NONCONFORMING FINITE ELEMENT APPROXIMATIONS OF THE STEKLOV EIGENVALUE PROBLEM AND ITS LOWER BOUND APPROXIMATIONS

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Abstract. The paper deals with error estimates and lower bound approximations of the Steklov eigenvalue problems on convex or concave domains by nonconforming finite element methods. We consider four types of nonconforming finite elements: Crouzeix-Raviart, $Q_1^{\rm rot}$, $EQ_1^{\rm rot}$ and enriched Crouzeix-Raviart. We first derive error estimates for the nonconforming finite element approximations of the Steklov eigenvalue problem and then give the analysis of lower bound approximations. Some numerical results are presented to validate our theoretical results.

Keywords: Steklov eigenvalue problem, nonconforming finite element, error estimate, lower bound of the eigenvalues

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1. Introduction

Steklov eigenvalue problems arise in a number of applications such as surface waves [9], stability of mechanical oscillators immersed in a viscous fluid [16], the vibration modes of a structure in contact with an incompressible fluid [10], the antiplane shearing on a system of collinear faults under slip-dependent friction law [13], vibrations of a pendulum [1], eigenoscillations of mechanical systems with boundary conditions containing frequency [23].

The analysis of the conforming finite element methods for the Steklov eigenvalue problems has been given by Bramble and Osborn [12], Andreev and Todorov [3].

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Furthermore, a posteriori error estimator for the linear finite element approximation has been proposed and analyzed by Armentano and Padra [6]. The boundary element methods for the Steklov eigenvalue problems have also been given by Han and Guan [21], Han, Guan and He [22], Huang and Lü [25], and Tang, Guan and Han [32]. The extrapolation method applied to the Steklov eigenvalue problem has been analyzed in [27]. Recently, the nonconforming finite element methods for the Steklov eigenvalue problems have also been analyzed by Yang, Li and Li [36] on the convex domain. So the first aim of this paper is to extend the error estimates of the Steklov eigenvalue problems by nonconforming finite element methods to the convex and concave domains.

The eigenvalue is a number, and thus it is credible if we get both the upper and lower bounds. For the Steklov eigenvalue problems, due to the Rayleigh quotient and minimum-maximum principle, it is natural to get the upper bounds by conforming finite element methods. For the lower bounds, Beattie and Goerisch [8], Goerisch and Albrecht [19], and Goerisch and He [20] give a type of variation method by choosing special trial functions (means on special domains) to get the lower bounds of the eigenvalues which needs solving eigenvalue problem twice and having some a priori information of the eigenvalues. But for the lower bounds by nonconforming finite element methods, only recently the work by Yang, Li and Li [36] gives some results of lower approximating on the convex domains. So the second aim of this paper is to analyze the lower bound approximations of Steklov eigenvalue problems by nonconforming finite element methods on the general domains. Besides three types of well-known nonconforming finite elements, a new type of nonconforming element which has better property of lower bound approximations will be introduced and analyzed for the Steklov eigenvalue problems.

In this paper we are concerned with the model problem

$$\begin{cases} -\Delta u + u = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = \lambda u & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded polygonal domain and $(\partial/\partial\nu)$ is the outward normal derivative on $\partial\Omega$.

The corresponding weak form of the problem (1.1) is:

Find $\lambda \in \mathbb{R}$ and $u \in H^1(\Omega)$ such that $\|u\|_b = 1$ and

$$a(u,v) = \lambda b(u,v) \quad \forall\, v \in H^1(\Omega), \tag{1.2}$$

where

$$a(u,v) = \int_\Omega (\nabla u \nabla v + uv)\,\mathrm{d}x\,\mathrm{d}y,$$
$$b(u,v) = \int_{\partial\Omega} uv\,\mathrm{d}s, \quad \|u\|_b = b(u,u)^{1/2}.$$

Evidently the bilinear form $a(\cdot,\cdot)$ is symmetric, continuous and coercive over the product space $H^1(\Omega) \times H^1(\Omega)$.

From [9] and [12] we know the eigenvalue problem (1.2) has an eigenvalue sequence $\{\lambda_j\}$:

$$0 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \lambda_k \leqslant \ldots, \quad \lim_{k\to\infty} \lambda_k = \infty,$$

and the associated eigenfunctions

$$u_1, u_2, \ldots, u_j, \ldots,$$

where $b(u_i, u_j) = \delta_{ij}$.

Let $\mathcal{T}_h$ be a shape-regular decomposition of $\overline{\Omega}$ into triangles or rectangles. The diameter of a cell $K \in \mathcal{T}_h$ is denoted by h_K. The mesh diameter h describes the maximum diameter of all cells $K \in \mathcal{T}_h$. Let $\mathcal{E}_h$ denote the edge set of $\mathcal{T}_h$ and $\mathcal{E}_h = \mathcal{E}_h^i \cup \mathcal{E}_h^b$, where $\mathcal{E}_h^i$ denotes the interior edge set and $\mathcal{E}_h^b$ denotes the edge set lying on the boundary $\partial\Omega$.

In this paper, we consider four types of nonconforming finite elements: Crouzeix-Raviart (CR), Q_1 rotation (Q_1^{rot}), Extension Q_1 rotation (EQ_1^{rot}) and the Enriched Crouzeix-Raviart (ECR) elements.

- The CR element space, proposed by Crouzeix and Raviart [17], is defined by

$$V^h = \Big\{ v \in L^2(\Omega) : \ v|_K \in \mathrm{span}\{1, x, y\},$$
$$\int_F v|_{K_1}\,\mathrm{d}s = \int_F v|_{K_2}\,\mathrm{d}s \text{ if } K_1 \cap K_2 = F \Big\}.$$

- The Q_1^{rot} element space, proposed by Rannacher and Turek [31] and Arbogast and Chen [4], is defined by

$$V^h = \Big\{ v \in L^2(\Omega) : \ v|_K \in \mathrm{span}\{1, x, y, x^2 - y^2\},$$
$$\int_F v|_{K_1}\,\mathrm{d}s = \int_F v|_{K_2}\,\mathrm{d}s \text{ if } K_1 \cap K_2 = F \Big\}.$$

- The EQ_1^{rot} element space, proposed by Lin, Tobiska, and Zhou [29], is defined by

$$V^h = \Big\{ v \in L^2(\Omega) \colon \ v|_K \in \mathrm{span}\{1, x, y, x^2, y^2\},$$
$$\int_F v|_{K_1} \,\mathrm{d}s = \int_F v|_{K_2} \,\mathrm{d}s \text{ if } K_1 \cap K_2 = F \Big\}.$$

- The ECR element space, proposed by Hu, Huang and Lin [24], and Lin et al. [30], is defined by

$$V^h = \Big\{ v \in L^2(\Omega) \colon \ v|_K \in \mathrm{span}\{1, x, y, x^2 + y^2\},$$
$$\int_F v|_{K_1} \,\mathrm{d}s = \int_F v|_{K_2} \,\mathrm{d}s \text{ if } K_1 \cap K_2 = F \Big\}.$$

Here Q_1^{rot} and EQ_1^{rot} elements are defined on rectangular meshes.

All the above nonconforming elements possess the following common properties:

(1) The space of shape functions contains the complete polynomials of degree 1;

(2) $v \in V^h$ is integrally continuous at the common edge F between the neighboring elements K_1 and K_2, i.e.,

$$\int_F v|_{K_1} \,\mathrm{d}s = \int_F v|_{K_2} \,\mathrm{d}s \text{ if } K_1 \cap K_2 = F;$$

(3) $V^h \not\subset H^1(\Omega)$, $V^h \subset L^2(\Omega)$, and $\delta V^h \subset L^2(\partial\Omega)$, where δV^h denotes the trace of V^h on the boundary $\partial\Omega$.

The nonconforming finite element approximation of (1.2) is defined as follows: Find $\lambda_h \in \mathbb{R}$ and $u_h \in V^h$ with $\|u_h\|_b = 1$ such that

$$a_h(u_h, v_h) = \lambda_h b(u_h, v_h) \quad \forall\, v_h \in V^h, \tag{1.3}$$

where

$$a_h(u_h, v_h) = \sum_{K \in \mathcal{T}_h} \int_K (\nabla u_h \nabla v_h + u_h v_h) \,\mathrm{d}x \,\mathrm{d}y.$$

Based on the bilinear form $a_h(\cdot,\cdot)$, we can define the following norm on $V + V_h$

$$\|v_h\|_h^2 = a_h(v_h, v_h).$$

Obviously, $a_h(\cdot,\cdot)$ is uniformly V^h-elliptic.

We know from [9] and [12] that the eigenvalue problem (1.3) has eigenvalues

$$0 < \lambda_{1,h} \leqslant \lambda_{2,h} \leqslant \ldots \leqslant \lambda_{k,h} \leqslant \ldots \leqslant \lambda_{N,h},$$

and the corresponding eigenfunctions

$$u_{1,h}, u_{2,h}, \ldots, u_{k,h}, \ldots, u_{N,h},$$

where $b(u_{i,h}, u_{j,h}) = \delta_{ij}$, $1 \leqslant i, j \leqslant N := \dim \delta V^h$.

In the following, we use the standard notation ([7], [15], and [33]) for the Sobolev spaces $H^m(\Omega)$ (standard interpolation spaces for a real number m) and their associated norms $\|\cdot\|_m$ and seminorms $|\cdot|_m$ for $m \geqslant 0$. The Sobolev space $H^0(\Omega)$ coincides with $L^2(\Omega)$, in which case the norm and the inner product are denoted by $\|\cdot\|_0$ and $(\cdot,\cdot)$, respectively. Throughout this paper, C denotes a generic positive constant independent of h, which may not be the same at each occurrence.

The rest of this paper is organized as follows. In Section 2, we analyze the error estimates for the corresponding source problem by nonconforming finite element methods. Then the error estimates of the eigenvalue problem are given in Section 3. Section 4 is devoted to analyzing the lower bound of the eigenvalues by nonconforming finite element methods. Some numerical results are given in Section 5 to validate our theoretical results and some concluding remarks are stated in the final section.

2. Nonconforming finite element approximations of the corresponding source problem

In order to analyze the error estimates of eigenpair approximations by nonconforming finite elements, we need to consider the following source problem associated with the eigenvalue problem (1.1):

Find $u \in H^1(\Omega)$ such that

$$a(u, v) = b(f, v) \quad \forall\, v \in H^1(\Omega), \tag{2.1}$$

and the corresponding discrete source problem (2.2) associated with (1.3):

Find $u_h \in V^h$ such that

$$a_h(u_h, v_h) = b(f, v_h) \quad \forall\, v_h \in V^h. \tag{2.2}$$

Lemma 2.1 ([12, (4.10)], [10, Proposition 4.4])**.** *For the Steklov source problem* (2.1), *if* $f \in L^2(\partial\Omega)$, *then* $u \in H^{1+r/2}(\Omega)$, $r \in (\frac{1}{2}, 1]$, *and*

$$\|u\|_{1+r/2} \leqslant C\|f\|_b. \tag{2.3}$$

Furthermore, if $f \in H^{1/2}(\partial\Omega)$, *we have* $u \in H^{1+r}(\Omega)$ *and*

$$\|u\|_{1+r} \leqslant C\|f\|_{1/2,\partial\Omega}. \tag{2.4}$$

In order to deduce the convergence order, we define the interpolation operators of the four types of nonconforming finite elements.

▷ For the CR element and Q_1^{rot} element, the interpolation operator $I_h\colon H^1(\Omega) \to V^h$ is defined by

$$\int_F I_h v \,\mathrm{d}s = \int_F v \,\mathrm{d}s \quad \forall\, F \in \mathcal{E}_h \text{ and } \forall\, v \in H^1(\Omega). \tag{2.5}$$

▷ For the ECR and EQ_1^{rot} element, the interpolation operator $I_h\colon H^1(\Omega) \to V^h$ is defined by the above equality (2.5) and

$$\int_K I_h v \,\mathrm{d}x \,\mathrm{d}y = \int_K v \,\mathrm{d}x \,\mathrm{d}y \quad \forall\, K \in \mathcal{T}_h \text{ and } \forall\, v \in H^1(\Omega). \tag{2.6}$$

According to the interpolation theory [15], we have the error estimates

$$\|u_j - I_h u_j\|_0 \leqslant Ch^{1+r}|u_j|_{1+r}, \quad 0 < r \leqslant 1, \tag{2.7}$$

$$\|u_j - I_h u_j\|_h \leqslant Ch^{r}|u_j|_{1+r}, \quad 0 < r \leqslant 1. \tag{2.8}$$

In order to give error estimates, we introduce the following trace inequality.

Lemma 2.2 ([33, Lemma 7.1.1], [36, Lemma 2.2])**.** *For any* $w \in H^s(K)$,

$$\int_{\partial K} |w|^2 \,\mathrm{d}s \leqslant C\{h_K^{-1}\|w\|_{0,K}^2 + h_K^{2s-1}|w|_{r,K}^2\} \quad (\tfrac{1}{2} \leqslant s \leqslant 1),$$

where the positive constant C *is independent of* w *and the diameter* h_K *of* K.

In the error estimate analysis of nonconforming finite element methods, we always need to define the L^2-projection operator on the edge $F \in \mathcal{E}_h$:

$$P_0^F f = \frac{1}{|F|}\int_F f \,\mathrm{d}s, \quad R_0^F f = f - P_0^F f, \tag{2.9}$$

and on the element $K \in \mathcal{T}_h$:

$$P_0^K f = \frac{1}{|K|}\int_K f \,\mathrm{d}x \,\mathrm{d}y, \quad R_0^K f = f - P_0^K f. \tag{2.10}$$

The operators P_0^F and P_0^K have the following properties.

Lemma 2.3. *If $w \in H^s(K)$, the following error estimate holds:*

$$\|R_0^K w\|_{0,K} \leqslant Ch^s |w|_{s,K}, \quad 0 \leqslant s \leqslant 1. \tag{2.11}$$

For any $f \in L^2(F)$ we have the inequalities

$$\|P_0^F f\|_{0,F} \leqslant \|f\|_{0,F}, \tag{2.12}$$

$$\|R_0^F f\|_{0,F} \leqslant \|f - v\|_{0,F} \quad \forall\, v \in P_0(K). \tag{2.13}$$

In order to analyze error estimates of nonconforming finite elements on concave domains, we also introduce the following trace inequality.

Lemma 2.4 ([11, Corollary 3.3], [14, Lemma 2.1])**.** *Let $K \in \mathcal{T}_h$, $F \in \partial K$, and $0 < \varepsilon < \frac{1}{2}$. Then for any $w \in H^{1+\varepsilon}(K)$ with $\Delta w \in L^2(K)$ there exists a positive constant C independent of w such that*

$$\|\nabla w \cdot \nu\|_{\varepsilon - 1/2, F} \leqslant C(\|\nabla w\|_{\varepsilon, K} + h_K^{1-\varepsilon} \|\Delta w\|_{0,K}).$$

We define the consistency error term of nonconforming finite elements as [15], [33]

$$E_h(u, v) = a_h(u, v) - b(f, v). \tag{2.14}$$

Theorem 2.1. *Let u be a solution of (2.1) and $u \in H^{1+r}(\Omega)$. Then $E_h(u, v)$ can be estimated by*

$$E_h(u, v) \leqslant Ch^r |u|_{1+r} \Big(\sum_{K \in \mathcal{T}_h} |v|_{1,K}^2 \Big)^{1/2} \quad \forall\, v \in H^1(\Omega) + V^h. \tag{2.15}$$

Let u be a solution of (2.1) and $u \in H^{1+r/2}(\Omega)$. Then the following estimate holds:

$$E_h(u, v) \leqslant Ch^{r/2} |u|_{1+r/2} \Big(\sum_{K \in \mathcal{T}_h} |v|_{1,K}^2 \Big)^{1/2} \quad \forall\, v \in H^1(\Omega) + V^h. \tag{2.16}$$

P r o o f. By Green's formula we have

$$\begin{aligned} E_h(u, v) &= a_h(u, v) - b(f, v) \\ &= \sum_{K \in \mathcal{T}_h} \int_K (\nabla u \nabla v + uv) \, \mathrm{d}x \, \mathrm{d}y - \int_{\partial\Omega} fv \, \mathrm{d}s \\ &= \int_\Omega (-\Delta u + u) v \, \mathrm{d}x \, \mathrm{d}y + \sum_{K \in \mathcal{T}_h} \int_{\partial K} \frac{\partial u}{\partial \nu} v \, \mathrm{d}s - \int_{\partial\Omega} \frac{\partial u}{\partial \nu} v \, \mathrm{d}s. \end{aligned}$$

Since u is a solution of (2.1), we have $\int_\Omega(-\Delta u+u)v\,\mathrm{d}x\,\mathrm{d}y=0$. Thus

$$E_h(u,v)=\sum_{K\in\mathcal{T}_h}\int_{\partial K}\frac{\partial u}{\partial\nu}\,v\,\mathrm{d}s-\int_{\partial\Omega}\frac{\partial u}{\partial\nu}\,v\,\mathrm{d}s=\sum_{F\in\mathcal{E}_h^i}\int_F\frac{\partial u}{\partial\nu}\,[v]\,\mathrm{d}s \tag{2.17}$$
$$=\sum_{F\in\mathcal{E}_h^i}\int_F\frac{\partial(u-\{I_hu\})}{\partial\nu}\,[v]\,\mathrm{d}s+\sum_{F\in\mathcal{E}_h^i}\int_F\frac{\partial(\{I_hu\})}{\partial\nu}\,[v]\,\mathrm{d}s$$
$$=\sum_{F\in\mathcal{E}_h^i}\int_F\frac{\partial(u-\{I_hu\})}{\partial\nu}\,[v]\,\mathrm{d}s,$$

where $[v]$ denotes the jump of v on F, $[v]=(v|_{K^+}-v|_{K^-})|_F$, $\{v\}$ denotes the average of v on F, $\{v\}=\frac{1}{2}(v|_{K^+}+v|_{K^-})|_F$, and we also use the property $\partial_\nu\{I_hu\}|_F=\mathrm{const}$ and

$$\int_F[v]\,\mathrm{d}s=0\quad\forall\,F\in\mathcal{E}_h^i\ \text{ and }\ \forall\,v\in H^1(\Omega)+V^h.$$

Now we estimate $\int_F(\partial(u-\{I_hu\})/\partial\nu)[v]\,\mathrm{d}s$, $F\in\mathcal{E}_h^i$ on the right-hand side of (2.17). From (2.10) we have

$$\int_F\frac{\partial(u-\{I_hu\})}{\partial\nu}\,[v]\,\mathrm{d}s=\int_F\frac{\partial(u-\{I_hu\})}{\partial\nu}\,R_0^F([v])\,\mathrm{d}s, \tag{2.18}$$

where we use

$$P_0^F([v])=\frac{1}{|F|}\int_F[v]\,\mathrm{d}s=0.$$

Applying the Cauchy-Schwarz inequality, we find that

$$\Big|\int_F\frac{\partial(u-\{I_hu\})}{\partial\nu}\,[v]\,\mathrm{d}s\Big| \tag{2.19}$$
$$=\Big|\int_F\frac{\partial(u-\{I_hu\})}{\partial\nu}\,R_0^F([v])\,\mathrm{d}s\Big|$$
$$\leqslant\Big\{\int_F\Big(\frac{\partial(u-\{I_hu\})}{\partial\nu}\Big)^2\,\mathrm{d}s\Big\}^{1/2}\Big\{\int_F(R_0^F([v]))^2\,\mathrm{d}s\Big\}^{1/2}.$$

By Lemma 2.2 we have

$$\int_F\Big(\frac{\partial(u-\{I_hu\})}{\partial\nu}\Big)^2\,\mathrm{d}s \tag{2.20}$$
$$\leqslant\frac{1}{2}\int_{F\cap K^+}\Big(\frac{\partial(u-I_hu)}{\partial\nu^+}\Big)^2\,\mathrm{d}s+\frac{1}{2}\int_{F\cap K^-}\Big(\frac{\partial(u-I_hu)}{\partial\nu^-}\Big)^2\,\mathrm{d}s$$
$$\leqslant C\{h_K^{-1}\|\nabla(u-I_hu)\|_{0,K^+\cup K^-}^2+h_K^{2r-1}\|\nabla(u-I_hu)\|_{r,K^+\cup K^-}^2\}$$
$$\leqslant Ch_K^{2r-1}\|u\|_{1+r,K^+\cup K^-}^2.$$

From (2.13), Lemma 2.2 with $r=1$ and (2.11) with $s=1$ we derive

$$\begin{aligned}(2.21)\quad \int_F (R_0^F([v]))^2\,\mathrm{d}s &= \int_F ([v]-P_0^F([v]))^2\,\mathrm{d}s\\ &= \int_F \big\{(v^+-P_0^F(v^+))-(v^--P_0^F(v^-))\big\}^2\,\mathrm{d}s\\ &\leqslant 2\Big\{\int_F (v^+-P_0^F(v^+))^2\,\mathrm{d}s+\int_F (v^--P_0^F(v^-))^2\,\mathrm{d}s\Big\}\\ &\leqslant 2\Big\{\int_F (v^+-P_0^{K^+}(v^+))^2\,\mathrm{d}s+\int_F (v^--P_0^{K^-}(v^-))^2\,\mathrm{d}s\Big\}\\ &\leqslant C\big\{(h_{K^+}^{-1}\|R_0^{K^+}(v^+)\|_{0,K^+}^2+h_{K^+}|R_0^{K^+}(v^+)|_{1,K^+}^2)\\ &\quad+(h_{K^-}^{-1}\|R_0^{K^-}(v^-)\|_{0,K^-}^2+h_{K^-}|R_0^{K^-}(v^-)|_{1,K^-}^2)\big\}\\ &\leqslant C\{h_{K^+}|v^+|_{1,K^+}^2+h_{K^-}|v^-|_{1,K^-}^2\},\end{aligned}$$

where $v^+=v|_{K^+}$, $v^-=v|_{K^-}$. Substituting (2.20) and (2.21) into (2.19), we get

$$(2.22)\qquad \Big|\int_F \frac{\partial(u-\{I_hu\})}{\partial\nu}\,[v]\,\mathrm{d}s\Big| \leqslant Ch_K^r|u|_{1+r,K^+\cup K^-}|v|_{1,K^+\cup K^-}.$$

Thus substituting (2.22) into (2.17) and the Cauchy-Schwarz inequality lead to (2.15).

Now we come to proving (2.16) for $r<1$. From (2.19), Lemma 2.4, and (2.21) we have

$$\begin{aligned}(2.23)\quad &\Big|\int_F \frac{\partial(u-\{I_hu\})}{\partial\nu}\,[v]\,\mathrm{d}s\Big|\\ &= \Big|\int_F \frac{\partial(u-\{I_hu\})}{\partial\nu}\,R_0^F([v])\,\mathrm{d}s\Big|\\ &\leqslant \Big\|\frac{\partial(u-\{I_hu\})}{\partial\nu}\Big\|_{r-1/2,F}\|R_0^F([v])\|_{1-r/2,F}\\ &\leqslant C(\|\nabla(u-I_hu)\|_{r/2,K^+\cup K^-}+h_K^{1-r/2}\|\Delta(u-I_hu)\|_{0,K^+\cup K^-})\\ &\quad\times h_K^{(r-1)/2}\|R_0^F([v])\|_{0,F}\\ &= C(h_K^{r/2}\|\nabla(u-I_hu)\|_{r/2,K^+\cup K^-}+h_K\|\Delta(u-I_hu)\|_{0,K^+\cup K^-})\\ &\quad\times h_K^{-1/2}\|R_0^F([v])\|_{0,F}\\ &\leqslant Ch_K^{r/2}\|u\|_{1+r/2,K^+\cup K^-}\|v\|_{1,K^+\cup K^-}.\end{aligned}$$

Substituting (2.23) into (2.17), we arrive at (2.16) by the Cauchy-Schwarz inequality. When $r=1$, (2.16) can be obtained directly by Lemma 2.2 and the same proof as for (2.15). □

Now we can state the error estimates of the nonconforming finite element approximation which are the main result of this section.

Theorem 2.2. *Let* $u \in H^{1+r}(\Omega)$ *be the solution of the source problem* (2.1) *and* u_h *the corresponding nonconforming finite element approximation defined by* (2.2). *Then we have the estimate*

$$\|u_h - u\|_h \leqslant Ch^r |u|_{1+r}. \tag{2.24}$$

Furthermore, the following error estimate in $\|\cdot\|_b$ *holds:*

$$\|u_h - u\|_b \leqslant Ch^{3r/2} |u|_{1+r}. \tag{2.25}$$

P r o o f. First, from the Strang lemma we have

$$\|u - u_h\|_h \leqslant C\Big(\inf_{v_h \in V^h} \|u - v_h\|_h + \sup_{0 \neq w_h \in V^h} \frac{|E_h(u, w_h)|}{\|w_h\|_h}\Big). \tag{2.26}$$

By the interpolation error estimate (2.8), we obtain

$$\inf_{v_h \in V^h} \|u - v_h\|_h \leqslant \|u - I_h u\|_h \leqslant Ch^r |u|_{1+r}$$

and we have the following estimate for consistent error by (2.15):

$$\sup_{0 \neq w_h \in V^h} \frac{|E_h(u, w_h)|}{\|w_h\|_h} \leqslant Ch^r |u|_{1+r}.$$

Combining the above two inequalities and (2.26), we arrive at (2.24).

Using the method developed by Nitsche (1974), Lascaux and Lesaint (1975) [15], we have the estimate

$$\begin{aligned} \|u - u_h\|_b \leqslant \sup_{0 \neq g \in L^2(\partial\Omega)} \frac{1}{\|g\|_b} \inf_{v \in V^h} \{ & C\|u - u_h\|_h \|\varphi - v\|_h \\ & + E_h(u, \varphi - v) + E_h(\varphi, u - u_h)\}, \end{aligned} \tag{2.27}$$

where $\varphi \in H^{1+r/2}(\Omega)$ is the unique solution of the auxiliary problem

$$a(v, \varphi) = b(g, v) \quad \forall\, v \in H^1(\Omega),$$

where $g \in L^2(\partial\Omega)$ acts as the load function.

Using (2.15) and the interpolation error estimate

$$\|\varphi - I_h\varphi\|_h \leqslant Ch^{r/2}\|\varphi\|_{1+r/2} \leqslant Ch^{r/2}\|g\|_b, \tag{2.28}$$

we get

$$|E_h(u, \varphi - I_h\varphi)| \leqslant Ch^r|u|_{1+r}\|\varphi - I_h\varphi\|_h \leqslant Ch^{3r/2}|u|_{1+r}\|g\|_b. \tag{2.29}$$

Combining (2.16) and (2.24) leads to

$$|E_h(\varphi, u - u_h)| \leqslant Ch^{3r/2}|u|_{1+r}\|g\|_b. \tag{2.30}$$

From (2.24), (2.29), (2.30) and taking $v = I_h\varphi$ in (2.27), we obtain (2.25). □

Theorem 2.3. *Let $u \in H^{1+r/2}(\Omega)$ be the solution of the source problem* (2.1) *and u_h the corresponding nonconforming finite element approximation defined by* (2.2). *Then we have the estimate*

$$\|u_h - u\|_h \leqslant Ch^{r/2}|u|_{1+r/2}. \tag{2.31}$$

Furthermore, the following error estimate in $\|\cdot\|_b$ holds:

$$\|u_h - u\|_b \leqslant Ch^r|u|_{1+r/2}. \tag{2.32}$$

Proof. The proof is the same as the proof of Theorem 2.2 except that we use (2.16) instead of (2.15). □

3. Nonconforming finite element approximations of the Steklov eigenvalue problem

In order to derive the error estimates of eigenpair approximations by nonconforming finite element methods, we define the solution operators A and T and their corresponding discrete versions A_h and T_h.

Concerning the problem (2.1), we define the operator $A\colon L^2(\partial\Omega) \to H^{1+r/2}(\Omega) \subset H^1(\Omega)$ as

$$a(Af, v) = b(f, v) \quad \forall\, v \in H^1(\Omega). \tag{3.1}$$

Based on the definition of A, we can define $T\colon L^2(\partial\Omega) \to H^{1/2+r/2}(\partial\Omega)$ by

$$Tf = (Af)', \tag{3.2}$$

where the prime denotes the trace on $\partial\Omega$.

So the eigenvalue problem (1.2) can be written in the operator form [9], [12]:

$$\lambda T u = u. \tag{3.3}$$

Then we define the corresponding discrete pair of operators $A_h\colon\ L^2(\partial\Omega) \to V^h$ and $T_h\colon\ L^2(\partial\Omega) \to \delta V^h \subset L^2(\partial\Omega)$ such that

$$a_h(A_h f, v) = b(f, v) \quad \forall\, v \in V^h, \tag{3.4}$$

and

$$T_h f = (A_h f)'. \tag{3.5}$$

Similarly, the discrete eigenvalue problem (1.3) can be written as

$$\lambda_h T_h u_h = u_h. \tag{3.6}$$

Lemma 3.1. *The operators T and T_h are self-adjoint operators. The following approximation property holds:*

$$\|T_h - T\|_b \to 0 \quad \textit{as } h \to 0, \tag{3.7}$$

and the operator T is compact.

Proof. First, for any $f, g \in L^2(\partial\Omega)$ we have

$$b(Tf, g) = b(g, Tf) = a(Ag, Af) = a(Af, Ag) = b(f, Ag) = b(f, Tg).$$

This means the operator T is a self-adjoint operator. In a similar way we can also prove that T_h is also self-adjoint.

With (2.32), the following estimate holds:

$$\begin{aligned}\|T_h - T\|_b &= \sup_{0 \neq g \in L^2(\partial\Omega)} \frac{\|T_h g - Tg\|_b}{\|g\|_b} = \sup_{0 \neq g \in L^2(\partial\Omega)} \frac{\|A_h g - Ag\|_b}{\|g\|_b} \\ &\leqslant \sup_{g \in L^2(\partial\Omega)} \frac{Ch^r \|Ag\|_{1+r/2}}{\|g\|_b} \leqslant Ch^r \to 0 \quad (h \to 0).\end{aligned}$$

This is the desired result (3.7). Since T_h is a finite rank operator, T is a compact operator. □

Now, we are in the position to prove the error estimates of the nonconforming finite element approximations to the exact eigenpair. Let λ_j denote the jth eigenvalue of T, let $M(\lambda_j)$ be the corresponding eigenspace spanned by eigenfunctions of T according to λ_j and let $\delta M(\lambda_j)$ denote the trace of $M(\lambda_j)$ on $\partial\Omega$.

We state the order-preserving convergence which comes from [18], [35], [36].

Lemma 3.2 ([2], [18], [35], [36]). *Let λ_j be the jth eigenvalue of* (1.2), *and $\lambda_{j,h}$ the jth eigenvalue of* (1.3) *with the corresponding eigenfunctions $u_{j,h}$ and $\|u_{j,h}\|_b = 1$. Then there exists $u_j \in M(\lambda_j)$ with $\|u_j\|_b = 1$ and*

$$\lambda_{j,h} - \lambda_j = \frac{\lambda_j \lambda_{j,h}}{b(u_j, u_{j,h})}\, b((T - T_h)u_j, u_j) + R_1, \tag{3.8}$$

$$\|u_{j,h} - u_j\|_b \leqslant C\lambda_j^2 \|(T - T_h)u_j\|_b, \tag{3.9}$$

$$\|u_{j,h} - u_j\|_h = \lambda_j \|Au_j - A_h u_j\|_h + R_2, \tag{3.10}$$

where $|R_1| \leqslant C\|(T - T_h)u_j\|_b^2$ and $|R_2| \leqslant C\|(T - T_h)u_j\|_b$.

Theorem 3.1. *Under the assumptions of Lemma 3.2, the following error estimates hold:*

$$|\lambda_{j,h} - \lambda_j| \leqslant C\lambda_j^2 h^{2r} \|u_j\|_{1/2,\partial\Omega}^2, \tag{3.11}$$

$$\|u_j - u_{j,h}\|_b \leqslant C\lambda_j^2 h^{3r/2} \|u_j\|_{1/2,\partial\Omega}, \tag{3.12}$$

$$\|u_j - u_{j,h}\|_h \leqslant C\lambda_j h^r \|u_j\|_{1/2,\partial\Omega}, \tag{3.13}$$

where C is a constant independent of h and λ_j.

Proof. From (2.24), (2.25), and (2.4), we have

$$\|Au_j - A_h u_j\|_h \leqslant Ch^r \|u_j\|_{1/2,\partial\Omega}, \tag{3.14}$$

$$\|Tu_j - T_h u_j\|_b \leqslant Ch^{3r/2} \|u_j\|_{1/2,\partial\Omega}. \tag{3.15}$$

Some calculations lead to

$$\begin{aligned} b(Tu_j - T_h u_j, u_j) &= b(Tu_j, u_j) - b(T_h u_j, u_j) \\ &= a_h(Au_j, Au_j) - a_h(A_h u_j, A_h u_j) \\ &= a_h(Au_j - A_h u_j, Au_j) + a_h(A_h u_j, Au_j - A_h u_j) \\ &= 2a_h(Au_j - A_h u_j, Au_j) - a_h(Au_j - A_h u_j, Au_j - A_h u_j). \end{aligned}$$

For the first term we have

$$\begin{aligned} &a_h(Au_j - A_h u_j, Au_j) \\ &\quad = a_h(Au_j - A_h u_j, Au_j) - b(Au_j - A_h u_j, u_j) + b(Tu_j - T_h u_j, u_j) \\ &\quad = E_h(Au_j - A_h u_j, Au_j) + b(Tu_j - T_h u_j, u_j). \end{aligned}$$

Then

$$b(Tu_j - T_h u_j, u_j) = -2E_h(Au_j - A_h u_j, Au_j) + a_h(Au_j - A_h u_j, Au_j - A_h u_j),$$

which together with (2.15), (2.24), and (2.4) yields

$$|b(Tu_j - T_h u_j, u_j)| \leqslant Ch^{2r} \|u_j\|^2_{1/2,\partial\Omega}. \tag{3.16}$$

Therefore, substituting (3.15) and (3.16) into (3.8), we obtain (3.11). Similarly, substituting (3.15) into (3.9), and (3.14) and (3.15) into (3.10), we obtain (3.12) and (3.13), respectively. □

4. Lower bounds of eigenvalues

Motivated by the recent interesting results about lower bounds of eigenvalues by nonconforming finite element methods ([5], [24], [28], [34], [37], [38], [39]), we also consider the lower bounds of the Steklov eigenvalue problems. First we need the following eigenvalue error expansion.

Lemma 4.1 ([5], [39]). *Let* $(\lambda_j, u_j) \in \mathbb{R} \times H^1(\Omega)$ *be an eigenpair of* (1.2) *and let* $(\lambda_{j,h}, u_{j,h}) \in \mathbb{R} \times V^h$ *be the corresponding approximation defined by* (1.3). *Then*

$$\begin{aligned} \lambda_j - \lambda_{j,h} = {} & \|u_j - u_{j,h}\|^2_h - \lambda_{j,h}\|v - u_{j,h}\|^2_b \\ & + \lambda_{j,h}(\|v\|^2_b - \|u_j\|^2_b) + 2a_h(u_j - v, u_{j,h}) \quad \forall\, v \in V^h. \end{aligned} \tag{4.1}$$

P r o o f. Since $\|u_j\|_b = \|u_{j,h}\|_b = 1$, $a_h(u_j, u_j) = \lambda_j$, and $a_h(u_{j,h}, u_{j,h}) = \lambda_{j,h}$, we have

$$\begin{aligned} \lambda_j + \lambda_{j,h} &= a_h(u_j - u_{j,h}, u_j - u_{j,h}) + 2a_h(u_j, u_{j,h}) \\ &= \|u_j - u_{j,h}\|^2_h + 2a_h(v, u_{j,h}) + 2a_h(u_j - v, u_{j,h}) \\ &= \|u_j - u_{j,h}\|^2_h + 2\lambda_{j,h} b(v, u_{j,h}) + 2a_h(u_j - v, u_{j,h}) \\ &= \|u_j - u_{j,h}\|^2_h - \lambda_{j,h}\|v - u_{j,h}\|^2_b + \lambda_{j,h}\|u_{j,h}\|^2_b \\ &\quad + \lambda_{j,h}\|v\|^2_b + 2a_h(u_j - v, u_{j,h}) \\ &= \|u_j - u_{j,h}\|^2_h - \lambda_{j,h}\|v - u_{j,h}\|^2_b + 2\lambda_{j,h} \\ &\quad + \lambda_{j,h}(\|v\|^2_b - \|u_j\|^2_b) + 2a_h(u_j - v, u_{j,h}). \end{aligned}$$

Then (4.1) can be obtained and we complete the proof. □

Lemma 4.2. *Let $u \in H^{1+r}(\Omega)$. Then we have the estimate*

$$|a_h(u - I_h u, v_h)| = \left|\int_\Omega (u - I_h u) v_h \,\mathrm{d}x\,\mathrm{d}y\right| \leqslant C h^s |u|_{1+r} \|v_h\|_d \quad \forall\, v_h \in V^h, \tag{4.2}$$

where $s = 1 + r$, $d = 0$ for CR and Q_1^{rot} elements, and $s = 2 + r$, $d = h$ for ECR and EQ_1^{rot} elements.

Proof. Using Green's formula, we obtain [5], [28], [30]

$$\sum_{K \in \mathcal{T}_h} \int_K \nabla(u - I_h u) \nabla v_h \,\mathrm{d}x\,\mathrm{d}y = 0 \quad \forall\, v_h \in V^h.$$

Thus

$$\begin{aligned} a_h(u - I_h u, v_h) &= \sum_{K \in \mathcal{T}_h} \int_K \nabla(u - I_h u) \nabla v_h \,\mathrm{d}x\,\mathrm{d}y + \int_\Omega (u - I_h u) v_h \,\mathrm{d}x\,\mathrm{d}y \\ &= \int_\Omega (u - I_h u) v_h \,\mathrm{d}x\,\mathrm{d}y \quad \forall\, v_h \in V^h. \end{aligned} \tag{4.3}$$

For CR and Q_1^{rot} elements, we have

$$\left|\int_\Omega (u - I_h u) v_h \,\mathrm{d}x\,\mathrm{d}y\right| \leqslant C h^{1+r} |u|_{1+r} \|v_h\|_0.$$

This shows that (4.2) holds for $s = 1 + r$ and $d = 0$.

On the other hand, for ECR and EQ_1^{rot} elements, we introduce a piecewise constant interpolation operator Π_0. Then

$$\begin{aligned} \left|\int_\Omega (u - I_h u) v_h \,\mathrm{d}x\,\mathrm{d}y\right| &= \left|\int_\Omega (u - I_h u) \Pi_0 v_h \,\mathrm{d}x\,\mathrm{d}y + \int_\Omega (u - I_h u)(v_h - \Pi_0 v_h) \,\mathrm{d}x\,\mathrm{d}y\right| \\ &= \left|\int_\Omega (u - I_h u)(v_h - \Pi_0 v_h) \,\mathrm{d}x\,\mathrm{d}y\right| \leqslant C h^{2+r} |u|_{1+r} \|v_h\|_h. \end{aligned}$$

This shows that (4.2) holds for $s = 2 + r$ and $d = h$. $\square$

Lemma 4.3. *Let* $u_j \in H^{1+r}(\Omega)$ *be an eigenfunction of* (1.1). *Then the following estimate holds:*

$$\|u_j - I_h u_j\|_b \leqslant Ch^{3r/2}\|u_j\|_{1+r}. \tag{4.4}$$

Proof. For any $g \in L^2(\partial\Omega)$ we have

$$\begin{aligned} b(g, u_j - I_h u_j) &= a(Ag, u_j) - a_h(A_h g, I_h u_j) \\ &= a_h(Ag, u_j - I_h u_j) + a_h(Ag - A_h g, I_h u_j) \\ &= a_h(Ag, u_j - I_h u_j) + a_h(Ag, I_h u_j) - b(g, I_h u_j) \\ &= a_h(Ag, u_j - I_h u_j) + E_h(Ag, u_j - I_h u_j). \end{aligned} \tag{4.5}$$

From (2.24), (2.31), and (4.2) we get

$$\begin{aligned} |a_h(Ag, u_j - I_h u_j)| &= |a_h(Ag - A_h g, u_j - I_h u_j) + a_h(A_h g, u_j - I_h u_j)| \\ &\leqslant Ch^{3r/2}\|u_j\|_{1+r}\|g\|_b. \end{aligned}$$

Combining (2.3), (2.16), and (2.8) leads to the estimate

$$|E_h(Ag, u_j - I_h u_j)| \leqslant Ch^{3r/2}|u|_{1+r}\|g\|_b.$$

Substituting the above two inequalities into (4.5), we obtain

$$b(g, u_j - I_h u_j) \leqslant Ch^{3r/2}\|u_j\|_{1+r}\|g\|_b \quad \forall\, g \in L^2(\partial\Omega).$$

This means we have (4.4) and complete the proof. □

Theorem 4.1. *Under the conditions of Lemma* 4.2, *if* h *is sufficiently small, then*

$$\lambda_j - \lambda_{j,h} = \|u_j - u_{j,h}\|_h^2 + 2\int_\Omega (u_j - I_h u_j)u_{j,h}\,\mathrm{d}x\,\mathrm{d}y + R, \tag{4.6}$$

where $|R| \leqslant C(h^{1+3r/2} + h^{3r})$.

Proof. Taking $v = I_h u_j$ in (4.1), we estimate the second, third and fourth terms on the right-hand side of (4.1). From (4.4) and (3.12) we have

$$\|I_h u_j - u_{j,h}\|_b \leqslant \|I_h u_j - u_j\|_b + \|u_j - u_{j,h}\|_b \leqslant C\lambda_j^2 h^{3r/2}\|u_j\|_{1/2,\partial\Omega}.$$

In addition, we introduce the piecewise constant interpolation operator I_0 on $\partial\Omega$. Then, from (4.4), we have

$$\begin{aligned}|\|I_h u_j\|_b^2 - \|u_j\|_b^2| &= |2b(I_h u_j - u_j, u_j) - b(I_h u_j - u_j, u_j - I_h u_j)| \\ &\leqslant 2|b(I_h u_j - u_j, u_j - I_0 u_j)| + Ch^{3r}\|u_j\|_{1+r}^2 \\ &\leqslant C\|u_j - I_h u_j\|_b \|u_j - I_0 u_j\|_b + Ch^{3r}\|u_j\|_{1+r}^2 \\ &\leqslant Ch^{3r/2}\|u_j\|_{1+r} h\|u_j\|_{1,\partial\Omega} + Ch^{3r}\|u_j\|_{1+r}^2 \\ &\leqslant C(h^{1+3r/2} + h^{3r})\|u_j\|_{1+r}^2.\end{aligned}$$

Thus from the previous estimates and (4.3) we obtain (4.6). □

Corollary 4.1. *For ECR and EQ_1^{rot} elements, if $\|u_j - u_{j,h}\|_h \geqslant C(h^{1/2+3r/4-\gamma} + h^{3/2r-\gamma})$ (γ is an arbitrary small positive number), we have*

$$\lambda_{j,h} \leqslant \lambda_j, \tag{4.7}$$

when h is small enough.

P r o o f. From (4.2) and $|R| \leqslant C(h^{1+3r/2} + h^{3r})$ we know that the second and the third terms on the right-hand side of (4.6) are infinitesimals of higher order than the order of the first term. So the sign of the right-hand side of (4.6) is determined by the first term. Thus (4.7) holds. □

Corollary 4.2. *For CR and Q_1^{rot} elements, let us assume that there exists a positive constant C_1 independent of h and λ_j such that $\|u_j - u_{j,h}\|_h^2 \geqslant C_1\lambda_j^2 h^{2r}$. Then we have that*

$$\lambda_{j,h} \leqslant \lambda_j, \tag{4.8}$$

when $1/2 < r < 1$, or the eigenvalue λ_j is large enough and h small enough for $r = 1$.

P r o o f. When $1/2 < r < 1$, we can obtain (4.8) by an analysis similar to the proof of Corollary 4.1 concluding that the first term $\|u_j - u_{j,h}\|_h^2$ is the dominant term.

When $r = 1$ and the eigenvalue λ_j is large enough and h small enough the proof is the same as that of Corollary 4.2 in [36]. □

R e m a r k 4.1. Our lower bound results for the eigenvalue require the lower bound of the discretization error of the eigenfunction by the finite element method. In Křížek, Roos and Chen [26], the lower bound of the linear and bilinear elements has been obtained firstly. In our situation, different with the conforming elements, the nonconforming elements considered in this paper have the orthogonality property of the corresponding interpolation:

$$\sum_{K\in\mathcal{T}_h}\int_K \nabla(u-I_hu)\nabla v_h\,\mathrm{d}x\,\mathrm{d}y=0\quad\forall\, v_h\in V^h.$$

We define the following semi-norm in H^1+V^h:

$$|v|_{1,h}^2=\sum_{K\in\mathcal{T}_h}\int_K |\nabla v|^2\,\mathrm{d}x\,\mathrm{d}y.$$

Then the following estimate holds:

$$\begin{aligned}|u-I_hu|_{1,h}^2&\leqslant\left|\sum_{K\in\mathcal{T}_h}\int_K \nabla(u-I_hu)\nabla(u-I_hu)\,\mathrm{d}x\,\mathrm{d}y\right|\\&=\left|\sum_{K\in\mathcal{T}_h}\int_K \nabla(u-I_hu)\nabla(u-u_h)\,\mathrm{d}x\,\mathrm{d}y\right|\\&\leqslant|u-I_hu|_{1,h}|u-u_h|_{1,h}.\end{aligned}$$

This yields

$$|u-u_h|_{1,h}\geqslant|u-I_hu|_{1,h}. \tag{4.9}$$

So in order to get the lower bound of the discretization error of the eigenfunction approximation by nonconforming elements, we only need to estimate the lower bound of the interpolation error. In this way, we can obtain the assumptions in Corollary 4.1 and 4.2. These results will appear soon.

5. Numerical results

In this section we give two numerical examples to illustrate the theoretical results derived in this paper. The first example is defined on the unit square and the other one on the L shape domain.

5.1. Numerical results on square domain

The first example is to consider the problem (1.1) on the domain $\Omega=(0,1)\times(0,1)$. We use four types of nonconforming finite element methods to solve the Steklov eigenvalue problem. Each level of the computing meshes for CR and ECR elements are generated by the Delaunay methods. The meshes for Q_1^{rot} and EQ_1^{rot} elements are all rectangular meshes. The corresponding numerical results are shown in Tabs. 1, 2, 3, and 4.

h	$\lambda_{1,h}$	$\lambda_{2,h}$	$\lambda_{3,h}$	$\lambda_{4,h}$
0.2	0.24021996706	1.4833779747	1.4837060811	2.0488192219
0.1	0.24011871553	1.4891264232	1.4892258566	2.0691540091
0.05	0.24008801770	1.4915364371	1.4915376961	2.0792813571
0.025	0.24008132236	1.4921016468	1.4921019657	2.0817452102
0.0125	0.24007963788	1.4922519846	1.4922520044	2.0824176986
Trend	$\searrow$	$\nearrow$	$\nearrow$	$\nearrow$

Table 1. CR element for the Steklov eigenvalue problem on unit square.

h	$\lambda_{1,h}$	$\lambda_{2,h}$	$\lambda_{3,h}$	$\lambda_{4,h}$
0.2	0.24007905476	1.4832406183	1.4835453433	2.0484910082
0.1	0.24007901073	1.4891044540	1.4892020418	2.0691254277
0.05	0.24007906375	1.4915309007	1.4915321274	2.0792735726
0.025	0.24007907882	1.4920976970	1.4920980544	2.0817416011
0.0125	0.24007908367	1.4922516952	1.4922517151	2.0824174452
Trend	$\nearrow$	$\nearrow$	$\nearrow$	$\nearrow$

Table 2. ECR element for the Steklov eigenvalue problem on unit square.

h	$\lambda_{1,h}$	$\lambda_{2,h}$	$\lambda_{3,h}$	$\lambda_{4,h}$
$1/8 \times 1/8$	0.24022921553	1.4902910985	1.4902910985	2.0587073322
$1/16 \times 1/16$	0.24011658591	1.4916252415	1.4916252415	2.0757319710
$1/32 \times 1/32$	0.24008845713	1.4921094845	1.4921094845	2.0807938684
$1/64 \times 1/64$	0.24008142796	1.4922515168	1.4922515168	2.0821676688
$1/128 \times 1/128$	0.24007967101	1.4922898159	1.4922898159	2.0825251579
Trend	$\searrow$	$\nearrow$	$\nearrow$	$\nearrow$

Table 3. Q_1^{rot} element for the Steklov eigenvalue problem on unit square.

h	$\lambda_{1,h}$	$\lambda_{2,h}$	$\lambda_{3,h}$	$\lambda_{4,h}$
$1/8 \times 1/8$	0.24007899921	1.4902177346	1.4902177346	2.0586553833
$1/16 \times 1/16$	0.24007904431	1.4916067329	1.4916067329	2.0757187294
$1/32 \times 1/32$	0.24007907251	1.4921048465	1.4921048465	2.0807905398
$1/64 \times 1/64$	0.24007908185	1.4922503566	1.4922503566	2.0821668354
$1/128 \times 1/128$	0.24007908447	1.4922895257	1.4922895257	2.0825249495
Trend	$\nearrow$	$\nearrow$	$\nearrow$	$\nearrow$

Table 4. EQ_1^{rot} element for the Steklov eigenvalue problem on unit square.

From the numerical results on the square domain, only for ECR and EQ_1^{rot} elements we can obtain lower bound approximations for each eigenvalue but for CR and Q_1^{rot} we obtain lower bound approximations only for sufficient large eigenvalues, which validates the results presented in Corollaries 4.1 and 4.2.

5.2. Numerical results on L shape domain

The second example is to consider the problem (1.1) on the L shape domain $\Omega = (-1,1)\times(-1,1)\setminus[-1,0]\times[-1,0]$. We also use four types of nonconforming finite element methods to solve the Steklov eigenvalue problem. Each level of the computing meshes for CR and ECR elements are also generated by the Delaunay methods. The meshes for Q_1^{rot} and EQ_1^{rot} elements are also rectangular meshes. The corresponding numerical results are shown in Tabs. 5, 6, 7, and 8.

h	$\lambda_{1,h}$	$\lambda_{2,h}$	$\lambda_{3,h}$	$\lambda_{4,h}$
0.4	0.34187686015	0.60455170710	0.97587945075	1.6559729552
0.2	0.34153539184	0.61154990008	0.98157021650	1.6771168478
0.1	0.34144698689	0.61492596140	0.98343509208	1.6872111659
0.05	0.34142342797	0.61616348440	0.98405859602	1.6907532008
0.025	0.34141779461	0.61656376012	0.98422268566	1.6917253859
Trend	$\searrow$	$\nearrow$	$\nearrow$	$\nearrow$

Table 5. CR element for the Steklov eigenvalue problem on L shape domain.

h	$\lambda_{1,h}$	$\lambda_{2,h}$	$\lambda_{3,h}$	$\lambda_{4,h}$
0.4	0.34115571455	0.60399355635	0.97530283904	1.6545456348
0.2	0.34133960789	0.61140129364	0.98142984916	1.6769133370
0.1	0.34139532086	0.61488843783	0.98340299514	1.6871801671
0.05	0.34141095827	0.61615457838	0.98405119353	1.6907472152
0.025	0.34141477640	0.61656161050	0.98422090237	1.6917241457
Trend	$\nearrow$	$\nearrow$	$\nearrow$	$\nearrow$

Table 6. ECR element for the Steklov eigenvalue problem on L shape domain.

h	$\lambda_{1,h}$	$\lambda_{2,h}$	$\lambda_{3,h}$	$\lambda_{4,h}$
$1/4\times1/4$	0.34219848597	0.61264182772	0.98118784370	1.6703651787
$1/8\times1/8$	0.34161018614	0.61501267622	0.98325733469	1.6845229993
$1/16\times1/16$	0.34146438263	0.61610489382	0.98398720715	1.6898735024
$1/32\times1/32$	0.34142810103	0.61656024783	0.98420070247	1.6914738739
$1/64\times1/64$	0.34141905352	0.61674346334	0.98425834079	1.6919097147
Trend	$\searrow$	$\nearrow$	$\nearrow$	$\nearrow$

Table 7. Q_1^{rot} element for the Steklov eigenvalue problem on L shape domain.

h	$\lambda_{1,h}$	$\lambda_{2,h}$	$\lambda_{3,h}$	$\lambda_{4,h}$
$1/4 \times 1/4$	0.34138721370	0.61207522624	0.98072422788	1.6700984490
$1/8 \times 1/8$	0.34140753674	0.61487052906	0.98313965996	1.6844530544
$1/16 \times 1/16$	0.34141373023	0.61606933092	0.98395766662	1.6898557890
$1/32 \times 1/32$	0.34141543854	0.61655135635	0.98419330927	1.6914694301
$1/64 \times 1/64$	0.34141588792	0.61674124053	0.98425649197	1.6919086027
Trend	$\nearrow$	$\nearrow$	$\nearrow$	$\nearrow$

Table 8. EQ_1^{rot} element for the Steklov eigenvalue problem on L shape domain.

Similarly, from the numerical results, only for ECR and EQ_1^{rot} elements we can obtain lower bound approximations for each eigenvalue but for CR and Q_1^{rot} we can obtain lower bound approximations only for sufficiently large eigenvalues, which validate the results presented in Corollaries 4.1 and 4.2. Notice that even on the L shape domain, we find the convergence order of the first eigenvalue is full.

6. Concluding remarks

In this paper we consider the nonconforming finite elements for the Steklov eigenvalue problems both on convex and concave domains. The lower bound approximation of the eigenvalues are also analyzed for four types of nonconforming finite elements: CR, ECR, Q_1^{rot}, and EQ_1^{rot}. Based on our analysis, for ECR and EQ_1^{rot} elements we can obtain lower bounds of the eigenvalues both on convex and concave domains. This is also the first paper giving the analysis of the ECR element for the Steklov eigenvalue problem. Especially, since the ECR element is defined on general triangular meshes, it can be used in the adaptive finite element method. This should be our future work.

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