ON PRESERVATION UNDER UNIVARIATE WEIGHTED DISTRIBUTIONS

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Abstract. We derive some new results for preservation of various stochastic orders and aging classes under weighted distributions. The corresponding reversed preservation properties as straightforward conclusions of the obtained results for the direct preservation properties, are developed. Damage model of Rao, residual lifetime distribution, proportional hazards and proportional reversed hazards models are discussed as special weighted distributions to try some of our results.

Keywords: weighted distribution; preservation; stochastic ordering; aging classes

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1. Introduction and preliminaries

Weighted distributions are useful to model data in situations where the distribution of the observed data does not coincide with the original distribution of the data. A number of such instances were explained and described by Rao [17] and Patil and Rao [16]. Recently, the study of some reliability aspects of weighted distributions has attracted the attention of many researchers (cf. Kochar and Gupta [10], Nanda and Jain [12], Navarro et al. [14] and Pakes et al. [15] among others). Numerous research works have also been devoted to investigate the properties of weighted distributions in the context of stochastic orderings and aging classes (cf. Bartoszewicz and Skolimowska [4], Misra et al. [11], Błażej [5], Bartoszewicz [3] and Izadkhah et al. [7]). One of the main problems in some of these works was the problem of preservation of stochastic orders and aging classes under univariate weighted distributions. For example, using a representation of weighted distributions Błażej [5] and Bartoszewicz [3] obtained some results for preservation of several stochastic orders and aging classes. By appealing to some bivariate characterizations of stochastic orders Misra et al. [11] derived a similar kind of results. Also, using some well-known characterizations of aging classes by means of stochastic orders and using the concept of the totally positivity (cf. Karlin [9]), Izadkhah et al. [6] presented some achievements for preservation of a number of aging classes under weighting.

In this paper, using a technical lemma given in Barlow and Proschan [2], we develop a complete study to get the preservation of several univariate stochastic orders and aging classes under weighted distributions. In this context, a new approach will be introduced, although some of the results obtained are similar to the previous results in the literature (cf. Misra et al. [11]). In addition, according to Izadkhah et al. [6], some special weighted distributions are proposed which are applied to some practical situations, and we examine the derived results for those special cases (cf. Ahmad and Kayid [1]). Another direction of this paper, which has not been investigated in the literature before, is to provide the reversed preservation property of weighted distributions. Throughout the paper, examples are also given to explain some useful facts. The notation $I_A(t)$ stands for the indicator function of any set A in $\mathbb{R}$. It will be also assumed that $\overset{\text{st}}{=}$ denotes the equality of distributions.

Let X and Y be two random variables with absolutely continuous cumulative distribution functions (cdf) F and G, probability density functions (pdf) f and g, and survival functions (sf) $\overline{F}$ and $\overline{G}$, respectively. Assume further that $u_X = \sup\{x\colon F(x) < 1\}$ and $u_Y = \sup\{x\colon G(x) < 1\}$ are the respective upper bounds of X and Y, and $l_X = \inf\{x\colon F(x) > 0\}$ and $l_Y = \inf\{x\colon G(x) > 0\}$ are their corresponding lower bounds. For two nonnegative weight functions w_1 and w_2, the random variables X_{w_1} and Y_{w_2} are called the weighted random variables associated with X and Y, which have probability density functions (cf. Jain et al. [8])

$$f_1(x) = \frac{w_1(x)f(x)}{\eta_1} \quad \text{and} \quad g_1(x) = \frac{w_2(x)g(x)}{\eta_2}, \quad x \in \mathbb{R},$$

respectively, where $0 < \eta_1 = E(w_1(X)) < \infty$ and $0 < \eta_2 = E(w_2(Y)) < \infty$. The distribution functions of X_{w_1} and Y_{w_2} are, respectively, obtained as

$$F_1(x) = \frac{A_1(x)F(x)}{\eta_1} \quad \text{and} \quad G_1(x) = \frac{A_2(x)G(x)}{\eta_2}, \quad x \in \mathbb{R},$$

and their corresponding survival functions as

$$\overline{F}_1(x) = \frac{B_1(x)\overline{F}(x)}{\eta_1} \quad \text{and} \quad \overline{G}_1(x) = \frac{B_2(x)\overline{G}(x)}{\eta_2}, \quad x \in \mathbb{R},$$

where $A_1(x) = E(w_1(X) \mid X \leqslant x)$, $A_2(x) = E(w_2(Y) \mid Y \leqslant x)$, $B_1(x) = E(w_1(X) \mid X > x)$, and $B_2(x) = E(w_2(Y) \mid Y > x)$.

The random variable $X_t = (X - t \mid X > t)$ for $t < u_X$ is called the residual life of X having sf $\overline{F}_t(x) = \overline{F}(t+x)/\overline{F}(t)$, $x \in (0, \infty)$. Also, the random variable $X_{(t)} = (t - X \mid X \leqslant t)$ for $t > l_X$ is known as the reversed residual life or the inactivity time of X, which has sf $\overline{F}_{(t)}(x) = F(t-x)/F(t)$, $x \in [0, \infty)$. The foregoing characteristics are similarly defined for the random variable Y. The hazard rates (hr) of X and Y are, respectively, given by $r_F(x) = f(x)/\overline{F}(x)$, $x \in (-\infty, u_X)$, and $r_G(x) = g(x)/\overline{G}(x)$, $x \in (-\infty, u_Y)$. The reversed hazard rates (rh) of X and Y are defined as $q_F(x) = f(x)/F(x)$, $x \in (l_X, \infty)$, and $q_G(x) = g(x)/G(x)$, $x \in (l_Y, \infty)$, respectively. The mean of random variables X_x and Y_x, called the mean residual lifetimes (mrl) of X and Y, are given, respectively, by

$$m_F(x) = \begin{cases} \displaystyle\int_x^\infty \frac{\overline{F}(t)}{\overline{F}(x)}\,\mathrm{d}t, & x < u_X, \\ 0, & x \geqslant u_X, \end{cases} \quad \text{and} \quad m_G(x) = \begin{cases} \displaystyle\int_x^\infty \frac{\overline{G}(t)}{\overline{G}(x)}\,\mathrm{d}t, & x < u_Y, \\ 0, & x \geqslant u_Y. \end{cases}$$

The reversed mean residual lifetimes (rmr) of X and Y are, respectively, defined as the mathematical expectations of $X_{(x)}$ and $Y_{(x)}$, and are given by

$$\alpha_F(x) = \begin{cases} \displaystyle\int_{l_X}^x \frac{F(t)}{F(x)}\,\mathrm{d}t, & x > l_X, \\ 0, & x \leqslant l_X, \end{cases} \quad \text{and} \quad \alpha_G(x) = \begin{cases} \displaystyle\int_{l_Y}^x \frac{G(t)}{G(x)}\,\mathrm{d}t, & x > l_Y, \\ 0, & x \leqslant l_Y. \end{cases}$$

According to Shaked and Shanthikumar [18] and Nanda et al. [13], we have the following partial orders to be used throughout the paper. We use the convention that $a/0 = \infty$ for $a > 0$, and also $0/0 = 0$. The random variable X is smaller than Y in:

(i) Usual stochastic order ($X \leqslant_{\mathrm{st}} Y$), if $\overline{F}(x) \leqslant \overline{G}(x)$ for all $x \in \mathbb{R}$, or equivalently, if $F(x) \geqslant G(x)$ for all $x \in \mathbb{R}$.

(ii) Hazard rate order ($X \leqslant_{\mathrm{hr}} Y$), if $r_F(x) \geqslant r_G(x)$ for all $x \in \mathbb{R}$, or equivalently, if $[f(x)\overline{G}(x) - g(x)\overline{F}(x)] \geqslant 0$ for all $x \in \mathbb{R}$.

(iii) Reversed hazard rate order ($X \leqslant_{\mathrm{rh}} Y$), if $q_F(x) \leqslant q_G(x)$ for all $x \in \mathbb{R}$, or equivalently, if $[g(x)F(x) - f(x)G(x)] \geqslant 0$ for all $x \in \mathbb{R}$.

(iv) Mean residual life order ($X \leqslant_{\mathrm{mrl}} Y$), if $m_F(x) \leqslant m_G(x)$ for all $x \in \mathbb{R}$, or equivalently, if $[\overline{F}(x) \int_x^\infty \overline{G}(t)\,\mathrm{d}t - \overline{G}(x) \int_x^\infty \overline{F}(t)\,\mathrm{d}t] \geqslant 0$ for all $x \in \mathbb{R}$.

(v) Reversed mean residual life order ($X \leqslant_{\mathrm{rmr}} Y$), if $\alpha_F(x) \geqslant \alpha_G(x)$ for all $x \in \mathbb{R}$, or equivalently, if $[G(x) \int_0^x F(t)\,\mathrm{d}t - F(x) \int_0^x G(t)\,\mathrm{d}t] \geqslant 0$ for all $x \in \mathbb{R}$.

(vi) Increasing convex order ($X \leqslant_{\mathrm{icx}} Y$), if $\int_x^\infty \overline{F}(t)\,\mathrm{d}t \leqslant \int_x^\infty \overline{G}(t)\,\mathrm{d}t$ for all $x \in \mathbb{R}$.

(vii) Increasing concave order ($X \leqslant_{\mathrm{icv}} Y$), if $\int_{-\infty}^x G(t)\,\mathrm{d}t \leqslant \int_{-\infty}^x G(t)\,dt$ for all $x \in \mathbb{R}$.

The following aging classes are defined as in Shaked and Shanthikumar [18]. The random variable X is said to have:

(i) Decreasing Mean Residual Life (DMRL) property, if the function $m_F(x)$ is decreasing in $x \in \mathbb{R}$, or equivalently, if $\int_x^\infty \overline{F}(t)\,\mathrm{d}t$ is log-concave on S_X.

(ii) Increasing Reversed Mean Residual life (IRMR) property, if the function $\alpha_F(x)$ is increasing in $x \in \mathbb{R}$, or equivalently, if $\int_{l_X}^x F(t)\,\mathrm{d}t$ is log-concave on S_X.

The nonnegative random variable X is said to have:

(iii) New Better than Used (NBU) property, if $\overline{F}(x)\overline{F}(y) \geqslant \overline{F}(x+y)$ for all x, y on $[0,\infty)$.

(iv) New Better than Used in Expectation (NBUE) property, if $m_F(0) \geqslant m_F(x)$ for all $x \in [0,\infty)$.

(v) New Better than Used in Convex order (NBUC) property, if $X_t \leqslant_{\mathrm{icx}} X$ for all $t \geqslant 0$.

2. Preservation of stochastic orders

In this section, we establish our main results for preservation of some stochastic orders under weighted distributions. Then, the reversed implication that stochastic orders of weighted distributions imply the stochastic orders of parent distributions, will be discussed. By applying the following lemma and using the fact that the parent distribution can be regarded, at least theoretically, as a weighted version of the weighted distribution, each result for preservation under weighting can be translated to the reversed direction. It is to be mentioned here that the problem of the reversed preservation under weighting is important, because it provides information about the original distribution of a weighted data set via a mathematical implication. Let $\eta = E(w(X))$, $B(x) = E(w(X) \mid X > x)$, $A(y) = E(w(X) \mid X \leqslant y)$ and $C(x,y) = E(w(X) \mid x < X \leqslant y)$. Suppose that X_w is the weighted version of X with the weight function w which has pdf f_w and cdf F_w.

Lemma 1. *Let $T = X_w$ and for $\nu(x) = 1/w(x)$ let $C^*(x,y) = E(\nu(T) \mid x < T \leqslant y)$. Then $C^*(x,y) = 1/C(x,y)$ for all $x \leqslant y \in \mathbb{R}$. Furthermore, $T_\nu =_{\mathrm{st}} X$, where T_ν is the weighted version of T with weight function ν.*

Proof. Note that

$$C(x,y) = \eta \frac{F_w(y) - F_w(x)}{F(y) - F(x)}.$$

For all $x \leqslant y \in \mathbb{R}$, we get

$$\begin{aligned} C^*(x,y) &= \int_x^y \frac{\nu(t) f_w(t)}{F_w(y) - F_w(x)}\,\mathrm{d}t = \frac{\int_x^y f(t)\,\mathrm{d}t}{\eta(F_w(y) - F_w(x))} \\ &= \frac{1}{\eta}\frac{F(y) - F(x)}{F_w(y) - F_w(x)} = \frac{1}{C(x,y)}. \end{aligned}$$

Denote $\eta^* = E(\nu(T))$, $B^*(x) = E(\nu(T) \mid T > x)$ and $A^*(y) = E(\nu(T) \mid T \leqslant y)$. Then, using the identity $C^*(x,y) = 1/C(x,y)$, by letting $y \to \infty$, $x \to \infty$ and $(x,y) \to (-\infty, \infty)$, one at a time, we obtain, respectively, $B^*(x) = 1/B(x)$, $A^*(y) = 1/A(y)$, and $\eta^* = 1/\eta$. $\square$

Now, we will pay our attention to the well-known usual stochastic order. Preservation properties of this order were considered in Theorem 3.1 of Misra et al. [11] and Theorem 11 of Bartoszewicz and Skolimowska [4]. The following result is a characterization of the usual stochastic order.

Theorem 1. *Let w_i be non-increasing (non-decreasing) for some $i = 1, 2$; and let $w_2(x)/\eta_2 \leqslant (\geqslant) w_1(x)/\eta_1$ for all $x \in \mathbb{R}$. Then*

$$X \leqslant_{\mathrm{st}} Y \Leftrightarrow X_{w_1} \leqslant_{\mathrm{st}} Y_{w_2}.$$

Proof. Suppose that w_i is non-increasing for some $i = 1, 2$; and let $w_2(t)/\eta_2 \leqslant w_1(t)/\eta_1$ for all $t \in \mathbb{R}$. Then, for all $x \in \mathbb{R}$, we get

$$\begin{aligned} F_1(x) - G_1(x) &= \int_{-\infty}^{x} (f_1(t) - g_1(t))\,\mathrm{d}t \\ &= \int_{-\infty}^{x} \Big(\frac{w_1(t)}{\eta_1} f(t) - \frac{w_2(t)}{\eta_2} g(t)\Big)\,\mathrm{d}t \\ &\geqslant \int_{-\infty}^{x} \frac{w_i(t)}{\eta_i}(f(t) - g(t))\,\mathrm{d}t \\ &= \int_{-\infty}^{\infty} h_i(t)\,\mathrm{d}W(t), \end{aligned}$$

where $h_i(t) = w_i(t) I_{(-\infty,x]}(t)/\eta_i$, and $W(t) = F(t) - G(t)$. Obviously, $h_i(t)$ is non-negative and by assumption it is non-increasing in t. We know that $X \leqslant_{\mathrm{st}} Y$ gives $\int_{-\infty}^{x} \mathrm{d}W(t) \geqslant 0$ for all $x \in \mathbb{R}$. Hence, Lemma 7.1 (b) in Barlow and Prochan [2] directly provides the proof. $\square$

The case when $w_1 = w_2 = w$ gives the following corollary.

Corollary 1. *Let w be a monotone function for which $E(w(X)) = E(w(Y))$. Then $X \leqslant_{\mathrm{st}} Y$ if and only if $X_w \leqslant_{\mathrm{st}} Y_w$.*

Example 1. Let X and Y denote the lifetimes of two devices having cumulative distribution functions F and G, respectively. Take $w(x) = I_{(t_0,\infty)}(x)$, where $t_0 \in S_X \cap S_Y$ is a time point at which $F(t_0) = G(t_0)$. Then, observe that $X_w \stackrel{\mathrm{st}}{=} (X \mid X > t_0)$ and $Y_w \stackrel{\mathrm{st}}{=} (Y \mid Y > t_0)$. From Corollary 1, by the known properties of the usual stochastic order, $X \leqslant_{\mathrm{st}} Y$ if and only if $X_{t_0} \leqslant_{\mathrm{st}} Y_{t_0}$, where X_{t_0} and Y_{t_0} are the residual lifetimes associated with X and Y, respectively. Similarly, by taking $w(x) = I_{(0,t_0]}(x)$ such that $F(t_0) = G(t_0)$, we conclude that $X \leqslant_{\mathrm{st}} Y$ if and only if $X_{(t_0)} \geqslant_{\mathrm{st}} Y_{(t_0)}$, where $X_{(t_0)}$ and $Y_{(t_0)}$ are the reversed residual lifetimes of X and Y, respectively.

Now, we discuss the problem of preservation of the hazard rate order and the reversed hazard rate order under weighting. For similar results we refer the readers to Misra et al. [11].

Theorem 2. *Let w_2/w_1 be non-decreasing and let w_i be non-decreasing for some $i = 1, 2$. Then $X \leqslant_{\mathrm{hr}} Y$ implies $X_{w_1} \leqslant_{\mathrm{hr}} Y_{w_2}$.*

Proof. Denote $\mathrm{d}W_x(t) = w(x,t)\,\mathrm{d}t$ with $w(x,t) = [f(x)g(t) - f(t)g(x)]I_{[x,\infty)}(t)$ for all $x \in \mathbb{R}$ and for all $t \in \mathbb{R}$. The hazard rate order between X and Y can be translated to $\int_y^\infty \mathrm{d}W_x(t) \geqslant 0$ for all $y \leqslant x \in \mathbb{R}$. Note that $X \leqslant_{\mathrm{hr}} Y$ is equivalent to $f(x)/g(x) \geqslant \overline{F}(x)/\overline{G}(x)$ for all $x \in \mathbb{R}$, and on the other hand $X \leqslant_{\mathrm{hr}} Y$ provides that $\overline{F}(x)/\overline{G}(x)$ is non-increasing for $x \in \mathbb{R}$. Therefore, $X \leqslant_{\mathrm{hr}} Y$ implies that $f(x)/g(x) \geqslant \overline{F}(y)/\overline{G}(y)$ for all $x \leqslant y \in \mathbb{R}$. In fact, this means that $\int_y^\infty \mathrm{d}W_x(t) \geqslant 0$ for all $x \leqslant y \in \mathbb{R}$. As a result, $X \leqslant_{\mathrm{hr}} Y$ implies that $\int_y^\infty \mathrm{d}W_x(t) \geqslant 0$ for all $x \in \mathbb{R}$, and for all $y \in \mathbb{R}$. The assumption that w_2/w_1 is non-decreasing yields $w_1(x)w_2(t) \geqslant w_1(t)w_2(x)$ for all $t \geqslant x \in \mathbb{R}$. Now, one has

$$\begin{aligned}
f_1(x)\overline{G}_1(x) - g_1(x)\overline{F}_1(x) &= \int_x^\infty [f_1(x)g_1(t) - g_1(x)f_1(t)]\,\mathrm{d}t \\
&= \int_x^\infty \Big(\frac{w_1(x)w_2(t)f(x)g(t)}{\eta_1\eta_2} - \frac{w_1(t)w_2(x)g(x)f(t)}{\eta_1\eta_2}\Big)\,\mathrm{d}t \\
&\geqslant \int_x^\infty \frac{w_i(t)w_{3-i}(x)}{\eta_1\eta_2}[f(x)g(t) - f(t)g(x)]\,\mathrm{d}t \\
&= \int_{-\infty}^\infty h_i(t)[f(x)g(t) - f(t)g(x)]I_{[x,\infty)}(t)\,\mathrm{d}t \\
&= \int_{-\infty}^\infty h_i(t)\,\mathrm{d}W_x(t),
\end{aligned}$$

where $h_i(t) = (\eta_1\eta_2)^{-1}w_i(t)w_{3-i}(x)$. From the assumption of w_i being non-decreasing, we observe that h_i is a non-decreasing function. Hence, Lemma 7.1 (a) of Barlow and Proschan [2] completes the proof. □

By applying Lemma 1 to Theorem 2 we derive the following corollary.

Corollary 2. *If w_2/w_1 is non-increasing and at least one of w_i's for $i = 1, 2$ is non-increasing, then $X_{w_1} \leqslant_{\mathrm{hr}} Y_{w_2}$ implies $X \leqslant_{\mathrm{hr}} Y$.*

Example 2 (Damage model of Rao). Suppose that Z_1 and Z_2 are two random variables with density functions h_1 and h_2, respectively. We assume that Z_1 is independent of X and that Z_2 is independent of Y. Visualize the random variable X_{w_1} as it records the amount of observation X only if $Z_1 = X$ (whenever the observed values of Z_1 and of X are equal) and imagine that Y_{w_2} records the amount of observation Y only if $Z_2 = Y$. Rao [17] showed that this is a situation, where the weighted distributions can be applied (see also Patil and Rao [16] for more details). Here, the weight functions are $w_1 = h_1$ and $w_2 = h_2$. If at least one of Z_i's for $i = 1, 2$ has a non-increasing density such that $Z_1 \leqslant_{\mathrm{lr}} Z_2$, then according to Corollary 2, $X_{w_1} \leqslant_{\mathrm{hr}} Y_{w_2}$ implies $X \leqslant_{\mathrm{hr}} Y$.

The following theorem is analogously derived.

Theorem 3. *Let w_2/w_1 be non-decreasing and also let w_i be non-increasing for some $i = 1, 2$. Then $X \leqslant_{\mathrm{rh}} Y$ implies $X_{w_1} \leqslant_{\mathrm{rh}} Y_{w_2}$.*

Proof. As in the proof of Theorem 2, by appealing to the fact that $X \leqslant_{\mathrm{rh}} Y$ is equivalent to $\int_{-\infty}^{x} [f(t)g(x) - f(x)g(t)]\,\mathrm{d}t \geqslant 0$ for all $x \in \mathbb{R}$, and further because it is equivalent to $G(x)/F(x)$ being non-decreasing in x, we obtain that $\int_{-\infty}^{y} \mathrm{d}W_x(t) \geqslant 0$ for all $x, y \in \mathbb{R}$, where $\mathrm{d}W_x(t) = w(x,t)\,\mathrm{d}t$ with $w(x,t) = [f(t)g(x) - f(x)g(t)]I_{(-\infty,x)}(t)$. The result will be obtained by using Lemma 7.1 (b) of Barlow and Proschan [2]. □

Corollary 3. *If w_2/w_1 is non-increasing and at least one of w_i's for $i = 1, 2$ is non-decreasing, then $X_{w_1} \leqslant_{\mathrm{rh}} Y_{w_2}$ implies $X \leqslant_{\mathrm{rh}} Y$.*

We now concentrate on the preservation of the mean residual life order and the reversed mean residual life order under weighted distributions (see also Theorem 2.2 in Izadkhah et al. [7] for similar results).

Theorem 4. *Let B_2/B_1 be non-decreasing such that B_i is non-decreasing for some $i = 1, 2$. Let $X \leqslant_{\mathrm{mrl}} Y$. Then $X_{w_1} \leqslant_{\mathrm{mrl}} Y_{w_2}$.*

Proof. To start, we have

$$\begin{aligned} X \leqslant_{\mathrm{mrl}} Y &\Leftrightarrow \int_x^{\infty} [\overline{G}(t)\overline{F}(x) - \overline{G}(x)\overline{F}(t)]\,\mathrm{d}t \geqslant 0 \quad \forall\, x \in \mathbb{R}, \\ &\Leftrightarrow \int_{-\infty}^{\infty} [\overline{G}(t)\overline{F}(x) - \overline{G}(x)\overline{F}(t)]I_{[x,\infty)}(t)\,\mathrm{d}t \geqslant 0 \quad \forall\, x \in \mathbb{R}, \\ &\Leftrightarrow \int_y^{\infty} \mathrm{d}W_x(t) \geqslant 0 \quad \forall\, y \leqslant x \in \mathbb{R}, \end{aligned}$$

where $\mathrm{d}W_x(t) = [\overline{G}(t)\overline{F}(x) - \overline{G}(x)\overline{F}(t)]I_{[x,\infty)}(t)\,\mathrm{d}t$. Note that

$$X \leqslant_{\mathrm{mrl}} Y \Leftrightarrow \frac{\int_x^\infty \overline{F}(t)\,\mathrm{d}t}{\overline{F}(x)} \leqslant \frac{\int_x^\infty \overline{G}(t)\,\mathrm{d}t}{\overline{G}(x)} \quad \forall\, x \in \mathbb{R},$$
$$\Leftrightarrow \frac{\int_x^\infty \overline{F}(t)\,\mathrm{d}t}{\int_x^\infty \overline{G}(t)\,\mathrm{d}t} \leqslant \frac{\overline{F}(x)}{\overline{G}(x)} \quad \forall\, x \in \mathbb{R}.$$

In addition,

$$X \leqslant_{\mathrm{mrl}} Y \Leftrightarrow \frac{\int_x^\infty \overline{F}(t)\,\mathrm{d}t}{\int_x^\infty \overline{G}(t)\,\mathrm{d}t} \leqslant \frac{\int_y^\infty \overline{F}(t)\,\mathrm{d}t}{\int_y^\infty \overline{G}(t)\,\mathrm{d}t} \quad \forall\, x \leqslant y \in \mathbb{R}.$$

Therefore, $X \leqslant_{\mathrm{mrl}} Y$ yields

$$\frac{\int_y^\infty \overline{F}(t)\,\mathrm{d}t}{\int_y^\infty \overline{G}(t)\,\mathrm{d}t} \leqslant \frac{\overline{F}(x)}{\overline{G}(x)} \quad \forall\, x \leqslant y \in \mathbb{R}.$$

The above inequality holds if and only if $\int_y^\infty \mathrm{d}W_x(t) \geqslant 0$ for all $x \leqslant y \in \mathbb{R}$. Thus, $X \leqslant_{\mathrm{mrl}} Y$ implies that $\int_y^\infty \mathrm{d}W_x(t) \geqslant 0$ for all $x \in \mathbb{R}$ and for all $y \in \mathbb{R}$. Because B_2/B_1 is non-decreasing, we get for all $x \in \mathbb{R}$,

$$\begin{aligned}
&\int_x^\infty [\overline{G}_1(t)\overline{F}_1(x) - \overline{G}_1(x)\overline{F}_1(t)]\,\mathrm{d}t \\
&\qquad = \int_x^\infty \Big(\frac{B_2(t)B_1(x)}{\eta_1\eta_2}\overline{G}(t)\overline{F}(x) - \frac{B_2(x)B_1(t)}{\eta_1\eta_2}\overline{G}(x)\overline{F}(t)\Big)\,\mathrm{d}t \\
&\qquad \geqslant \int_x^\infty \frac{B_{3-i}(x)B_i(t)}{\eta_1\eta_2}[\overline{G}(t)\overline{F}(x) - \overline{G}(x)\overline{F}(t)]\,\mathrm{d}t \\
&\qquad = \int_{-\infty}^\infty \frac{B_{3-i}(x)B_i(t)}{\eta_1\eta_2}[\overline{G}(t)\overline{F}(x) - \overline{G}(x)\overline{F}(t)]I_{[x,\infty)}(t)\,\mathrm{d}t \\
&\qquad = \int_{-\infty}^\infty h_i(t)\,\mathrm{d}W_x(t),
\end{aligned}$$

where $h_i(t) = (\eta_1\eta_2)^{-1}B_{3-i}(x)B_i(t)$, by the second assumption, is non-decreasing in t. By Lemma 7.1 (a) of Barlow and Proschan [2] we deduce that $\int_{-\infty}^\infty h_i(t) \times \mathrm{d}W_x(t) \geqslant 0$ for all $x \in \mathbb{R}$. □

Corollary 4. *If B_2/B_1 is non-increasing and if B_i is non-increasing for some $i = 1, 2$, then $X_{w_1} \leqslant_{\mathrm{mrl}} Y_{w_2}$ implies $X \leqslant_{\mathrm{mrl}} Y$.*

Theorem 5. *Let A_i be non-increasing for some $i = 1, 2$ and let A_2/A_1 be non-decreasing. Then $X \leqslant_{\mathrm{rmr}} Y$ implies $X_{w_1} \leqslant_{\mathrm{rmr}} Y_{w_2}$.*

P r o o f. We know that $X \leqslant_{\mathrm{rmr}} Y$ yields $\int_{-\infty}^{x} [F(t)G(x) - F(x)G(t)] \, \mathrm{d}t \geqslant 0$ for all $x \in \mathbb{R}$. Also, $X \leqslant_{\mathrm{rmr}} Y$ is equivalent to $\int_{-\infty}^{x} F(t) \, \mathrm{d}t / \int_{-\infty}^{x} G(t) \, \mathrm{d}t$ being non-increasing in $x \in \mathbb{R}$. Therefore, in a manner similar to the discussion made in the proof of Theorem 4, we get $\int_{-\infty}^{y} \mathrm{d}W(t) \geqslant 0$ for all $x \in \mathbb{R}$ and for all $y \in \mathbb{R}$, where $\mathrm{d}W(t) = [F(t)G(x) - F(x)G(t)] I_{(-\infty,x]}(t) \, \mathrm{d}t$. Again, Lemma 7.1 (b) of Barlow and Proschan [2] completes the proof. $\square$

Corollary 5. *If A_i is non-decreasing for some $i = 1, 2$ such that A_2/A_1 is non-increasing, then $X_{w_1} \leqslant_{\mathrm{rmr}} Y_{w_2}$ implies $X \leqslant_{\mathrm{rmr}} Y$.*

We now study conditions under which the increasing convex order and the increasing concave order are preserved by weighting.

Theorem 6. *Let B_i be non-decreasing for some $i = 1, 2$ and let $B_2(x)/\eta_2 \geqslant B_1(x)/\eta_1$ for all $x \in \mathbb{R}$. Then $X \leqslant_{\mathrm{icx}} Y$ implies $X_{w_1} \leqslant_{\mathrm{icx}} Y_{w_2}$.*

P r o o f. By imposing the second assumption, we get, for all $x \in \mathbb{R}$,

$$\begin{aligned}
\int_{x}^{\infty} [\overline{G}_1(t) - \overline{F}_1(t)] \, \mathrm{d}t &= \int_{x}^{\infty} \Big(\frac{B_2(t)}{\eta_2} \overline{G}(t) - \frac{B_1(t)}{\eta_1} \overline{F}(t) \Big) \, \mathrm{d}t \\
&\geqslant \int_{x}^{\infty} \frac{B_i(t)}{\eta_i} [\overline{G}(t) - \overline{F}(t)] \, \mathrm{d}t \\
&= \int_{-\infty}^{\infty} \frac{B_i(t) I_{[x,\infty)}(t)}{\eta_i} [\overline{G}(t) - \overline{F}(t)] \, \mathrm{d}t.
\end{aligned}$$

Now, take $h_i(t) = \eta_i^{-1} B_i(t) I_{[x,\infty)}(t)$, which by assumption is increasing in t for all x. Because of $X \leqslant_{\mathrm{icx}} Y$, we have that $\int_{x}^{\infty} [\overline{G}(t) - \overline{F}(t)] \, \mathrm{d}t \geqslant 0$ for all $x \in \mathbb{R}$. Lemma 7.1 (a) of Barlow and Proschan [2] is again applicable and gives the proof. $\square$

Corollary 6. *If A_i is non-increasing for some $i = 1, 2$ such that $B_2(x)/\eta_2 \leqslant B_1(x)/\eta_1$ for all $x \in \mathbb{R}$, then $X_{w_1} \leqslant_{\mathrm{icx}} Y_{w_2}$ implies $X \leqslant_{\mathrm{icx}} Y$.*

Parallelly with the result of Theorem 6 we have the following result.

Theorem 7. *Let A_i be non-increasing for some $i = 1, 2$ and let $A_1(x)/\eta_1 \geqslant A_2(x)/\eta_2$ for all $x \in \mathbb{R}$. Then $X \leqslant_{\mathrm{icv}} Y$ implies $X_{w_1} \leqslant_{\mathrm{icv}} Y_{w_2}$.*

P r o o f. The proof is obtained as the proof of Theorem 6, by knowing that $X \leqslant_{\mathrm{icv}} Y$ is equivalent to $\int_{-\infty}^{x} [F(t) - G(t)] \, \mathrm{d}t \geqslant 0$ for all $x \in \mathbb{R}$, and then applying Lemma 7.1 (b) of Barlow and Proschan [2]. $\square$

Corollary 7. *If A_i is non-decreasing for some $i = 1, 2$ and if $A_1(x)/\eta_1 \leqslant A_2(x)/\eta_2$ for all $x \in \mathbb{R}$, then $X_{w_1} \leqslant_{\rm icv} Y_{w_2}$ gives $X \leqslant_{\rm icv} Y$.*

In order for conditions of Theorems 6 and 7 to be well satisfied, it is noticeable here that a sufficient condition to get $B_2/\eta_2 \geqslant B_1/\eta_1$ is that the function B_2/B_1 is non-decreasing and also a sufficient condition for $A_1/\eta_1 \geqslant A_2/\eta_2$ to be valid is that A_2/A_1 is non-decreasing. Besides, if w_1 and w_2 are, respectively, non-decreasing and non-increasing, then B_1 and A_2 have the same monotonic properties accordingly (see Remark 2.2 in Misra et al. [11] for more detailed discussions).

3. Preservation of aging classes

In this section, using some representations of aging classes via stochastic orders we develop some results for preservation of several aging classes under weighting. Parallelly, some relevant characterizations are given. We will focus only on positive aging classes. Results for negative aging classes which indeed are dual classes for the classes that were defined in Section 1.1 can be similarly derived.

Theorem 8. *Let w be non-increasing (non-decreasing) and let $w(x)/\eta \leqslant (\geqslant)$ $w(x+t)/B(t)$ for all $x \geqslant 0$ and for all $t \geqslant 0$. Then X is NBU if and only if X_w is NBU.*

Proof. We know that X is NBU if and only if $X_t \leqslant_{\rm st} X$ for all $t > 0$. From Lemma 2.1 (ii) in Izadkhah et al. [6], $E(w(X_t + t)) = B(t)$. Taking $w_1(x) = w(x+t)$ and $w_2(x) = w(x)$, under the assumptions $X_t \leqslant_{\rm st} X$ is equivalent to $(X_t)_{w_1} \leqslant_{\rm st} X_{w_2}$ for all $t > 0$. By Lemma 2.2 of Izadkhah et al. [6], $(X_t)_{w_1} \stackrel{\rm st}{=} (X_w - t \mid X_w > t)$ for all $t > 0$. Thus, it follows that $(X_w - t \mid X_w > t) \leqslant_{\rm st} X_w$ for all $t > 0$, which means that X_w is NBU. $\square$

It is to be mentioned that when $l_X = 0$, if $w(t+x)/B(t)$ is increasing in t for all $x \geqslant 0$, then $w(x)/\eta \leqslant w(x+t)/B(t)$ holds true for all $x \geqslant 0$ and for all $t \geqslant 0$.

Theorem 9. *Let B be non-decreasing and let $w(x)E(X_w) \geqslant B(x)E(X)$ for all $x > 0$. Then $X \in$ NBUE implies $X_w \in$ NBUE.*

Proof. First, notice that $X \in$ NBUE if and only if $\int_x^\infty [f(t)E(X) - \overline{F}(t)]\,\mathrm{d}t \geqslant 0$ for all $t > 0$. We have, for all $x > 0$, that

$$\int_x^\infty [f_w(t)E(X_w) - \overline{F}_w(t)]\,\mathrm{d}t = \int_x^\infty \Big(\frac{w(t)E(X_w)}{\eta} f(t) - \frac{B(t)\overline{F}(t)}{\eta}\Big)\,\mathrm{d}t$$

$$\geqslant \int_x^\infty \frac{B(t)}{\eta}[E(X)f(t)-\overline{F}(t)]\,\mathrm{d}t = \int_0^\infty \frac{B(t)I_{[x,\infty)}(t)}{\eta}[E(X)f(t)-\overline{F}(t)]\,\mathrm{d}t,$$

where the inequality follows from the second assumption. Set $h(t)=\eta^{-1}B(t)I_{[x,\infty)}(t)$. From the first assumption, h is non-decreasing in t for all $x \geqslant 0$. Lemma 7.1 (a) of Barlow and Proschan [2] is applicable providing the proof of the theorem. $\square$

Corollary 8. *If B is non-increasing and if $w(x)E(X_w) \leqslant B(x)E(X)$ for all $x > 0$, then $X_w \in$ NBUE implies $X \in$ NBUE.*

Theorem 10. *Let X be* DMRL. *If B is non-decreasing and w/B is non-decreasing, then X_w is* DMRL.

Proof. We know that X is DMRL if and only if the mean residual life of X i.e., $m_F(x) = \int_x^\infty \overline{F}(t)\,\mathrm{d}t/\overline{F}(x)$ is non-increasing in x. We get

$$\frac{\mathrm{d}}{\mathrm{d}x}m_F(x) = \frac{f(x)}{\overline{F}(x)}\frac{\int_x^\infty \overline{F}(t)\,\mathrm{d}t}{\overline{F}(x)} - 1.$$

Thus, X is DMRL if and only if for all $x \in \mathbb{R}$,

$$\int_x^\infty [\overline{F}(x)f(t) - \overline{F}(t)f(x)]\,\mathrm{d}t \geqslant 0, \tag{3.1}$$

which is equivalent to $\int_y^\infty \mathrm{d}W_x(t) \geqslant 0$ for all $y \leqslant x \in \mathbb{R}$, where $\mathrm{d}W_x(t) = w(x,t)\,\mathrm{d}t$ where $w(x,t) = [\overline{F}(x)f(t) - \overline{F}(t)f(x)]I_{[x,\infty)}(t)$. Note that, by using (3.1) and by definition, we have

$$\begin{aligned} X \text{ is DMRL} &\Leftrightarrow \frac{\int_x^\infty \overline{F}(t)\,\mathrm{d}t}{\overline{F}(x)} \geqslant \frac{\int_y^\infty \overline{F}(t)\,\mathrm{d}t}{\overline{F}(y)} \quad \forall\, x \leqslant y \in \mathbb{R},\\ &\Leftrightarrow \frac{\overline{F}(x)}{f(x)} \geqslant \frac{\int_x^\infty \overline{F}(t)\,\mathrm{d}t}{\overline{F}(x)} \quad \forall\, x \in \mathbb{R}.\end{aligned}$$

Using the above inequalities, we obtain

$$\begin{aligned} X \text{ is DMRL} &\Rightarrow \frac{\overline{F}(x)}{f(x)} \geqslant \frac{\int_y^\infty \overline{F}(t)\,\mathrm{d}t}{\overline{F}(y)} \quad \forall\, x \leqslant y \in \mathbb{R},\\ &\Leftrightarrow \int_y^\infty [\overline{F}(x)f(t) - \overline{F}(t)f(x)]\,\mathrm{d}t \geqslant 0 \quad \forall\, x \leqslant y \in \mathbb{R},\\ &\Leftrightarrow \int_y^\infty \mathrm{d}W_x(t) \geqslant 0 \quad \forall\, x \leqslant y \in \mathbb{R}.\end{aligned}$$

Therefore, $X \in$ DMRL provides that $\int_y^\infty \mathrm{d}W_x(t) \geqslant 0$ for all $x \in \mathbb{R}$ and for all $y \in \mathbb{R}$. It can be here written by assumption that

$$\begin{aligned}
&\int_x^\infty [\overline{F}_w(x)f_w(t) - \overline{F}_w(t)f_w(x)]\,\mathrm{d}t \\
&\quad = \int_x^\infty \Big(\frac{B(x)w(t)f(t)\overline{F}(x)}{\eta^2} - \frac{B(t)w(x)f(x)\overline{F}(t)}{\eta^2}\Big)\,\mathrm{d}t \\
&\quad \geqslant \int_x^\infty \frac{B(t)w(x)}{\eta^2}[\overline{F}(x)f(t) - \overline{F}(t)f(x)]\,\mathrm{d}t \\
&\quad = \int_{-\infty}^\infty \frac{B(t)w(x)}{\eta^2}[\overline{F}(x)f(t) - \overline{F}(t)f(x)]I_{[x,\infty)}(t)\,\mathrm{d}t = \int_{-\infty}^\infty h(t)\,\mathrm{d}W_x(t),
\end{aligned}$$

where $h(t) = \eta^{-2}B(t)w(x)$, which is non-decreasing by assumption. Hence, Lemma 7.1 (a) of Barlow and Proschan [2] can be applied to obtain the proof. □

Corollary 9. *If B is non-increasing and if w/B is non-decreasing, then $X_w \in$ DMRL implies $X \in$ DMRL.*

Example 3 (Proportional hazards model). Consider the model $\overline{G}(x) = [\overline{F}(x)]^\theta$, $\theta > 0$. This model is referred to as the PHR model in the literature. The cdf G is easily shown to be a weighted version of F induced by the weight function $w(x) = [\overline{F}(x)]^{\theta-1}$ from which we get $B(x) = \theta^{-1}[\overline{F}(x)]^{\theta-1}$. It can be readily seen that if $\theta \in (0,1]$, then the assumptions of Theorem 10 hold. Thus, if F has the DMRL property then G has the DMRL property.

Theorem 11. *Let X be IRMR, let A be non-increasing and also let A/w be non-decreasing. Then X_w is IRMR.*

Proof. We know that X is IRMR if and only if the reversed mean residual life of X i.e., the function α given by $\alpha(x) = \int_{-\infty}^x F(t)\,\mathrm{d}t/F(x)$ is non-decreasing in x. It is obvious that

$$\frac{\mathrm{d}}{\mathrm{d}x}\alpha(x) = 1 - \frac{f(x)}{F(x)}\frac{\int_{-\infty}^x F(t)\,\mathrm{d}t}{F(x)}.$$

So, X is IRMR if and only if

$$\int_{-\infty}^x [F(x)f(t) - F(t)f(x)]\,\mathrm{d}t \geqslant 0 \quad \forall\, x \in \mathbb{R}, \tag{3.2}$$

which means that $\int_{-\infty}^y \mathrm{d}W_x(t) \geqslant 0$ for all $x \leqslant y \in \mathbb{R}$, where $\mathrm{d}W_x(t) = [F(x)f(t) - F(t)f(x)]I_{(-\infty,x]}(t)\,\mathrm{d}t$. Furthermore,

$$X \text{ is IRMR} \Leftrightarrow \frac{\int_{-\infty}^y F(t)\,\mathrm{d}t}{F(y)} \leqslant \frac{\int_{-\infty}^x F(t)\,\mathrm{d}t}{F(x)} \quad \forall\, y \leqslant x \in \mathbb{R}.$$

Therefore, using the above equivalence relation and via (3.2), if X is IRMR then

$$\frac{\int_{-\infty}^{y} F(t)\,\mathrm{d}t}{F(y)} \leqslant \frac{F(x)}{f(x)} \quad \forall\, y \leqslant x \in \mathbb{R},$$

which is equivalent to $\int_{-\infty}^{y} \mathrm{d}W_x(t) \geqslant 0$ for all $y \leqslant x \in \mathbb{R}$. That is $\int_{-\infty}^{y} \mathrm{d}W_x(t) \geqslant 0$ for all $x \in \mathbb{R}$ and for all $x \in \mathbb{R}$. In other direction, we deduce by assumption that, for all $x \in \mathbb{R}$,

$$\begin{aligned}
\int_{-\infty}^{x} &[F_w(x)f_w(t) - F_w(t)f_w(x)]\,\mathrm{d}t \\
&= \int_{-\infty}^{x} \Big(\frac{A(x)w(t)F(x)f(t)}{\eta^2} - \frac{A(t)w(x)f(x)F(t)}{\eta^2}\Big)\,\mathrm{d}t \\
&\geqslant \int_{-\infty}^{x} \frac{A(t)w(x)}{\eta^2}[F(x)f(t) - F(t)f(x)]\,\mathrm{d}t \\
&= \int_{-\infty}^{\infty} \frac{A(t)w(x)}{\eta^2}[F(x)f(t) - F(t)f(x)]I_{(-\infty,x]}(t)\,\mathrm{d}t = \int_{-\infty}^{\infty} h(t)\,\mathrm{d}W_x(t),
\end{aligned}$$

where $h(t) = \eta^{-2}A(t)w(x)$, which is non-increasing by assumption. At the end, by Lemma 7.1 (b) of Barlow and Proschan [2] it follows that $\int_{-\infty}^{\infty} h(t)\,\mathrm{d}W_x(t) \geqslant 0$ for all $x \in \mathbb{R}$, which completes the proof. $\square$

Corollary 10. *If A is non-decreasing and if A/w is non-increasing, then $X_w \in$ IRMR implies $X \in$ IRMR.*

E x a m p l e 4 (Proportional reversed hazards model). The model of $G(x) = [F(x)]^{\theta}$, $\theta > 0$, is well-known in the literature as the PRHR model. The distribution function G is a weighted version of F with the weight $w(x) = [F(x)]^{\theta-1}$, which gives $A(x) = \theta^{-1}[F(x)]^{\theta-1}$. It can be readily seen that if $\theta \in (0,1]$, then the assumptions of Theorem 11 hold. As a result, if F is IRMR then G is IRMR.

Theorem 12. *Let $X \in$ NBUC. If B is non-decreasing in x, and $B(x)/\eta \geqslant B(t+x)/B(t)$ for all $x \geqslant 0$ and for all $t \geqslant 0$. Then $X_w \in$ NBUC.*

P r o o f. Let $t \geqslant 0$ be fixed. Then, $X \in$ NBUC implies that $X_t \leqslant_{\mathrm{icx}} X$ for all $t \geqslant 0$. By an application of Lemma 2.1 in Izadkhah et al. [6], for $w_1(x) = w(t+x)$ and $w_2(x) = w(x)$, we have $B_1(x) = B(t+x)$, $\eta_1 = B(t)$ and also we know that $B_2(x) = B(x)$ and that $\eta_2 = \eta$. Now, Theorem 6 gives $X_w \geqslant_{\mathrm{icx}} (X_t)_{w_1}$. From Lemma 2.2 of Izadkhah et al. [6], $(X_t)_{w_1} \overset{\mathrm{st}}{=} (X_w - t \mid X_w > t)$ for all $t \geqslant 0$. Hence, $X_w \geqslant_{\mathrm{icx}} (X_w - t \mid X_w > t)$, for all $t \geqslant 0$, which provides the proof directly. $\square$

Corollary 11. *If B is non-increasing and $B(x)/\eta \leqslant B(t+x)/B(t)$ for all $t \geqslant 0$ and for all $x \geqslant 0$, then $X_w \in$ NBUC yields $X \in$ NBUC.*

Assume that $l_X = 0$ and that B is log-concave on $(0, \infty)$. Then $B(x)/\eta \geqslant B(t+x)/B(t)$ for all $t \geqslant 0$ and for all $x \geqslant 0$. It is to be mentioned here that if w is non-decreasing then B is also non-decreasing as discussed after Corollary 7.

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