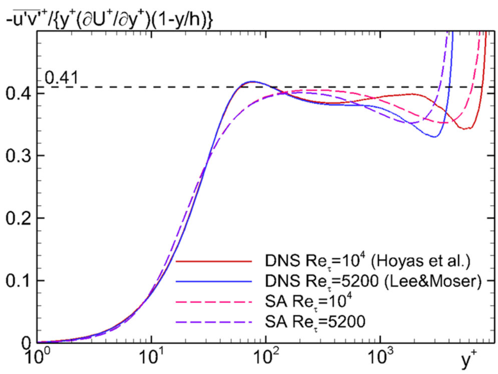


Fig. 3 Channel results (Spalart and Abe 2021). Left, turbulent and total shear stresses; center, turbulent kinetic energy; right, dissipation. Solid, DNS; dashed, k-ϵ. Red, $Re_\tau = 2000$; black, $Re_\tau = 10,000$. κ is set to 0.444. Courtesy H. Abe

would be small. The principal fact is that the DNS results have essentially no dependence on Reynolds number up to $y^+ \approx 300$; in this respect, they satisfy the GLW. In the log layer, the SA and DNS results are similar, that is, both have a trend towards a constant. In SA, that is 0.41; in DNS, as already mentioned the level is not close enough to constant to set a firm value for κ, for instance to decide between 0.39 and 0.40. This has been an unfortunate feature of DNS datasets, and we attribute it to the errors due to insufficient time-averaging. Still, our position is that DNS does not contradict the GLW for the eddy viscosity and therefore for the entire SA model, and that this fact is simply associated with the velocity Law of the Wall. An improved f_{v1} function would need to asymptote to the chosen value of κ as $y^+ \to \infty$, and the DNS knowledge base is clearly not adequate to generate such a function, whether using ML or not.

Fig. 4 Channel results for the SA model. Courtesy A. Garbaruk



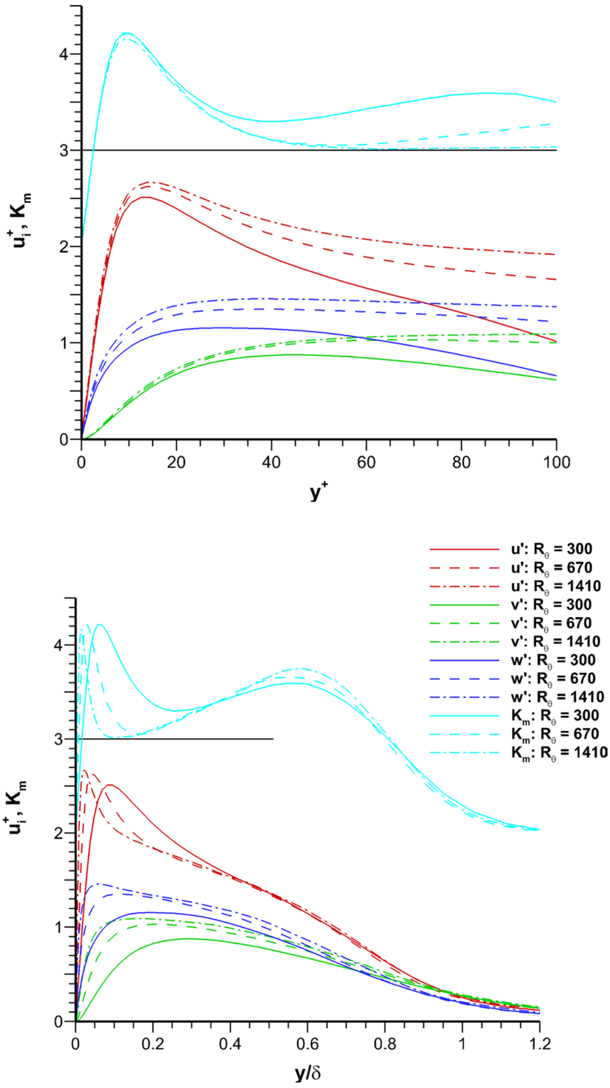


Fig. 5 Boundary-layer DNS results for root-mean-square of velocity components and mean-velocity derivative, $K_m \equiv \kappa y^+ dU^+/dy^+$ with $\kappa = 0.41$ (Spalart 1988). Left, versusy^+; right, versus y/δ. Red, $\overline{u'^2}^{1/2}$; blue, $\overline{w'^2}^{1/2}$; green, $\overline{v'^2}^{1/2}$; cyan, K_m, displaced upwards by 2 units; black, $K_m = 1$. Solid, $Re_\theta = 300$; - - -,$Re_\theta = 670$;- --- -, $Re_\theta = 1410$. Courtesy G. Coleman

The figure shows that, fortuitously, the original SA model does not encounter the Structural Limitation, because it predicts only the Reynolds shear stress. Its nonlinear versions such as QCR2020 which provide other stresses of course suffer from the Structural Limitation (Rumsey et al. 2020).

Finally, boundary-layer DNS results are presented in Fig. 5. The conflict with the GLW is strong all the way to the wall, especially for $\overline{w'^2}$ in that it is not a function of y^+; this was clear in DNS already in 1988, and recent results on more powerful computers have confirmed it. Consequently, the wall value of dissipation ϵ^+_{wall} is Reynolds-number dependent (recall Fig. 3c), contrary to the GLW, as are many other quantities. Its values for Re_θ equal to 300, 670, and 1410 respectively are 0.207, 0.235, and 0.258; this is not precisely a logarithmic dependence since the steps are 0.028 and 0.023, but it does not suggest that the rise could saturate soon. The rise for channel flow in Fig. 3c was much slower, since it was only by 0.027 for a factor of 5 in Reynolds number. Thus, consensus DNS numbers have not been firmly established, even making assumptions for an approximate correspondence between Re_{δ^*} and Re_τ. Figure 5b does not reveal the existence of a plateau for $\overline{u'^2}$, but here also the Reynolds number may not be sufficient to anticipate the behaviour at engineering Reynolds numbers; see discussion below. On the other hand, the experiments (not shown) all have very much the same trend versus y. The Reynolds-number dependence of the peak $\overline{u'^2}$ as measured by DeGraff and Eaton (2000) continues that in Fig. 5 (although we note that the measured value is about 8% higher than the DNS value at the same Re_θ).

Figure 5a strongly support our description above of a rapid approach to the Velocity Law of the Wall in Nature, in marked contrast to the Reynolds stresses at the same y^+. The precise behaviour of the flows as they approach the velocity Law of the Wall has been the subject of many studies (Smits 2022) and in particular Klewicki et al. (2009 and references therein); these authors pointed out the slow approach of DNS results to the log law, and how a close correspondence begins at a y^+ of several hundred, which is later than had been often considered since the 1970's. The point is also made by Spalart and Abe (2021). We believe that such subtle corrections to the velocity LW will only add to the failure of the GLW we have introduced here.

Before concluding, we mention an apparent paradox of the GLW ansatz, namely that the velocity U itself does not satisfy it; applying the reasoning in raw form, U "should" in logic have $\alpha = 1$, $\beta = 0$, and reach a constant U^+ in the CSL. Our earlier locality argument in favour of dU/dy rather than U is vague; a more relevant statement is that the model equations only involve dU/dy. These $N+1$ combined equations are solved, and U itself then results from "post-processing" by integration. This is curious nevertheless. There is a faint possibility that other quantities could also have a logarithmic y-dependence even in the CSL, but we have no sign of that so far.

Concretely, U does approach the GLW behaviour, in the following sense. According to the classical defect law, starting in the log layer, it satisfies $U = U_b + u_\tau g\left(\frac{y}{h}\right)$ where U_b is the bulk velocity and g a function, independent of Reynolds number. As Re_τ increases, $\frac{u_\tau}{U_b}$ tends to 0, and therefore at fixed $\frac{y}{h}$ we have $lim_{Re\to\infty} U = U_b$: the velocity is, increasingly, uniform from the log layer to the centreline, consistent with $\beta = 0$. This argument predates turbulence modelling, but both model and experimental results agree with it. This is a "plug flow" with boundary layers, as well explained by Pullin et al. (2013), but still U does not follow a function $f\left(y^+\right)Cu_\tau$ with $\lim_{y^+\to\infty} f\left(y^+\right) = 1$.

7 Discussion

We hope to have presented the GLW conjecture in enough detail for the reader to have formed an opinion about the chances that it is fully correct, and that identifying Classical Turbulence Models is a useful step. The Structural Limitation is increasingly relevant in an era of raised expectations from the models, fuelled by the prospect of Machine Learning. From the point of view of an engineer, in a boundary layer with slow variation in x, the mission of the model is limited to the function $\overline{u'v'}$(y) and poor accuracy for $\overline{u'^2}$ has little impact on the mean flow field. With more rapid variations and especially if subjected to a shock wave, the mission now includes $\overline{u'^2}(x, y)$ and $\overline{v'^2}(x, y)$; $\overline{u'^2}$ (along with $\overline{w'^2}$) is precisely one of the two Reynolds stresses which Classical Models will permanently struggle with if the thesis of the present paper is correct.

The interest is not limited to stresses; some colleagues hope to use DNS data for modelling various budget terms of the stresses, especially the pressure terms. A very reasonable first step in ML would be to train a new model or model version on the public-domain database from channel-flow DNS, probably using a Neural Network (NN), and including all four Reynolds stresses in the error measure. In fact, this may already have been attempted, and not have been successful enough to publish. The Reynolds number Re_τ now has a wide dynamic range in DNS studies, of [180,10^4], but practical values are much larger than 10^4, so that using the new model in engineering would constitute a strong extrapolation, which risks a situation of overfitting. The Deep NN is liable to effectively use the molecular viscosity over the whole domain in y instead of only the viscous region to obtain a better agreement for $\overline{u'^2}(y)$ over a finite range of Re_τ, with severe consequences outside that interval. Careful ML operators have tools to detect this, for instance by training up to 5,000 and then testing at 10,000, and they should certainly use them.

The channel training could be restricted to $\overline{u'v'}$, but a flow such as the Periodic Hill would be a likely next candidate, and in it such a restriction would be wrong (to begin with, this stress depends on the axes used).

Our claim of a Structural Limitation could be obviated by a mathematical counterexample, such as a published model which is compatible with Navier–Stokes solvers and properly invariant but does not obey the GLW. It also could be obviated by the finding that the GLW in fact applies to Nature, but only at Reynolds numbers much higher than those reached so far. It is clear that Direct Numerical Simulations will be permanently unable to address this. However, high-Reynolds-number experimental results definitely conflict with the feature of Fig. 5b that the stresses steadily decrease with y in the core of the boundary layer. These experimental results show a second peak of $\overline{u'^2}$ (Smits 2022). This is not a clear trend towards the plateau required by the GLW, but still it can suggest that all DNS results and experimental results before the Superpipe are unfortunately in a “low-Reynolds-number regime.” Establishing that extremely high Reynolds numbers revive the GLW would please the theoretician, since after all generalizing the Law of the Wall is a logical step to take, but it is unclear how we would use it in industry.

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Declarations

Conflict of interests TThe author has no such interests..

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