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PERIODIC SOLUTION TO A MULTISPECIES PREDATOR-PREY COMPETITION DYNAMIC SYSTEM WITH BEDDINGTON-DEANGELIS FUNCTIONAL RESPONSE AND TIME DELAY

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Abstract. In this paper, we are concerned with a delayed multispecies competition predator-prey dynamic system with Beddington-DeAngelis functional response. Some sufficient conditions which guarantee the existence of a positive periodic solution for the system are obtained by applying the Mawhin coincidence theory. The interesting thing is that the result is related to the delays, which is different from the corresponding ones known from literature (the results are delay-independent).

Keywords: multispecies predator-prey model; competition dynamic system; positive periodic solution; Beddington-DeAngelis functional; time delays response

MSC 2010: 92D25, 34C25

1. Introduction

The dynamic relationship between predators and their preys has long been and will continue to be one of the dominant themes in both ecology and mathematical ecology due to its universal existence and importance. Recently, many authors have studied the predator-prey system with the Beddington-DeAngelis functional response which was first proposed by Beddington [1] and DeAngelis Goldstein and O'Neill [5], independently. Although they have made much progress in the study

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of predator-prey models with the Beddington-DeAngelis functional response, such models are not well studied yet in the sense that all the existing results are based on the assumption that the predator preys on one prey. This assumption is rarely the case in real life. Naturally, a more realistic and interesting model should take into account the predator preying on more than one prey. Zeng and Fan [14] proposed a more reasonable model in real life which takes on the form

$$(1.1)\qquad \begin{cases} \dot{x}_i(t) = x_i(t)\Big[a_i(t) - \sum_{j=1}^{n} b_{ij}(t)x_j(t) - \dfrac{c_i(t)y(t)}{\alpha_i(t)+\beta_i(t)x_i(t)+\gamma_i(t)y(t)}\Big], \\ \hfill i = 1,2,\ldots,n, \\ \dot{y}(t) = y(t)\Big[-d(t) + \sum_{j=1}^{n} \dfrac{f_j(t)x_j(t)}{\alpha_j(t)+\beta_j(t)x_j(t)+\gamma_j(t)y(t)}\Big]. \end{cases}$$

By applying the comparison theorem they obtained a result on the existence of almost periodic solutions. As was pointed out by Kuang [8], any model of species dynamics without delays is an approximation at best. More detailed arguments on the importance and usefulness of time-delays in realistic models may also be found in the classical books of Macdonald [12] and Gopalsamy [7]. Some excellent results for delayed biological systems have been obtained by many researchers such as Beretta and Kuang [2], Lu [11], Cai, Huang and Chen [4].

Motivated by the above reasons and considering that the delay may occur in the competition among preys, in this paper we consider the delayed differential system with Beddington-DeAngelis functional response

$$(1.2)\qquad \begin{cases} \dot{x}_i(t) = x_i(t)\Big[a_i(t) - \sum_{j=1}^{n} b_{ij}(t)x_j(t-\tau_j(t)) - \dfrac{c_i(t)y(t)}{\alpha_i(t)+\beta_i(t)x_i(t)+\gamma_i(t)y(t)}\Big], \\ \hfill i = 1,2,\ldots,n, \\ \dot{y}(t) = y(t)\Big[-d(t) + \sum_{j=1}^{n} \dfrac{f_j(t)x_j(t)}{\alpha_j(t)+\beta_j(t)x_j(t)+\gamma_j(t)y(t)}\Big], \end{cases}$$

with initial conditions

$$(1.3)\qquad \begin{aligned} x_i(\theta) &= \varphi_i(\theta), \quad \theta \in [-\tau, 0], \quad \varphi_i(\theta) \in C([-\tau,0],\mathbb{R}_+), \quad i = 1,2,\ldots,n, \\ y(\theta) &= \psi(\theta), \quad \theta \in [-\tau, 0], \quad \psi(\theta) \in C([-\tau,0],\mathbb{R}_+), \end{aligned}$$

where $x_i(t)$, $y(t)$ denote the size of prey and predator populations at time t, respectively, $a_i(t)$, $b_{ij}(t)$, $c_i(t)$, $d(t)$, $f_i(t)$, $\alpha_i(t)$, $\beta_i(t)$, $\gamma_i(t)$: $\mathbb{R} \to [0,+\infty)$ $(i, j = 1,2,\ldots,n)$ are continuous positive periodic functions with a period T,

$$\tau = \max\{\tau_j(t),\ t \in [0,T],\ j = 1,2,\ldots,n\},$$

$\tau_j(t)$ is a nonnegative and continuously differentiable periodic function with the period T on $\mathbb{R}$. Obviously, system (1.1) is a special case of systems (1.2)–(1.3) if one chooses $\tau_j(t) \equiv 0$ in systems (1.2)–(1.3).

We define

$$\overline{f} = \frac{1}{T}\int_0^T f(t)\,\mathrm{d}t, \quad f^L = \min_{t\in[0,T]} f(t), \quad f^M = \max_{t\in[0,T]} f(t),$$

where f is a continuous T-periodic function.

The aim of this paper is to obtain sufficient conditions for the existence of positive periodic solutions for system (1.2) by using the Mawhin coincidence theorem and some analysis approaches. It is interesting that the result obtained in this paper is related to the delay $\tau_j(t)$ $(j = 1, 2, \dots, n)$ (or delay-dependent), which makes it different from the previous works [4], [9], [10], [13] that are delay-independent.

2. Lemmas

In order to present sufficient conditions guaranteeing the existence of a positive periodic solution for the system (1.2), we first introduce the coincidence degree theorem.

Let X and Y be two Banach spaces, $L\colon \operatorname{Dom} L \subset X \to Y$ a linear map, and $N\colon X \to Y$ a continuous map. If $\dim \operatorname{Ker} L = \operatorname{codim} \operatorname{Im} L < +\infty$ and $\operatorname{Im} L \in Y$ is closed, then the operator L is called a Fredholm operator with index zero [6]. And if L is a Fredholm operator with index zero and there exist continuous projections $P\colon X \to X$ and $Q\colon Y \to Y$ such that $\operatorname{Im} P = \operatorname{Ker} L$, $\operatorname{Im} L = \operatorname{Ker} Q = \operatorname{Im}(I - Q)$, then $L|_{\operatorname{Dom} L \cap \operatorname{Ker} P}\colon (I - P)X \to \operatorname{Im} L$ has an inverse function; we denote it by K_p. Assume $\Omega \in X$ is any open set. If $QN(\overline{\Omega})$ is bounded and $K_p(I - Q)N(\overline{\Omega}) \in X$ is relative compact, then we say $N \in \overline{\Omega}$ is L-compact.

Now we recall the Mawhin coincidence theorem.

Lemma 2.1 ([6])**.** *Let both X and Y be Banach spaces, $L\colon \operatorname{Dom} L \subset X \to Y$ a Fredholm operator with index zero, $\Omega \in Y$ an open bounded set, and let $N\colon \overline{\Omega} \to X$ be L-compact on $\overline{\Omega}$. If all the following conditions hold:*

(C_1) *$Lx \neq \lambda Nx$ for $x \in \partial\Omega \cap \operatorname{Dom} L, \lambda \in (0, 1)$;*

(C_2) *$Nx \notin \operatorname{Im} L$ for $x \in \partial\Omega \cap \operatorname{Ker} L$;*

(C_3) *$\deg\{JQN, \Omega \cap \operatorname{Ker} L, 0\} \neq 0$, where $J\colon \operatorname{Im} Q \to \operatorname{Ker} L$ is an isomorphism;*

then the equation $Lx = Nx$ has at least one solution on $\overline{\Omega} \cap \operatorname{Dom} L$.

Lemma 2.2 ([11]). *If $\tau \in C^1(\mathbb{R},\mathbb{R})$, with $\tau(t+T) \equiv \tau(t)$ and $\dot\tau(t) < 1$ for $t \in [0,T]$, then the function $\varphi(t) = t - \tau(t)$ has a unique inverse $\varphi^{-1}(t)$ satisfying $\varphi \in C(\mathbb{R},\mathbb{R})$ with $\varphi^{-1}(s+T) \equiv \varphi^{-1}(s) + T$ for $s \in [0,T]$.*

Lemma 2.3 ([3]). *Suppose $g(t)$ is a differentiable continuous T-periodic function on $\mathbb{R}$ for $\forall t_0 \in \mathbb{R}$, $\forall t_1, t_2 \in [t_0, t_0+T]$, then*

$$g(t) \leqslant g(t_1) + \int_{t_0}^{t_0+T} |\dot g(t)|\,\mathrm{d}t,\ \ g(t) \geqslant g(t_2) - \int_{t_0}^{t_0+T} |\dot g(t)|\,\mathrm{d}t.$$

3. Existence of periodic solutions

Theorem 3.1. *Let the following conditions hold:*

(A_1) $\dot\tau_j(t) < 1$ $(j = 1,2,\ldots,n)$ *for* $t \in \mathbb{R}$;

(A_2) $\bar a_i - \sum\limits_{j=1, j\neq i}^{n} \overline{B}_{ij}\exp(H_j) - \big(\overline{c_i/\alpha_i}\big)\exp(\widetilde H) > 0$, $i = 1,2,\ldots,n$;

(A_3) *any of the following inequalities holds:*

$$\overline{f}_j\exp(S_j) - \bar d(\alpha_j^M + \beta_j^M\exp(S_j)) > 0, \quad j = 1,2,\ldots,n.$$

Then system (1.2) has at least one positive T-periodic solution, where $B_{ij}(t)$, H_j, $\widetilde H$, S_1 are defined in the following proof.

Proof. Since

$$x_i(t) = x_i(0)\exp\bigg\{\int_0^t \bigg[a_i(s) - \sum_{j=1}^n b_{ij}(s)x_j(s-\tau_j(s)) - \frac{c_i(s)y(s)}{\alpha_i(s)+\beta_i(s)x_i(s)+\gamma_i(s)y(s)}\bigg]\,\mathrm{d}s\bigg\},$$

$$y(t) = y(0)\exp\bigg\{\int_0^t \bigg[-d(s) + \sum_{j=1}^n \frac{f_j(s)x_j(s)}{\alpha_j(s)+\beta_j(s)x_j(s)+\gamma_j(s)y(s)}\bigg]\,\mathrm{d}s\bigg\},$$

the solution of system (1.2) remains positive for all $t \in \mathbb{R}$. We let

$$x_i(t) = \mathrm{e}^{u_i(t)},\ y(t) = \mathrm{e}^{v(t)},\ i = 1,2,\ldots,n. \tag{3.1}$$

On substituting (3.1) into system (1.2), this system can be reformulated in the form

$$\begin{cases} \dot u_i(t) = a_i(t) - \sum\limits_{j=1}^n b_{ij}(t)\exp\{u_j(t-\tau_j(t))\} - \dfrac{c_i(t)\exp\{v(t)\}}{\alpha_i(t)+\beta_i(t)\exp\{u_i(t)\}+\gamma_i(t)\exp\{v(t)\}}, \\ \dot v(t) = -d(t) + \sum\limits_{j=1}^n \dfrac{f_j(t)\exp\{u_j(t)\}}{\alpha_j(t)+\beta_j(t)\exp\{u_j(t)\}+\gamma_j(t)\exp\{v(t)\}}. \end{cases} \tag{3.2}$$

In order to apply Lemma 2.1 to the study of existence of positive periodic solutions to the above system, we set

$$X = Y = \{z(t) = (u(t), v(t))^\top = (u_1(t), u_2(t), \dots, u_n(t), v(t))^\top \in C(\mathbb{R}, \mathbb{R}^{n+1})\colon\ z(t+T) \equiv z(t)\},$$

equipped with the norm

$$\|z\| = \|(u(t), v(t))^\top\| = \sum_{i=1}^{n} \max_{t\in[0,T]} |u_i(t)| + \max_{t\in[0,T]} |v(t)|.$$

Then both X and Y are Banach spaces, where $\top$ is the transpose.

Take $z \in X$, the periodicity yields that both

$$\mathcal{F}_i(z,t) = a_i(t) - \sum_{j=1}^{n} b_{ij}(t) \exp\{u_j(t - \tau_j(t))\} - \frac{c_i(t)\exp\{v(t)\}}{\alpha_i(t) + \beta_i(t)\exp\{u_i(t)\} + \gamma_i(t)\exp\{v(t)\}} \in C(\mathbb{R}, \mathbb{R}),$$

$$\mathcal{G}(z,t) = -d(t) + \sum_{j=1}^{n} \frac{f_j(t)\exp\{u_j(t)\}}{\alpha_j(t) + \beta_j(t)\exp\{u_j(t)\} + \gamma_j(t)\exp\{v(t)\}} \in C(\mathbb{R}, \mathbb{R}),$$

are T-periodic. Define operators L, P, and Q as follows,

$$L\colon \operatorname{Dom} L \cap X \to Y, \quad Lz = \frac{\mathrm{d}z}{\mathrm{d}t}, \quad P(z) = z(0), \quad Q(z) = \frac{1}{T}\int_0^T z(t)\,\mathrm{d}t,$$

where $\operatorname{Dom} L = \{z;\ z \in X\colon\ z(t) \in C^1(\mathbb{R}, \mathbb{R}^{n+1})\}$, and define $N\colon\ X \to Y$ by

$$Nz = (\mathcal{F}(z,t), \quad \mathcal{G}(z,t))^\top = (\mathcal{F}_1(z,t), \mathcal{F}_2(z,t), \dots, \mathcal{F}_n(z,t), \mathcal{G}(z,t))^\top.$$

Then

$$\operatorname{Ker} L = \mathbb{R}^{n+1}, \quad \dim \operatorname{Ker} L = \operatorname{codim} \operatorname{Im} L,$$

and

$$\operatorname{Im} L = \left\{ z \in Y\colon\ \int_0^T z(t)\,\mathrm{d}t = 0 \right\}$$

is closed in Y, and both P, Q are continuous projections satisfying

$$\operatorname{Im} P = \operatorname{Ker} L, \quad \operatorname{Im} L = \operatorname{Ker} Q = \operatorname{Im}(I - Q).$$

So L is a Fredholm operator with index zero, which implies that L has a unique inverse. We denote by $K_p\colon \operatorname{Im} L \to \operatorname{Ker} P \cap \operatorname{Dom} L$ the inverse of L. By simple calculation, we obtain

$$K_p(z) = \int_0^t z(s)\,\mathrm{d}s - \frac{1}{T}\int_0^T\int_0^t z(s)\,\mathrm{d}s\,\mathrm{d}t.$$

Therefore,

$$\begin{aligned} QNz = \Big(\frac{1}{T}\int_0^T \mathcal{F}_1(z,s)\,\mathrm{d}s,\ \frac{1}{T}\int_0^T \mathcal{F}_2(z,s)\,\mathrm{d}s, \dots,& \\ \frac{1}{T}\int_0^T \mathcal{F}_n(z,s)\,\mathrm{d}s,\ \frac{1}{T}\int_0^T \mathcal{G}(z,s)\,\mathrm{d}s\Big)^{\top}& \end{aligned}$$

and

$$K_p(I-Q)Nz = (\eta_1,\ \eta_2, \dots, \eta_n, \gamma)^{\top},$$

where

$$\begin{aligned} \eta_i &= \int_0^T \mathcal{F}_i(z,s)\,\mathrm{d}s - \frac{1}{T}\int_0^T\int_0^t \mathcal{F}_i(z,s)\,\mathrm{d}s\,\mathrm{d}t - \Big(\frac{t}{T}-\frac{1}{2}\Big)\int_0^T \mathcal{F}_i(z,s)\,\mathrm{d}s, \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad i = 1,2,\dots,n, \\ \gamma &= \int_0^T \mathcal{G}(z,s)\,\mathrm{d}s - \frac{1}{T}\int_0^T\int_0^t \mathcal{G}(z,s)\,\mathrm{d}s\,\mathrm{d}t - \Big(\frac{t}{T}-\frac{1}{2}\Big)\int_0^T \mathcal{G}(z,s)\,\mathrm{d}s. \end{aligned}$$

Obviously, it is not difficult to check by the Lebesgue convergence theorem that both QN and $K_p(I-Q)N$ are continuous. By using the Arzela-Ascoli Theorem, we know that the operator $K_p(I-Q)N(\overline{\Omega})$ is compact and $QN(\overline{\Omega})$ is bounded for any open set $\Omega \in X$. So $N \in \Omega$ is L-compact on $\overline{\Omega}$.

In order to apply Lemma 2.1, we need to search for an appropriate open, bounded subset Ω. Corresponding to the operator equation $Lz = \lambda Nz$ for $\lambda \in (0,1)$, we have

$$\text{(3.3)}\qquad \begin{cases} \dot{u}_i(t) = \lambda\Big[a_i(t) - \sum\limits_{j=1}^{n} b_{ij}(t)\exp\{u_j(t-\tau_j(t))\} \\ \qquad\qquad - \dfrac{c_i(t)\exp\{v(t)\}}{\alpha_i(t)+\beta_i(t)\exp\{u_i(t)\}+\gamma_i(t)\exp\{v(t)\}}\Big], \\ \dot{v}(t) = \lambda\Big[-d(t) + \sum\limits_{j=1}^{n} \dfrac{f_j(t)\exp\{u_j(t)\}}{\alpha_j(t)+\beta_j(t)\exp\{u_j(t)\}+\gamma_j(t)\exp\{v(t)\}}\Big]. \end{cases}$$

Assume that $(u(t), v(t))^\top = (u_1(t), \dots, u_n(t), v(t))^\top \in X$ is a solution of (3.3) for a certain $\lambda \in (0,1)$. Integrating (3.3) over the interval $[0,T]$, we obtain

$$\tag{3.4} \int_0^T \sum_{j=1}^n b_{ij}(t)\exp\{u_j(t-\tau_j(t))\}\,\mathrm{d}t + \int_0^T \frac{c_i(t)\exp\{v(t)\}}{\alpha_i(t)+\beta_i(t)\exp\{u_i(t)\}+\gamma_i(t)\exp\{v(t)\}}\,\mathrm{d}t = \int_0^T a_i(t)\,\mathrm{d}t,$$

$$\tag{3.5} \int_0^T \sum_{j=1}^n \frac{f_j(t)\exp\{u_j(t)\}}{\alpha_j(t)+\beta_j(t)\exp\{u_j(t)\}+\gamma_j(t)\exp\{v(t)\}}\,\mathrm{d}t = \int_0^T d(t)\,\mathrm{d}t.$$

It follows from (3.3)–(3.5) that

$$\tag{3.6} \int_0^T |\dot u_i(t)|\,\mathrm{d}t < \int_0^T a_i(t)\,\mathrm{d}t + \int_0^T \left[\sum_{j=1}^n b_{ij}(t)\exp\{u_j(t-\tau_j(t))\} + \frac{c_i(t)\exp\{v(t)\}}{\alpha_i(t)+\beta_i(t)\exp\{u_i(t)\}+\gamma_i(t)\exp\{v(t)\}}\right]\mathrm{d}t = 2\bar a_i T,$$

$$\tag{3.7} \int_0^T |\dot v(t)|\,\mathrm{d}t < \int_0^T \mathrm{d}(t)\,\mathrm{d}t + \int_0^T \left[\sum_{j=1}^n \frac{f_j(t)\exp\{u_j(t)\}}{\alpha_j(t)+\beta_j(t)\exp\{u_j(t)\}+\gamma_j(t)\exp\{v(t)\}}\right]\mathrm{d}t = 2\bar d T.$$

Since $(u(t), v(t))^\top \in X$, there exist ξ_{i_*}, ξ_i^*, ζ_*, $\zeta^* \in [0,T]$ such that

$$\tag{3.8} u_i(\xi_{i_*}) = \min_{t\in[0,T]} u_i(t), \quad u_i(\xi_i^*) = \max_{t\in[0,T]} u_i(t),$$

$$\tag{3.9} v(\zeta_*) = \min_{t\in[0,T]} v(t), \quad v(\zeta^*) = \max_{t\in[0,T]} v(t).$$

In view of Lemma 2.2 and condition (A_1), we get

$$\tag{3.10} \begin{aligned} \int_0^T b_{ij}(t)\exp\{u_j(t-\tau_j(t))\}\,\mathrm{d}t &= \int_{-\tau_j(0)}^{T-\tau_j(T)} \frac{b_{ij}(\varphi_j^{-1}(t))\exp\{u_j(t)\}}{1-\dot\tau_j(\varphi_j^{-1}(t))}\,\mathrm{d}t \\ &= \int_0^T \frac{b_{ij}(\varphi_j^{-1}(t))\exp\{u_j(t)\}}{1-\dot\tau_j(\varphi_j^{-1}(t))}\,\mathrm{d}t \\ &= \int_0^T B_{ij}(t)\exp\{u_j(t)\}\,\mathrm{d}t, \end{aligned}$$

in which φ_j^{-1} is the inverse function of $\varphi_j = t-\tau_j(t)$, $B_{ij}(t) = \frac{b_{ij}(\varphi_j^{-1}(t))}{1-\dot\tau_j(\varphi_j^{-1}(t))}$. It follows from (3.4), (3.8), and (3.10) that

$$\int_0^T B_{ii}(t)\exp\{u_i(t)\}\,\mathrm{d}t < \bar a_i T,$$

which yields

$$u_i(\xi_{i_*}) < \ln \frac{\bar{a}_i}{\overline{B}_{ii}}. \tag{3.11}$$

From Lemma 2.3, (3.6), and (3.11), we get

$$u_i(t) \leqslant u_i(\xi_{i_*}) + \int_0^T |\dot{u}_i(t)| \,\mathrm{d}t < \ln \frac{\bar{a}_i}{\overline{B}_{ii}} + 2\bar{a}_i T \equiv H_i. \tag{3.12}$$

In view of (3.5) and (3.9), we obtain

$$\int_0^T \sum_{j=1}^n \frac{f_j(t)\exp(H_j)}{\gamma_j(t)\exp\{v(\zeta_*)\}} \,\mathrm{d}t > \bar{d}T,$$

which implies

$$v(\zeta_*) < \ln \sum_{j=1}^n \Big(\overline{\frac{f_j}{\gamma_j}}\Big) \frac{\exp(H_j)}{\bar{d}}. \tag{3.13}$$

From Lemma 2.3, (3.7), and (3.13), we obtain

$$v(t) \leqslant v(\zeta_*) + \int_0^T |\dot{v}(t)| \,\mathrm{d}t < \ln \sum_{j=1}^n \Big(\overline{\frac{f_j}{\gamma_j}}\Big) \frac{\exp(H_j)}{\bar{d}} + 2\bar{d}T \equiv \widetilde{H}. \tag{3.14}$$

On the other hand, from (3.4), (3.8), (3.10) and condition (A_2), we get

$$\overline{B}_{ii} T \exp\{u_i(\xi_i^*)\} + \sum_{j=1, j\neq i}^n \overline{B}_{ij} T \exp(H_j) + \Big(\overline{\frac{c_i}{\alpha_i}}\Big) T \exp(\widetilde{H}) > \bar{a}_i T,$$

$$u_i(\xi_i^*) > \ln \frac{1}{\overline{B}_{ii}} \Big[\bar{a}_i - \sum_{j=1, j\neq i}^n \overline{B}_{ij} \exp(H_j) - \Big(\overline{\frac{c_i}{\alpha_i}}\Big) \exp(\widetilde{H})\Big]. \tag{3.15}$$

We derive from Lemma 2.3, (3.7), and (3.15) that

$$\begin{aligned} u_i(t) &\geqslant u_i(\xi_i^*) - \int_0^T |\dot{u}_i(t)| \,\mathrm{d}t \\ &> \ln \frac{1}{\overline{B}_{ii}} \Big[\bar{a}_i - \sum_{j=1, j\neq i}^n \overline{B}_{ij} \exp(H_j) - \Big(\overline{\frac{c_i}{\alpha_i}}\Big) \exp(\widetilde{H})\Big] - 2\bar{a}_i T \equiv S_i, \end{aligned} \tag{3.16}$$

which, together with (3.12), yields

$$\max_{t\in[0,T]}|u_i(t)| < \max\Big\{\Big|\ln\frac{\bar a_i}{\overline{B}_{ii}}\Big| + 2\bar a_i T,\ \Big|\ln\frac{1}{\overline{B}_{ii}}\Big[\bar a_i - \sum_{j=1,j\neq i}^{n}\overline{B}_{ij}\exp(H_j) - \Big(\overline{\frac{c_i}{\alpha_i}}\Big)\exp(\widetilde{H})\Big]\Big| + 2\bar a_i T\Big\} \equiv R_i. \tag{3.17}$$

Similarly, from (3.5), (3.9), and (3.16), noticing that $t/(m+nt)$ $(m,n>0)$ is increasing for $t>0$, we have

$$\frac{\overline{f}_j T\exp(S_j)}{\alpha_j^M + \beta_j^M\exp(S_j) + \gamma_j^M\exp(\zeta^*)} < \bar dT, \quad j=1,2,\dots,n.$$

In view of condition (A_3), assuming without loss of generality, that when $j=j_0$ $(j_0=1,2,\dots,n)$, the inequality

$$\overline{f}_{j_0}\exp(S_{j_0}) - \bar d(\alpha_{j_0}^M + \beta_{j_0}^M\exp(S_{j_0})) > 0$$

holds, one has

$$v(\zeta^*) > \ln\frac{\overline{f}_{j_0}\exp(S_{j_0}) - \bar d(\alpha_{j_0}^M + \beta_{j_0}^M\exp(S_{j_0}))}{\bar d\gamma_{j_0}^M}. \tag{3.18}$$

Thus,

$$v(t) \geqslant v(\zeta^*) - \int_0^T|\dot v(t)|\,\mathrm{d}t > \ln\frac{\overline{f}_{j_0}\exp(S_{j_0}) - \bar d(\alpha_{j_0}^M + \beta_{j_0}^M\exp(S_{j_0}))}{\bar d\gamma_{j_0}^M} - 2\bar dT, \tag{3.19}$$

which together with (3.14) gives

$$\max_{t\in[0,T]}|v(t)| < \max\Big\{\Big|\ln\sum_{j=1}^{n}\Big(\overline{\frac{f_j}{\gamma_j}}\Big)\frac{\exp(H_j)}{\bar d}\Big| + 2\bar dT,\ \Big|\ln\frac{\overline{f}_{j_0}\exp(S_{j_0}) - \bar d(\alpha_{j_0}^M + \beta_{j_0}^M\exp(S_{j_0}))}{\bar d\gamma_{j_0}^M}\Big| + 2\bar dT\Big\} \equiv \widetilde{R}. \tag{3.20}$$

Clearly $R_i, \widetilde{R}$ in (3.17) and (3.20) are independent of λ. Set $M = \sum\limits_{i=1}^{n} R_i + \widetilde{R} + R_0$, where R_0 is taken sufficiently large such that each solution (if it exists) $z^* = (u^*, v^*)^\top = (u_1^*, \dots, u_n^*, v^*)^\top \in \mathbb{R}^{n+1}$ of the algebraic equation

$$QNz = (\tilde u_1, \dots, \tilde u_n, \tilde v)^\top = 0 \tag{3.21}$$

satisfies $\|z^*\| = \|(u^*, v^*)^\top\| = \sum\limits_{j=1}^{n} |u_j^*| + |v^*| < M$, in which

$$\tilde{u}_i = \bar{a}_i - \sum_{j=1}^{n} \bar{b}_{ij} \exp(u) - \Big(\overline{\frac{c_i}{\alpha_i + \beta_i \exp(u_i) + \gamma_i \exp(v)}}\Big) \exp(v),$$

$$\tilde{v} = -\bar{d} + \sum_{j=1}^{n} \Big(\overline{\frac{f_j}{\alpha_j + \beta_j \exp(u_j) + \gamma_j \exp(v)}}\Big) \exp(u_j).$$

We now take $\Omega = \{z = (u(t), v(t))^\top = (u_1(t), \ldots, u_n(t), v(t))^\top | z \in X, \|z\| < M\}$. It is clear that Ω verifies the condition (C_1) in Lemma 2.1. When $z = (u, v)^\top \in \partial\Omega \cap \operatorname{Ker} L = \partial\Omega \cap \mathbb{R}^{n+1}$, $z = (u, v)^\top$ is a constant vector in $\mathbb{R}^{n+1}$ with $\|z\| = \|(u, v)^\top\| = \sum\limits_{j=1}^{n} |u_j| + |v| = M$, then we have $QNz \neq 0$. This proves that condition (C_2) in Lemma 2.1 is satisfied.

Finally, we will show that condition (C_3) in Lemma 2.1 holds. Since

$$z = (u, v)^\top = (u_1, \ldots, u_n, v)^\top$$

is a constant vector in $\mathbb{R}^{n+1}$, by virtue of the mean value theorem there exist θ_i, $\tilde{\theta}_j$ such that

$$QNz = QN(u, v)^\top = (p_1 + q_1, \ldots, p_n + q_n, \tilde{p} + \tilde{q})^\top,$$

where

$$\begin{aligned} p_i &= \bar{a}_i - \bar{b}_{ii} \exp(u_i), \\ q_i &= -\sum_{j=1, j\neq i}^{n} \bar{b}_{ij} \exp(u_j) - \frac{\bar{c}_i \exp(v)}{\alpha_i(\theta_i) + \beta_i(\theta_i) \exp(u_i) + \gamma_i(\theta_i) \exp(v)}, \\ \tilde{p} &= \sum_{j=1}^{n} \frac{\overline{f}_j \exp(u_j)}{\alpha_j(\tilde{\theta}_j) + \beta_j(\tilde{\theta}_j) \exp(u_j) + \gamma_j(\tilde{\theta}_j) \exp(v)}, \quad \tilde{q} = 0. \end{aligned}$$

Define the homotopy φ: $\operatorname{Dom} L \times [0, 1] \to X$ by

$$\varphi((u, v)^\top, \mu) = (p_1, \ldots, p_n, \tilde{p})^\top + \mu(q_1, \ldots, q_n, \tilde{q})^\top,$$

where $\mu \in [0, 1]$ is a parameter, $(u, v)^\top \in \partial\Omega \cap \operatorname{Ker} L = \partial\Omega \cap \mathbb{R}^{n+1}$, $(u, v)^\top$ is a constant vector in $\mathbb{R}^{n+1}$ with $\|(u, v)^\top\| = \sum\limits_{j=1}^{n} |u_j| + |v| = M$. We will show that when $(u, v)^\top \in \partial\Omega \cap \operatorname{Ker} L$, then $\varphi((u, v)^\top, \mu) \neq 0$. Assume that the conclusion is not true, i.e., the constant vector $(u, v)^\top$ with $\|(u, v)^\top\| = \sum\limits_{j=1}^{n} |u_j| + |v| = M$ satisfies

$\varphi((u,v)^\top,\mu)=0$. Similarly to the arguments of (3.11)–(3.20) and from the definition of M, we have $\|(u,v)^\top\| = \sum\limits_{j=1}^{n} |u_j| + |v| < M$. Obviously, the algebraic equation

$$QN(u,v)^\top = \varphi((u,v)^\top,0) = 0$$

has a unique solution $(u^*,v^*)^\top = (u_1^*,\dots,u_n^*,v^*)^\top$. We select J, the isomorphism of $\operatorname{Im} Q$ onto $\operatorname{Ker} L$ as an identity map. So, due to the homotopy invariance theorem of topology degree we have

$$\begin{aligned}\deg\{JQN(u,v)^\top,\Omega\cap\operatorname{Ker}L,0\} &= \deg\{\varphi((u,v)^\top,1),\Omega\cap\operatorname{Ker}L,0\}\\ &= \deg\{\varphi((u,v)^\top,0),\Omega\cap\operatorname{Ker}L,0\} = \operatorname{sign}\{\det A\} = (-1)^{n+1}\neq 0,\end{aligned}$$

where

$$A=\begin{bmatrix} -\bar{b}_{11}\mathrm{e}^{u_1^*} & \dots & 0 & 0\\ \vdots & \ddots & \vdots & \vdots\\ 0 & \dots & -\bar{b}_{nn}\mathrm{e}^{u_n^*} & 0\\ * & * & * & -\sum\limits_{j=1}^{n}\frac{\overline{f}_j\gamma_j(\tilde{\theta}_j)\exp(u_j^*+v^*)}{[\alpha_j(\tilde{\theta}_j)+\beta_j(\tilde{\theta}_j)\exp(u_j^*)+\gamma_j(\tilde{\theta}_j)\exp(v^*)]^2}\end{bmatrix}.$$

By now we have verified all the requirements of the Mawhin coincidence theorem in Ω and hence the system (1.2) has at least one positive T-periodic solution. Therefore, the system (1.2) has at least one positive T-periodic solution. □

R e m a r k 3.1. From the condition (A_1) in Theorem 3.1, we can see that the result of this paper is related to the delays $\tau_j(t), j=1,2,\dots,n$, which makes it different from the corresponding ones of [4], [9], [10], [13], i.e., the results obtained in [4], [9], [10], [13] are not related to the delays (delay-independent). The time delays $\tau_j(t), j=1,2,\dots,n$, are very important for our results.

4. Examples and numeric simulations

When $\tau(t)$ is a nonnegative constant, we give two suitable examples together with their numeric simulations to verify the result by using MatLab.

E x a m p l e 4.1. As an application, we consider the following system:

$$\text{(4.1)}\quad \begin{cases} \dot{x} = x(t)\Big[3+0.5\sin t-(0.5+0.1\sin t)x(t-0.5)\\ \qquad\qquad -\dfrac{(0.8-0.2\cos t)y(t)}{(0.3-0.1\sin t)+(0.6+0.1\cos t)x(t)+y(t)}\Big],\\ \dot{y} = y(t)\Big[-(0.3-0.2\sin t)+\dfrac{(1.8-0.5\sin t)x(t)}{(0.3-0.1\sin t)+(0.6+0.1\cos t)x(t)+y(t)}\Big],\end{cases}$$

with initial conditions $\varphi_1(0)=2.5$, $\varphi(0)=0.5$.

It is not difficult to verify that the coefficients of system (4.1) satisfy the conditions in Theorem 3.1, so we see that the system (4.1) has at least one positive 2π-periodic solution. Its integral curves and orbits are shown in Figs. 1–4, respectively; we see that the predator-y and prey-x are persistent.

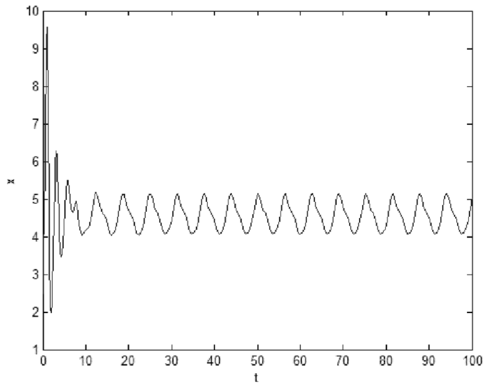

Fig. 1. The integral curve of prey(x)-time(t).

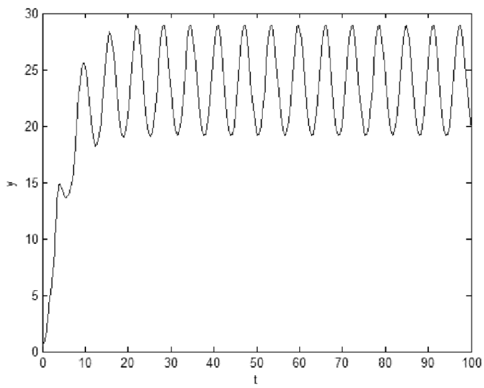

Fig. 2. The integral curve of predator(y)-time(t).

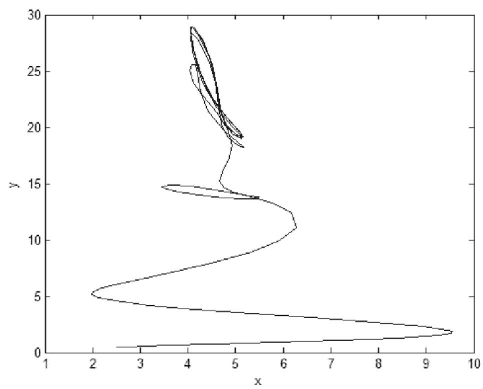

Fig. 3. The orbit of predator(y)-prey(x).

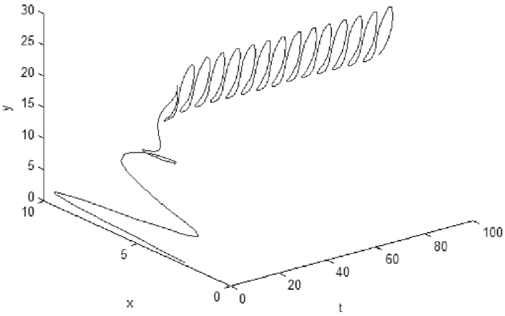

Fig. 4. The orbit of predator(y)-prey(x)-time(t).

Example 4.2. We consider a three-species competition system with time delays

$$(4.2)\quad \begin{cases} \dot{x}_1 = x_1(t)\Big[(2.5+0.1\cos t) - 0.3x_1(t-0.1) \\ \qquad\qquad -0.1x_2(t-0.1) - \dfrac{0.1y(t)}{0.2+0.1x_1(t)+0.3y(t)}\Big], \\ \dot{x}_2 = x_2(t)\Big[(2-0.2\sin t) - 0.2x_1(t-0.1) \\ \qquad\qquad -0.1x_2(t-0.1) - \dfrac{0.1y(t)}{0.3+0.2x_2(t)+0.3y(t)}\Big], \\ \dot{y} = y(t)\Big[-(0.5+0.1\cos t) \\ \qquad\qquad +\dfrac{3x_1(t)}{0.2+0.1x_1(t)+0.3y(t)} + \dfrac{x_2(t)}{0.3+0.2x_2(t)+0.3y(t)}\Big], \end{cases}$$

with initial conditions $\varphi_1(0)=2$, $\varphi_2(0)=2$, $\varphi(0)=1.5$.

It is obvious that (A_1), (A_2), and (A_3) hold. So by Theorem 3.1 we claim that the system (4.2) has at least one positive 2π-periodic solution. Its integral curves and orbits are shown in Figs. 5–8, respectively. From Figs. 5–8, we see that the predator-y and prey-x are persistent.

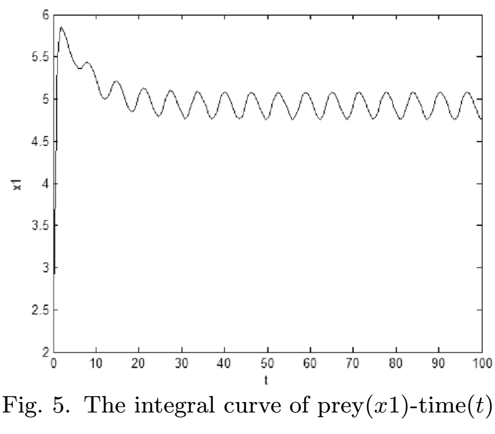

Fig. 5. The integral curve of prey($x1$)-time(t).

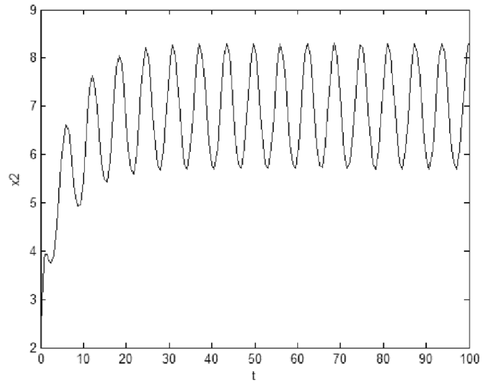

Fig. 6. The integral curve of prey($x2$)-time(t).

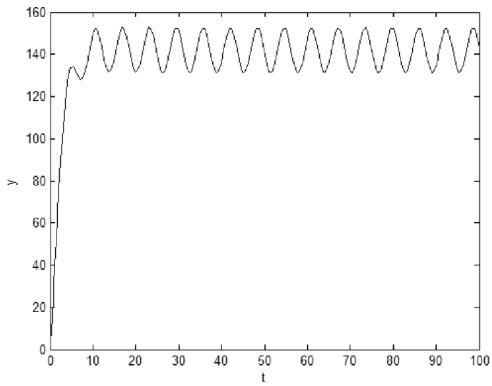

Fig. 7. The orbit of predator(y)-time(t).

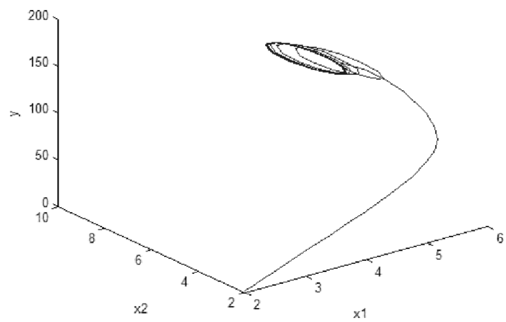

Fig. 8. The orbits of predator(y)-prey($x2$)-prey($x1$).

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