SOME REMARKS ON TWO-SCALE CONVERGENCE AND PERIODIC UNFOLDING*

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Abstract. The paper discusses some aspects of the adjoint definition of two-scale convergence based on periodic unfolding. As is known this approach removes problems concerning choice of the appropriate space for admissible test functions. The paper proposes a modified unfolding which conserves integral of the unfolded function and hence simplifies the proofs and its application in homogenization theory.

The article provides also a self-contained introduction to two-scale convergence and gives ideas for generalization to non-periodic homogenization.

Keywords: two-scale convergence, unfolding, homogenization

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1. Introduction

Two-scale convergence was introduced by Nguetseng [16] and further developed by Allaire [1] and others. For further detailed information on two-scale convergence e.g. the survey papers [13], [7] can be recommended.

In this paper we discuss classical two-scale convergence and the adjoint approach based on periodic unfolding called also dilation operator or two-scale transform, especially modification of the unfolding which conserves the integrals, see Section 4.

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The editor learnt with great sorrow that Nils E M Svanstedt, Professor of Mathematics at Chalmers University of Technology and the University of Gothenburg, unexpectedly passed away at the age of 53 on April 28, 2012.

The two-scale convergence enables to overcome the following problem:

Let $\{u_n\}$ and $\{v_n\}$ be two weakly converging sequences.
What is the limit of their product $\{u_n v_n\}$?

In general, the weak limit $\lim(u_n v_n)$ differs from the product $\lim u_n \lim v_n$ as is illustrated by the following simple example: In $L^2(0, 2\pi)$ the sequences $\{u_n\}$ and $\{v_n\}$ given by $u_n(x) = v_n(x) = \sin(nx)$ both have weak limit 0 by the Riemann-Lebesgue lemma, but the weak limit of the sequence of products $\{u_n v_n\}$ is equal to $\frac{1}{2}$. The phenomenon is caused by the fact that in the process of taking the weak limit the information about the local behavior (like oscillations or concentrations) of the sequences $\{u_n\}$ and $\{v_n\}$ is lost.

The problem introduced appears e.g. in homogenization theory which studies behavior of the solutions u^ε to a sequence of differential equations with periodic coefficients $a^\varepsilon(x)$ while the period ε of the coefficients goes to 0. In the typical model problem a sequence $\{u^\varepsilon\}$ of solutions to the following problems is studied

$$(1) \qquad -\operatorname{div}(a^\varepsilon(x)\nabla u^\varepsilon) = f \text{ in } \Omega, \quad u^\varepsilon = 0 \text{ on } \partial\Omega, \quad \varepsilon = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots,$$

where the relation $a^\varepsilon(x) = a(x/\varepsilon)$ for a periodic $a(y) \geqslant \alpha > 0$, $a \in L^\infty(\mathbb{R}^N)$ defines a weakly converging sequence $\{a^\varepsilon\}$ of coefficients. For a domain $\Omega \subset \mathbb{R}^N$ with a "good" boundary and $f \in L^2(\Omega)$ the unique weak solution $u^\varepsilon \in V = W_0^{1,2}(\Omega)$ exists. Since the sequence $\{u^\varepsilon\}$ is bounded, it contains a weakly converging subsequence of gradients $\{\nabla u^{\varepsilon'}\}$. The proof of the main result of homogenization, i.e. convergence of the solutions u^ε to the solution u^* of the so-called homogenized problem, starts with the weak formulation of the problem

$$(2) \qquad \text{Find } u^\varepsilon \in V \quad \text{such that} \quad \int_\Omega (\nabla v)^{\mathrm{T}} a^\varepsilon \nabla u^\varepsilon \,\mathrm{d}x = \int_\Omega v f \,\mathrm{d}x \quad \forall\, v \in V.$$

We now have to pass to the limit as $\varepsilon' \to 0$. The left-hand side integral contains a product of two weakly converging sequences $\{a^{\varepsilon'}\}$ and $\{\nabla u^{\varepsilon'}\}$. As the previous counterexample shows, it is not possible to pass directly to the limit.

This crucial problem in homogenization theory was first solved by a special choice of the sequence of weakly converging periodic test functions, see e.g. the book [3] by Bensoussan, Lions and Papanicolaou. Its substance was generalized to the non-periodic case by F. Murat and L. Tartar [14] in their "div-curl" lemma.

A straightforward approach to periodic homogenization problems appeared in two-scale convergence introduced by G. Nguetseng [16] and further developed by e.g. G. Allaire [1]. A sequence $\{u^\varepsilon\}$ of the variable x has the limit u^0 of double variables x and y, the local behavior is conserved in the second variable y. According to the classical definition the sequence $\{u^\varepsilon\}$ *two-scale converges* to the limit u^0 if the convergence

$$\int_\Omega u^\varepsilon(x)\varphi\Big(x, \frac{x}{\varepsilon}\Big)\,\mathrm{d}x \to \int_\Omega \int_Y u^0(x,y)\varphi(x,y)\,\mathrm{d}y\,\mathrm{d}x$$

holds for the so-called admissible test functions $\varphi(x,y)$ from a space $\mathcal{V}$ of functions being periodic in the variable y, see Definition 3.1. The space $\mathcal{V}$ cannot be the whole space $L^p(\Omega \times Y)$ of functions periodic in y, it also cannot be too small, see Section 3.

The adjoint approach is based on the so-called periodic unfolding. In contrast to Definition 3.1, where the x, y-variable test function φ is transformed into x-variable function and the limit analysis takes place in $L^p(\Omega)$, in the adjoint approach the sequence of x-variable functions u^ε is transformed into a sequence of x, y-variable functions $\mathcal{T}_\varepsilon u^\varepsilon$ and the limit analysis is carried out in $L^p(\Omega \times Y)$, see Definition 4.1 and 4.3.

The major advantage of this approach is that there are no problems with the so-called admissibility, i.e. regularity of the test functions. Also the strong two-scale convergence can be defined in a natural way. In addition, compactness and limit passage theorems will follow directly from the L^p-theory.

The periodic unfolding called also dilation appeared in [2], [5], the two-scale convergence based on periodic unfolding was introduced in [6], [8], [7] and [15]. The unfolding works well on the whole space $\mathbb{R}^N$, but in the case of a domain Ω with "incomplete" boundary cells the unfolding is not defined in these incomplete cells. Thus in [8], [7] the unfolding was modified by extending functions by zero in the incomplete cells, see Section 4. But this unfolding does not conserve the integrals. Extension of the unfolding (11) in [10], [11] solves the problem, simplifies the proofs and removes several difficulties and the necessity of introducing "unfolding criterion for integrals", see Section 4. For sequences bounded in $L^p(\Omega)$ the definitions of two-scale convergence are equivalent.

The paper aims also to give a self-contained introduction to two-scale convergence and its properties including proofs which can be simplified using integral conserving property. In the end ideas of generalization to non-periodic homogenization based on the integral conserving property are outlined.

2. Preliminaries

We start with formulation of notions used in periodic homogenization and two-scale convergence which are implicitly assumed but usually not explicitly specified.

Scale. A sequence $E = \{\varepsilon_n\}_{n=1}^{\infty}$ of small positive numbers ε_n tending to zero is called the *scale*. In this paper instead of a subscript $n \in \mathbb{N}$ all sequences will be denoted with a superscript ε_n from the scale E, but the n in ε_n is usually omitted and the sequences are denoted e.g. by $\{a^\varepsilon\}$, only. The notation comes from the periodic homogenization, where the sequence of problems with coefficients a^ε is studied and the parameter ε denotes magnitude of the period. Although in the literature it is usually not stated, the two-scale convergence is always defined with respect to a fixed scale E and speaking about a converging subsequence $\{u^{\varepsilon'}\}$, its two-scale convergence is taken with respect to a corresponding subscale $E' \subset E$.

Domain Ω, period Y and Y-periodic functions. In the following Ω will be a bounded domain in $\mathbb{R}^N$ with points $x = (x_1, \ldots, x_N)$ and with a "good" boundary, e. g. with a Lipschitz boundary $\partial\Omega$.

In the periodic homogenization Y denotes the basic period in $\mathbb{R}^N$ called also the unit cell with points $y = (y_1, \ldots, y_N)$. It can be any parallelepiped Y for the sake of simplicity of unit measure having the "paving property": the space $\mathbb{R}^N$ can be written as a union of disjoint $Y_\xi = Y + \xi$ which are the cells Y shifted by vectors ξ from a countable set Ξ in $\mathbb{R}^N$, i.e. $\mathbb{R}^N = \bigcup\limits_{\xi\in\Xi} Y_\xi$. In the model problem the basic cell Y is the N-dimensional unit cube $Y = \langle 0, 1)^N$, and the set of shifts Ξ is $\mathbb{Z}^N$.

We shall say that a function $a(y)$ is *Y-periodic*, if it is defined on $\mathbb{R}^N$ and satisfies $a(y+\xi) = a(y)$ for each $\xi \in \Xi$ and all $y \in \mathbb{R}^N$. In the model problem $a(y)$ is periodic in each variable y_i with period 1. If the function a depends, in addition, on another variable, say x, it is said to be Y-periodic in y.

Let us recall that taking a bounded measurable Y-periodic function $a(y)$ and the scale E, relation $a^\varepsilon(x) = a(x/\varepsilon) \equiv a(x_1/\varepsilon, \ldots, x_N/\varepsilon)$ defines a sequence of periodic functions with diminishing period. It converges in $L^p(\Omega)$ weakly if $1 \leqslant p < \infty$ and weakly-$*$ if $p = \infty$ to a constant function $\overline{a}$, which is the integral average of $a(y)$: $\overline{a}(x) = \int_Y a(y)\,\mathrm{d}y$. In the case when the unit cell Y does not have unit measure, the integral yielding the limit is, in addition, divided by the measure $|Y|$, i.e. $\overline{a} = |Y|^{-1}\int_Y a(y)\,\mathrm{d}y$. Let us remark that the mapping $x \mapsto x/\varepsilon$ is a special case of the so-called action on $\mathbb{R}^N$, see [17].

Spaces of periodic functions. The standard Lebesgue spaces of functions integrable in the pth power are denoted by $L^p(\Omega)$, $W^{k,p}(\Omega)$ means the Sobolev space of functions having partial derivatives up to order k in $L^p(\Omega)$ and $C^k(\overline{\Omega})$ denotes the space of functions with continuous derivatives up to order k in $\overline{\Omega}$.

Function spaces of Y-periodic functions will be denoted by $X_{\text{per}}(Y)$. Its elements $a(y)$ are defined on $\mathbb{R}^N$ and are periodic in y with period Y, their restriction to any bounded domain $G \subset \mathbb{R}^N$ is in $X(G)$, although the norm is taken over the cell Y only. The space $X_{\text{per}}(Y)$ can be "smaller" than functions of $X(Y)$ extended to $\mathbb{R}^N$ by periodicity. While $L^p_{\text{per}}(Y)$ can be identified with $L^p(Y)$, the space $C^1_{\text{per}}(Y)$ is a closed subspace of $C^1(\overline{Y})$, since its functions, in addition, have equal values and gradients on the opposite sides of the cell Y. A similar assertion holds for $W^{1,p}_{\text{per}}(Y)$.

Finally, spaces of abstract functions will be used. Let us remark that the Banach spaces $L^p(\Omega, L^p_{\text{per}}(Y))$, $L^p_{\text{per}}(Y, L^p(\Omega))$ and $L^p_{\text{per}}(\Omega \times Y)$ can be identified.

3. Classical definition of two-scale convergence

The two-scale convergence in $L^2(\Omega)$ was introduced in [16] and further worked out in [1]. In $L^p(\Omega)$, $p \in (1, \infty)$, see e.g. in [13], it is defined as follows:

Definition 3.1. A sequence of functions u^ε in $L^p(\Omega)$ two-scale converges to a limit $u^0 \in L^p(\Omega \times Y)$ with respect to the scale E if

$$\lim_{\varepsilon \to 0} \int_\Omega u^\varepsilon(x) \varphi\Big(x, \frac{x}{\varepsilon}\Big) \, \mathrm{d}x = \int_\Omega \int_Y u^0(x, y) \varphi(x, y) \, \mathrm{d}y \, \mathrm{d}x \tag{3}$$

for each test function $\varphi(x, y) \in \mathcal{V} = L^q(\Omega; C_{\text{per}}(Y))$, where $q = p/(p-1)$.

For the space $\mathcal{V}$ of *admissible functions* the space $C^0(\overline{\Omega}, L^p_{\text{per}}(Y))$ can be also used. If the definition requires that the sequence $\{u^\varepsilon\}$ is bounded in $L^p(\Omega)$, then the space $\mathcal{V} = C_0^\infty(\Omega) \otimes C^\infty_{\text{per}}(Y)$ of functions with compact support in Ω is sufficient. Some authors say that the sequence *converge two-scale strongly* if moreover $\|u^\varepsilon\|_{L^p(\Omega)} \to \|u^0\|_{L^p(\Omega \times Y)}$.

Problem of the space $\mathcal{V}$ of admissible test functions. The space $\mathcal{V}$ of test functions in the definition cannot be the whole $L^p(\Omega \times Y)$, since the test function $\varphi(x, y)$ is transformed into one variable function $\varphi(x, x/\varepsilon)$, i.e. it is a trace which for integrable and therefore only measurable function $\varphi(x, y) \in L^p(\Omega \times Y)$ is not defined. Thus some continuity of the test functions must be assumed.

The Banach space of admissible test functions $\mathcal{V}$ is usually supposed to contain functions satisfying Carathéodory conditions (measurability in one variable and continuity in the other variable) and two additional conditions

$$\Big\| v\Big(\cdot, \frac{\cdot}{\varepsilon}\Big) \Big\|_{L^p(\Omega)} \leqslant \|v\|_{\mathcal{V}}, \quad \Big\| v\Big(\cdot, \frac{\cdot}{\varepsilon}\Big) \Big\|_{L^p(\Omega)} \to \|v\|_{L^p(\Omega \times Y)}.$$

The first of these conditions is a consequence of the Scorza-Dragoni Theorem, see e.g. [9] and the second condition is proved in e.g. [1] for $\mathcal{V} = L^2(\Omega; C_{\rm per}(Y))$ and in [17] for the extension to the almost periodic and general deterministic cases. It turns out that the space $\mathcal{V}$ cannot be too small. E.g. in case of the space $\mathcal{V} = C_0^\infty(\Omega \times Y)$, the definition admits even sequences unbounded in $L^p(\Omega)$. Indeed, the sequence $u^\varepsilon(x) = \varepsilon^{-1} 1_{(0,\varepsilon)}(x)$, where $1_A(x) = 1$ for $x \in A$, otherwise $1_A(x) = 0$, see [18], satisfies the convergence (3) for all test functions, but is not bounded in $L^p((0,1))$, $p > 1$.

Thus some authors add to the definition the condition that the sequence $\{u^\varepsilon\}$ is bounded in $L^p(\Omega)$. This condition solves the problem—it is a consequence of an important lemma:

Let the sequence $\{u^\varepsilon\}$ be bounded in a Banach space X of integrable functions on Ω and let it converge to u^* in the sense of distributions, i.e. $\int_\Omega (u^\varepsilon - u^*)\varphi \,\mathrm{d}x \to 0$ for each $\varphi \in C_0^\infty(\Omega)$. Then $\{u^\varepsilon\}$ converges to u^* weakly in X.

4. Dual definitions based on the periodic unfolding

The alternative approach is based on the so-called periodic unfolding called also two-scale transform $\mathcal{T}_\varepsilon$. Each function u^ε of x-variable is transformed into a function $\mathcal{T}_\varepsilon u^\varepsilon$ of both x and y variables by the relation

$$(\mathcal{T}_\varepsilon u^\varepsilon)(x,y) = u^\varepsilon(t_\varepsilon(x,y)), \tag{4}$$

where the mapping $t_\varepsilon\colon \Omega \times Y \to \Omega$ will be described later. Then weak convergence of $\mathcal{T}_\varepsilon u^\varepsilon$ is tested in $L^p(\Omega \times Y)$, see [6], [8], [7], [15], [10], [11]:

The sequence $\{u^\varepsilon\} \subset L^p(\Omega)$ two-scale converges to u^0 in $L^p(\Omega)$ with respect to the scale E, if $\mathcal{T}_\varepsilon u^\varepsilon$ converge to u^0 weakly in $L^p(\Omega \times Y)$.

This approach removes difficulties with the space of test functions: both the limit and the test function can be taken from the maximal spaces: $u^0 \in L^p(\Omega; L^p_{\rm per}(Y))$ and φ in its dual space $L^q(\Omega; L^q_{\rm per}(Y))$. We need not take care of the space $\mathcal{V}$, admissibility and compatibility of the test functions as in the classical definition. The definitions in various papers differ in the unfolding near the boundary.

Periodic unfolding. The idea of periodic unfolding appeared in [2] and [5]. Since the system $\{Y_\xi\colon \xi \in \Xi\}$ of disjoint ξ-shifted cells Y covers $\mathbb{R}^N$, each point $x \in \mathbb{R}^N$ can be uniquely split into two parts $x = [x]_Y + \{x\}_Y$, where $[x]_Y \in \Xi$ is the shift ξ of the cell Y_ξ containing x and $\{x\}_Y \in Y$ is the relative position of x with respect to the cell.

In case of the unit cube period $Y = \langle 0,1)^N$, both vectors $[x]_Y = ([x_1], \dots, [x_N])$ and $\{x\}_Y = (\{x_1\}, \dots, \{x_N\})$ are vectors of integer and fraction parts of the coordinates x_i, i.e. $x_i = [x_i] + \{x_i\} \in \mathbb{R}$, where $[x_i] \in \mathbb{Z}$ and $\{x_i\} \in \langle 0,1)$.

Using the decomposition defined above to ε-scaled cells $Y_\xi^\varepsilon = \varepsilon Y_\xi$

$$x = \varepsilon\Big[\frac{x}{\varepsilon}\Big]_Y + \varepsilon\Big\{\frac{x}{\varepsilon}\Big\}_Y,$$

the unfolding can be defined by (4) with the so-called two-scale mapping $t_\varepsilon\colon \mathbb{R}^N \times Y \to \mathbb{R}^N$

$$t_\varepsilon(x,y) = \varepsilon\Big[\frac{x}{\varepsilon}\Big]_Y + \varepsilon y. \tag{5}$$

The unfolding defined by (5) was used in [6], [15]. It works well in case of $\Omega = \mathbb{R}^N$ or if the domain Ω can be written as the interior of a union of cells εY_ξ. Then, in addition, the unfolding has an important property: it conserves the integral of the functions:

$$\int_\Omega f(x)\,\mathrm{d}x = \iint_{\Omega\times Y} (\mathcal{T}_\varepsilon f)(x,y)\,\mathrm{d}x\,\mathrm{d}y. \tag{6}$$

But in general the domain Ω cannot be written in this way and in Ω around its boundary $\partial\Omega$ “incomplete” cells remain. In these cells the unfolded function $\mathcal{T}_\varepsilon u$ is not defined, since the range of the mapping t_ε overlaps Ω. Also the equality (6) does not hold, namely equality in Proposition 1 of [6] is not true.

Unfolding extended by zero. To remove the problem in [8] and [7] the unfolding was modified. Using the ε-scaled system $\{Y_\xi^\varepsilon = \varepsilon(Y+\xi),\ \xi \in \Xi\}$ the domain Ω was split into part Ω_ε containing the “complete” cells Y_ξ^ε and the remainder part Λ_ε containing the “incomplete” cells:

$$\Omega = \Omega_\varepsilon \cup \Lambda_\varepsilon \quad \text{where } \Omega_\varepsilon = \bigcup\{Y_\xi^\varepsilon\colon\ \xi \in \Xi,\ Y_\xi^\varepsilon \subset \Omega\} \ \text{ and } \ \Lambda_\varepsilon = \Omega \setminus \Omega_\varepsilon. \tag{7}$$

Then in [8], [7] outside of Ω_ε the unfolding (4) extends function u on Λ_ε by zero

$$(\mathcal{T}_\varepsilon u)(x,y) = \begin{cases} u\Big(\varepsilon\Big[\dfrac{x}{\varepsilon}\Big]_Y + \varepsilon y\Big) & \text{for } x \in \Omega_\varepsilon,\\ 0 & \text{for } x \in \Lambda_\varepsilon. \end{cases} \tag{8}$$

The unfolding $\mathcal{T}_\varepsilon$ is now well defined in Ω, nevertheless the important integral conservation equality (6) is lost. In [7] the problem was solved by introducing a new property: *The sequence f^ε is said to satisfy unfolding criterion for integrals* if

$$\lim_{\varepsilon\to 0}\int_{\Lambda_\varepsilon} f^\varepsilon(x)\,\mathrm{d}x = 0. \tag{9}$$

This property requires the difference $\int_\Omega f^\varepsilon(x)\,\mathrm{d}x - \iint_{\Omega\times Y} \mathcal{T}_\varepsilon(f^\varepsilon)(x,y)\,\mathrm{d}x\,\mathrm{d}y$ to go to zero, i.e. the integral conservation equality (6) holds in the limit. In [8], [7] this unfolding is used to define two-scale convergence for any domain Ω:

Definition 4.1. The sequence $\{u^\varepsilon\} \subset L^p(\Omega)$ *two-scale converges to* u^0 *in* $L^p(\Omega)$ *with respect to the scale* E if $\mathcal{T}_\varepsilon u^\varepsilon$ defined by (8) converge to u^0 weakly in $L^p(\Omega\times Y)$.

The new unfolding extended by identity. In this paper a modified extension of the unfolding operator $\mathcal{T}_\varepsilon$ is proposed, see also [10], [11], which removes the problems cited above: In the incomplete cells Λ_ε we define the two-scale mapping by putting $t_\varepsilon(x,y) = x$, i.e.

$$t_\varepsilon(x,y) = \begin{cases} \varepsilon\left[\dfrac{x}{\varepsilon}\right]_Y + \varepsilon y & \text{for } x \in \Omega_\varepsilon, \\ x & \text{for } x \in \Lambda_\varepsilon, \end{cases} \tag{10}$$

and thus the unfolding $\mathcal{T}_\varepsilon$ is extended by the identity:

$$(\mathcal{T}_\varepsilon u)(x,y) = \begin{cases} u\Big(\varepsilon\left[\dfrac{x}{\varepsilon}\right]_Y + \varepsilon y\Big) & \text{for } x \in \Omega_\varepsilon, \\ u(x) & \text{for } x \in \Lambda_\varepsilon. \end{cases} \tag{11}$$

Let us survey its properties.

Lemma 4.2. *Let* $\mathcal{T}_\varepsilon$ *be the unfolding defined by* (11). *Then for each* $\varepsilon > 0$ *we have:*

(a) *The unfolding* $\mathcal{T}_\varepsilon$ *is linear and multiplicative, i.e. for* $\alpha, \beta \in \mathbb{R}$ *and* $u, v\colon\ \Omega \to \mathbb{R}$

$$\mathcal{T}_\varepsilon(\alpha u + \beta v) = \alpha\mathcal{T}_\varepsilon(u) + \beta\mathcal{T}_\varepsilon(v), \quad \mathcal{T}_\varepsilon(uv) = \mathcal{T}_\varepsilon(u)\mathcal{T}_\varepsilon(v). \tag{12}$$

(b) *The unfolding* $\mathcal{T}_\varepsilon$ *conserves the integral, i.e. for* $f \in L^1(\Omega)$ *there is*

$$\iint_{\Omega\times Y} (\mathcal{T}_\varepsilon f)(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_\Omega f(x)\,\mathrm{d}x, \tag{13}$$

which implies that it is an isometry between $L^p(\Omega)$ *and* $L^p(\Omega\times Y)$*, i.e. for* $u \in L^p(\Omega)$

$$\|\mathcal{T}_\varepsilon u\|_{L^p(\Omega\times Y)} = \|u\|_{L^p(\Omega)}. \tag{14}$$

In case of $|Y| \neq 1$ *the unfolding operator* $\mathcal{T}_\varepsilon$ *multiplies the integrals by the measure* $|Y|$.

P r o o f. The first equality follows directly from (11). The second equality is a consequence of the measure conserving property of t_ε defined by (10): The full inverse image $(t_\varepsilon)^{-1}(M) \equiv \{(x,y) \in \Omega \times Y \colon t_\varepsilon(x,y) \in M\}$ of $M \subset \Omega$ in t_ε has the same Lebesgue measure, i.e. $|(t_\varepsilon)^{-1}(M)| = |M|$, since this holds for both the inner and the boundary parts of $M \cap Y^\varepsilon_\xi$. □

The new modified definition. The unfolding $\mathcal{T}_\varepsilon$ modified by (11) enables to introduce a new (see also [10], [11]), "the most convenient" definition of the two-scale convergence and also the strong two-scale convergence:

Definition 4.3. Let $\mathcal{T}_\varepsilon$ be the unfolding given by (11). Let E be a scale, $\{u^\varepsilon\}$ a sequence in $L^p(\Omega)$ and $u^0 \in L^p(\Omega \times Y)$.

(a) The sequence u^ε (weakly) two-scale converges to u^0 in $L^p(\Omega)$ with respect to the scale E if $\mathcal{T}_\varepsilon(u^\varepsilon)$ converge to u^0 weakly in $L^p(\Omega \times Y)$.

(b) The sequence u^ε strongly two-scale converges to u^0 in $L^p(\Omega)$ with respect to the scale E if $\mathcal{T}_\varepsilon(u^\varepsilon)$ converge to u^0 strongly in $L^p(\Omega \times Y)$.

Comparison of the definitions. The unfoldings (8), (11) and thus Definitions 4.1, 4.3 differ in the boundary stripe Λ_ε only, thus in the case when Λ_ε has zero measure both definitions coincide. In case of a general domain Ω when $|\Lambda_\varepsilon| > 0$ and $|\Lambda_\varepsilon| \to 0$ if the sequence u^ε is bounded in $L^p(\Omega)$, then the integral of u^ε over Λ_ε goes to zero and thus both definitions are equivalent. Nevertheless, for the sequence

$$u^\varepsilon(x) = |\Lambda_\varepsilon|^{-1} 1_{\Lambda_\varepsilon}(x)$$

(which is unbounded in $L^p(\Omega)$ for $p > 1$) the unfolding $\mathcal{T}_\varepsilon$ given by (8) yields $\mathcal{T}_\varepsilon u^\varepsilon \equiv 0$ and the convergence in Definition 4.1 is satisfied. In the unfolding given by (11) we have $\mathcal{T}_\varepsilon u^\varepsilon = u^\varepsilon$ and $\iint_{\Omega\times Y} \mathcal{T}_\varepsilon u^\varepsilon \,\mathrm{d}x\,\mathrm{d}y = 1$. Thus u^ε does not converge by Definition 4.3.

For a Y-periodic function ψ the unfolding of $\psi_\varepsilon(x) = \psi(x/\varepsilon)$ yields in $\Omega_\varepsilon \times Y$

$$\mathcal{T}_\varepsilon \psi_\varepsilon(x,y) = \psi_\varepsilon\Big(\varepsilon\Big[\frac{x}{\varepsilon}\Big] + \varepsilon y\Big) = \psi\Big(\Big[\frac{x}{\varepsilon}\Big] + y\Big) = \psi(y).$$

For previous bounded ψ and uniformly continuous $\varphi \in C^0(\overline{\Omega})$ the functions $\mathcal{T}_\varepsilon(\varphi\psi_\varepsilon)(x,y)$ converge uniformly to $\varphi(x)\psi(y)$ in $\overline{\Omega}_\varepsilon \times Y$. Using this fact one can prove that for u^ε bounded in $L^p(\Omega)$ Definitions 3.1, 4.1, and 4.3 are equivalent, see e.g. [7], Proposition 2.14.

Example 4.4.

(a) Let $f, g \in L^p(\Omega)$, $1 < p < \infty$ and $\psi \in L^\infty(Y_{\text{per}})$, such that $\int_Y \psi(y)\,\mathrm{d}y = 0$. Then

$$u^\varepsilon(x) = f(x)\psi\Big(\frac{x}{\varepsilon}\Big) + g(x)$$

is bounded in $L^p(\Omega)$. Since its unfolding $\mathcal{T}_\varepsilon u^\varepsilon$ yields $(\mathcal{T}_\varepsilon u^\varepsilon)(x, y) = f(x)\psi(y) + g(x)$ in $\Omega_\varepsilon \times Y$, u^ε strongly two-scale converges in $L^p(\Omega)$ with respect to the scale E to the limit

$$u^0(x, y) = f(x)\psi(y) + g(x).$$

The sequence converges to $g(x)$ in $L^p(\Omega)$ weakly, but not strongly (unless $f(x) \equiv 0$ or $\psi(y) \equiv 0$). The example shows that the local oscillations of u^ε, which are lost in the usual weak $L^p(\Omega)$ limit, are conserved in the strong two-scale limit.

(b) In the previous example the sequence was strongly two-scale converging. This was caused by the fact that the period of $\psi(x/\varepsilon)$ was "in resonance" with the scale $E = \{\varepsilon\}$. Considering a modified sequence $u^\varepsilon(x) = f(x)\psi(2x/\varepsilon) + g(x)$ and the same scale E, the sequence also strongly two-scale converges but the limit is $u^0(x, y) = f(x)\psi(2y) + g(x)$. The weak $L^p(\Omega)$ convergence and its limit is unchanged.

(c) If the period of the function ψ is not "in resonance" with the scale E, i.e. their ratio is irrational: e.g. $u^\varepsilon(x) = f(x)\psi(x/\sqrt{2}\varepsilon) + g(x)$, then the sequence $\{u^\varepsilon\}$ does not converge two-scale strongly but only two-scale weakly and its limit $u^0(x, y) = g(x)$ is independent of y, i.e. in the limit the local oscillations are again lost.

5. Properties of the two-scale convergence

The following results follow directly from the definition and the theory of L^p spaces.

Lemma 5.1. *Let E be a scale, $\{u^\varepsilon\}$ a sequence in $L^p(\Omega)$ and $u^0 \in L^p(\Omega \times Y)$. Then:*

(a) *Any sequence u^ε two-scale converging in $L^p(\Omega)$ is bounded in $L^p(\Omega)$.*

(b) *If the two-scale limit u^0 exists (weak or strong), then it is unique (as an element of $L^p(\Omega \times Y)$, i.e. up to a zero measure set).*

(c) *If a sequence $\{u^\varepsilon\}$ two-scale converges (weakly or strongly) to u^0 with respect to the scale E, then any subsequence $\{u^{\varepsilon'}\}$ of it two-scale converges (weakly or strongly) to the same limit u^0 with respect to the corresponding subscale E'.*

(d) *If the sequence* $\{u^\varepsilon\}$ *converges to* u^0 *two-scale strongly, then it also converges two-scale (weakly) to the same limit* u^0.

(e) *If the sequence* $\{u^\varepsilon\}$ *converges to* $u^0(x,y)$ *two-scale (strongly or weakly), then it also converges in* $L^p(\Omega)$ *weakly to* $u^* \in L^p(\Omega)$ *defined by* $u^*(x) = \int_Y u^0(x,y)\,\mathrm{d}y$.

(f) *If the sequence* u^ε *converges (strongly) to* u^* *in* $L^p(\Omega)$, *then it also converges two-scale (weakly and strongly) to* $u^0(x,y) = u^*(x)$ *with respect to any scale* E.

(g) *The relation between the convergences and two-scale convergences in* $L^p(\Omega)$ *can be expressed in the following diagram of implications:*

$$\text{strongly} \Longrightarrow \text{two-scale strongly} \Longrightarrow \text{two-scale (weakly)} \Longrightarrow \text{weakly.}$$

For applications the following compactness result is fundamental.

Theorem 5.2 (Compactness). *Let* $E = \{\varepsilon\}$ *be a scale and* $\{u^\varepsilon\}$ *be a bounded sequence in* $L^p(\Omega)$. *Then there exists a subscale* $E' = \{\varepsilon'\} \subset E$ *and a limit* $u^0 \in L^p(\Omega \times Y)$ *such that* $u^{\varepsilon'}$ *converge to* u^0 *two-scale weakly with respect to the subscale* E'.

Proof. Since the modified unfolding given by (11) satisfies Lemma 4.2 the proof is simple. For u^ε bounded in $L^p(\Omega)$, the unfolded $\mathcal{T}_\varepsilon u^\varepsilon$ is bounded in $L^p(\Omega \times Y)$ and thus there exists a subsequence—subscale $E' \subset E$ and a function $u^0 \in L^p(\Omega \times Y)$ such that $\mathcal{T}_{\varepsilon'} u^{\varepsilon'}$ converge weakly in $L^p(\Omega \times Y)$ to u^0. According to the Definition 4.3, $\{u^{\varepsilon'}\}$ converge to u^0 two-scale (weakly) with respect to the subscale E'. □

In many cases the following result enables to solve the problem mentioned in the introduction: passage to the limit in product of two weakly converging sequences if one of them is converging two-scale strongly and the second two-scale weakly.

Theorem 5.3 (Limit of the product of sequences). *Assume the sequence* $\{u^\varepsilon\}$ *converges to* u^0 *two-scale strongly and the sequence* $\{v^\varepsilon\}$ *converges to* v^0 *two-scale (weakly), both with respect to the same scale* E, *the former in* $L^p(\Omega)$ *and the latter in* $L^q(\Omega)$. *The exponents* $p, q, r \in (1,\infty)$ *are supposed to satisfy* $1/p + 1/q = 1/r < 1$. *Then the product* $u^\varepsilon v^\varepsilon$ *converges to the limit* $u^0 v^0 \equiv u^0(x,y)v^0(x,y)$ *two-scale (weakly) in* $L^r(\Omega)$.

In particular, for any $\varphi \in L^s(\Omega)$ *with* $s \in (1,\infty)$ *satisfying* $1/p + 1/q + 1/s = 1$ *we have:*

$$\int_\Omega u^\varepsilon(x) v^\varepsilon(x) \varphi(x)\,\mathrm{d}x \longrightarrow \iint_{\Omega\times Y} u^0(x,y) v^0(x,y) \varphi(x)\,\mathrm{d}x\,\mathrm{d}y. \tag{15}$$

P r o o f. Thanks to the equalities of Lemma 4.2 the value of the left-hand side of (15) is not changed under the unfolding:

$$L = \int_\Omega u^\varepsilon(x) v^\varepsilon(x)\varphi(x)\,\mathrm{d}x = \iint_{\Omega\times Y} \mathcal{T}_\varepsilon u^\varepsilon(x,y)\mathcal{T}_\varepsilon v^\varepsilon(x,y)\varphi(x)\,\mathrm{d}x\,\mathrm{d}y.$$

Adding and substracting the term $u^0\mathcal{T}_\varepsilon v^\varepsilon$, the last integral can be split into

$$L = \iint_{\Omega\times Y} (\mathcal{T}_\varepsilon u^\varepsilon - u^0)(x,y)\mathcal{T}_\varepsilon v^\varepsilon(x,y)\varphi(x)\,\mathrm{d}x\,\mathrm{d}y + \iint_{\Omega\times Y} u^0(x,y)\mathcal{T}_\varepsilon v^\varepsilon(x,y)\varphi(x)\,\mathrm{d}x\,\mathrm{d}y.$$

The first integral tends to zero, since it can be estimated by

$$\|\mathcal{T}_\varepsilon u^\varepsilon - u^0\|_{L^p(\Omega\times Y)} \cdot \|\mathcal{T}_\varepsilon v^\varepsilon\|_{L^q(\Omega\times Y)} \cdot \|\varphi\|_{L^s(\Omega)},$$

where $\mathcal{T}_\varepsilon v^\varepsilon$ is bounded and $\mathcal{T}_\varepsilon u^\varepsilon$ converges to u^0 strongly. Since $\mathcal{T}_\varepsilon v^\varepsilon$ converge weakly to v^0 and $v \mapsto \int u^0\varphi v$ acts as a continuous linear functional on $L^q(\Omega\times Y)$, the second integral tends to the right-hand side of (15). □

6. Unfolding of gradients and homogenization

In the homogenization problem (1) a sequence of gradients $\{\nabla u^\varepsilon\}$ appears. Due to (11) for $v \in W^{1,p}(\Omega)$ we have:

$$\mathcal{T}_\varepsilon(\nabla v) = \begin{cases} \dfrac{1}{\varepsilon}\nabla_y(\mathcal{T}_\varepsilon(v)) & \text{on } \Omega_\varepsilon \times Y, \\ \nabla v = \nabla_x(\mathcal{T}_\varepsilon(v)) & \text{on } \Lambda_\varepsilon \times Y. \end{cases}$$

Since $\mathcal{T}_\varepsilon(v)$ is independent of x in each cell of Ω_ε, the equality can be rewritten by means of the characteristic function 1_A of a set A ($1_A(x) = 1$ for $x \in A$, otherwise $1_A(x) = 0$)

$$\mathcal{T}_\varepsilon(\nabla v) = \frac{1}{\varepsilon}\nabla_y(\mathcal{T}_\varepsilon v) + \nabla_x(\mathcal{T}_\varepsilon v)1_{\Lambda_\varepsilon}. \tag{16}$$

Each sequence bounded in $W^{1,p}(\Omega)$ has gradient bounded in $L^p(\Omega,\mathbb{R}^N)$, thus using Theorem 5.2 the following compactness result can be proved:

Lemma 6.1. *Let E be a scale and $\{u^\varepsilon\}$ be a sequence bounded in $W^{1,p}(\Omega)$. Then there exists a subscale $E' \subset E$, function $u^* \in W^{1,p}(\Omega)$ and $u^0 \in L^p(\Omega; H^1_{\mathrm{per}}(Y))$ such that:*

(a) *$u^{\varepsilon'}$ converge to u^* weakly in $W^{1,p}(\Omega)$ and strongly in $L^p(\Omega)$ and*
(b) *$\nabla u^{\varepsilon'}$ converge to $\nabla u^* + \nabla_y u^0$ two-scale weakly in $L^p(\Omega)$.*

R e m a r k 6.2. The previous results can be applied to the homogenization problem (1). Passing to the limit as $\varepsilon' \to 0$ in (2) is possible due to Theorem 5.3 since a^ε is converging two-scale strongly to $a(y)$ and ∇u^ε is converging two-scale weakly thanks to Theorem 6.1.

7. Two-scale transform and general Σ-convergence

A key property in the construction of the unfolding operator or two-scale transform is the measure and integral conservation. The point of departure for general two-scale convergence, see [12] or [18], is a measure conserving map

$$\tau_\varepsilon\colon \mathcal{V} \subset L^p(\Omega \times Y) \longrightarrow \mathcal{W} \subset L^p(\Omega)$$

and general two-scale convergence is the sequential convergence with doubling of variables

$$\langle u^\varepsilon(x), \tau_\varepsilon v(x)\rangle_{\langle L^p(\Omega), L^q(\Omega)\rangle} \longrightarrow \langle u^0(x,y), v(x,y)\rangle_{\langle L^p(\Omega\times Y), L^q(\Omega\times Y)\rangle}.$$

The adjoint description introduces the inverse measure conserving map, where the doubling of variables takes place prior to the limit analysis, called unfolding or two-scale transform:

$$\mathcal{T}_\varepsilon\colon \mathcal{W} \to \mathcal{V}.$$

The sequential convergence is then the usual weak convergence in $L^p(\Omega \times Y)$, i.e.

$$\langle \mathcal{T}_\varepsilon u^\varepsilon(x,y), v(x,y)\rangle_{\langle L^p(\Omega\times Y), L^q(\Omega\times Y)\rangle} \to \langle u^0(x,y), v(x,y)\rangle_{\langle L^p(\Omega\times Y), L^q(\Omega\times Y)\rangle}.$$

By the measure conservation these two convergences are equivalent via the bilinear identity

$$\langle \mathcal{T}_\varepsilon u^\varepsilon(x,y), v(x,y)\rangle_{\langle L^p(\Omega\times Y), L^q(\Omega\times Y)\rangle} = \langle u^\varepsilon(x), \tau_\varepsilon v(x)\rangle_{\langle L^p(\Omega), L^q(\Omega)\rangle}.$$

With this description there is now no periodicity restriction for the unfolding operator. The action on the second scale by the unfolding operator must obey the measure preserving property but the focus can now be put on fundamental properties for this map. We recall that the map

$$\tau_\varepsilon\colon \mathcal{V} \to \mathcal{W}$$

maps elements in $L^p(\Omega \times Y)$ into a sequence of parameterized traces parameterized by $\varepsilon > 0$ in $L^p(\Omega)$, where we need to impose regularity in the variables (x, y) of

Carathéodory type in order to obtain test functions that work. Indeed, one needs the following two conditions to hold (see Section 3 above):

$$\|\tau_\varepsilon v\|_{L^p(\Omega)} \leqslant \|v\|_{\mathcal{V}}, \quad \|\tau_\varepsilon v\|_{L^p(\Omega)} \to \|v\|_{L^p(\Omega\times Y)}.$$

The case when $\tau_\varepsilon v$ is of the form $\tau_\varepsilon v(x) = v(x, H_\varepsilon(x))$ and H_ε is a group action on $\mathbb{R}^N$ is studied under the label λ-scale convergence in [19] and under the label of Σ-convergence in a series of papers by Nguetseng et al. see e.g. [17] and the references therein. If we choose v such that $v(x,\cdot)$ belongs to a suitable Banach algebra with mean value, then all the results from periodic homogenization carry over mutatis mutandis to a general deterministic setting. The first work on homogenization in algebras with mean values is due to Zhikov and Krivenko [20]. By considering the adjoint map $\mathcal{T}_\varepsilon$ one can avoid some measurability difficulties but the problem still remains to characterize $\mathcal{T}_\varepsilon$ for more general classes of admissible functions. For the periodic case an explicit characterization is possible. The crucial part here is to define a group action H_ε^* on $\mathbb{R}^N \times \mathbb{T}^N$. For simplicity we expose the one-dimensional case. As in Section 4, the symbol $[x]$ means the integer part of $x \in \mathbb{R}$. In the same way and notation as in (10) we define the group action H_ε^*:

$$H_\varepsilon^*(x,y) = \begin{cases} \varepsilon\left[\dfrac{x}{\varepsilon}\right] + \varepsilon y & \text{for } x \in \Omega_\varepsilon\text{-inner cells and for } y \in \mathbb{T}^1, \\ x & \text{for } x \in \Lambda_\varepsilon\text{-boundary cells and for } y \in \mathbb{T}^1. \end{cases} \tag{17}$$

This action called the *dilation action* defines explicitly the two-scale transform (unfolding)

$$\mathcal{T}_\varepsilon\colon \mathcal{W} \to \mathcal{V}$$

given by $\mathcal{T}_\varepsilon u^\varepsilon(x,y) = u^\varepsilon(H_\varepsilon^*(x,y))$ being the adjoint map to

$$\tau_\varepsilon\colon \mathcal{V} \to \mathcal{W}$$

given by $\tau_\varepsilon v(x) = v(x, x/\varepsilon)$ where $\mathcal{V} = L^2(\Omega; C_{\mathrm{per}}(Y))$.

The choice of the action H_ε is crucial. H_ε associates to algebras with mean value or ergodic algebras. For the deterministic case, see e.g. [17], H_ε is constructed as follows:

Let $\mathcal{H} = (H_\varepsilon)$ be an action group of the multiplicative group of positive reals $(\mathbb{R}^+, *)$ on the numerical space $\mathbb{R}^N$ satisfying the group properties

(i) $H_{\varepsilon_1} \circ H_{\varepsilon_2} = H_{\varepsilon_1\varepsilon_2}$ for all $\varepsilon_1, \varepsilon_2 \in \mathbb{R}^+$,

(ii) H_ε for $\varepsilon = 1$ is the identity mapping on $\mathbb{R}^N$.

We assume now that $\mathcal{H}$ has the following additional properties:

$(\mathrm{H})_1$ Each H_ε maps continuously $\mathbb{R}^N$ into itself.

(H)$_2$ $\lim_{\varepsilon\to 0} |H_\varepsilon(x)| = \infty$ for any $x \in \mathbb{R}^N \setminus \{0\}$.

(H)$_3$ The Lebesgue measure λ on $\mathbb{R}^N$ is quasi-invariant under $\mathcal{H}$, i.e. to each $\varepsilon > 0$ there is a $\gamma(\varepsilon) > 0$ such that $H_\varepsilon(\lambda) = \gamma(\varepsilon)\lambda$.

It is not trivial in what way to construct a dual action corresponding to $\mathcal{H} = (H_\varepsilon)$ with desired properties. We give the following example:

Example 7.1. For each fixed $\varepsilon > 0$ and $x \in \mathbb{R}^N$ let $H_\varepsilon(x) = x/\varepsilon$. This gives an action $\mathcal{H} = (H_\varepsilon)$ of the group $(\mathbb{R}^+, *)$ on $\mathbb{R}^N$. Further let $H_\varepsilon^*(x, y) = \varepsilon[x/\varepsilon] + \varepsilon y$ for $x \in \Omega_\varepsilon$-inner cells and for $y \in \mathbb{T}^1$. This gives the corresponding dual action $\mathcal{H}^* = (H_\varepsilon^*)$.

Let $u^\varepsilon(x) = u(H_\varepsilon(x))$ for a given $\varepsilon > 0$ and all $x \in \mathbb{R}^N$. For a function u from the space of bounded continuous complex functions on $\mathbb{R}^N$ denoted by $\mathcal{B}(\mathbb{R}^N)$ we have the following definition:

Definition 7.2. A function $u \in \mathcal{B}(\mathbb{R}^N)$ is said to have a *mean value property* if there exists a complex number $M(u)$ such that $u^\varepsilon \to M(u)$ in $L^\infty(\mathbb{R}^N)$-weak$*$ as $\varepsilon \to 0$. This class is denoted by Π^∞.

Example 7.3. We have $C_{\text{per}}(Y) \equiv C(\mathbb{T}^1) \subset \Pi^\infty$. Other important subsets of Π^∞ are the class of almost periodic functions or the class of functions with finite limit at infinity. See e.g. [17].

Example 7.4. If $v \in L^p(\Omega; C_{\text{per}}(Y)) \equiv L^p(\Omega; C(\mathbb{T}^1)) \subset L^p(\Omega; \Pi^\infty)$ then the mean value property of v yields the two-scale convergence

$$\langle u^\varepsilon(x), v(x, H_\varepsilon(x))\rangle_{\langle L^p(\Omega), L^q(\Omega)\rangle} \to \langle u^0(x, y), v(x, y)\rangle_{\langle L^p(\Omega\times Y), L^q(\Omega\times Y)\rangle},$$

where $u^0 \in L^p(\Omega; L^p(\mathbb{T}^1))$. By the construction above $\mathcal{T}_\varepsilon u^\varepsilon = u^\varepsilon H_\varepsilon^* \in L^p(\Omega; L^p(\mathbb{T}^1))$. The mean value property of u_ε^* now yields

$$\langle (u^\varepsilon H_\varepsilon^*)(x, y), v(x, y)\rangle_{\langle L^p(\Omega\times Y), L^q(\Omega\times Y)\rangle} \to \langle u^0(x, y), v(x, y)\rangle_{\langle L^p(\Omega\times Y), L^q(\Omega\times Y)\rangle}.$$

Here we have just indicated a few duality properties between two-scale convergence and periodic unfolding. It should be stressed that the explicit unfolding operator for general deterministic (non-periodic) structures is not known yet. In connection to this we like to point out that the two-scale convergence is also developed for the case of stochastic structures under the name of stochastic two-scale convergence in the mean by Bourgeat et al. in [4] but a corresponding stochastic unfolding method is not yet developed.

8. Conclusion

For bounded sequences in $L^p(\Omega)$, $p \in (1, \infty)$, all the three definitions of two-scale convergence 3.1, 4.1, and 4.3 are equivalent. The new modified Definition 4.3 introduces naturally also strong two-scale convergence and thus in many cases it enables to pass to the limits of a product of weakly converging sequences, see Theorem 5.3, particularly in the homogenization problems, where the coefficients a^ε converge two-scale strongly and the bounded solution gradients ∇u^ε converge two-scale weakly. The alternative approach in Definition 4.3 removes the restriction on test functions and simplifies the proofs. In comparison to [7] the notion of unfolding criterion for integrals is redundant.

Moreover, using the idea of two-scale mapping or two-scale transform only, the two-scale convergence can be generalized to non-periodic or even stochastic cases and in this way it extends homogenization even to non-periodic and stochastic media.

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