A CLASSICAL DECISION THEORETIC PERSPECTIVE ON WORST-CASE ANALYSIS

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Abstract. We examine worst-case analysis from the standpoint of classical Decision Theory. We elucidate how this analysis is expressed in the framework of Wald's famous Maximin paradigm for decision-making under strict uncertainty. We illustrate the subtlety required in modeling this paradigm by showing that information-gap's robustness model is in fact a Maximin model in disguise.

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1. Introduction

Worst-case analysis gives expression to what seems to be an instinctively natural, albeit a potentially costly, approach to managing uncertainty. Witness for instance the popular saying: *hope for the best, plan for the worst*!

Our objective in this paper is to examine the mathematical treatment that this seemingly intuitive concept is given in classical Decision Theory. In particular, we explain how worst-case analysis is captured in Wald's Maximin paradigm, and in what we term *Maximin models in disguise*.

The reason that it is important to make this clear is due to the centrality of the Maximin paradigm. This well established methodology for decision-making under uncertainty is supported by a substantial body of knowledge, both theoretical and practical, that has built up over the past eighty years. Hence, identifying Maximin models in disguise is not just an academic exercise—it has important practical modeling implications. In a word, it has the effect of illustrating that the art of creative modeling is an important element in the formulation of Maximin models.

We begin with a brief discussion of the concept of *worst-case analysis*, which includes its mathematical formulation in the context of input data problems that are subject to uncertainty. We then introduce the classical and mathematical programming formats of Wald's Maximin paradigm. This leads to a discussion of the modeling aspects of the Maximin paradigm, in particular the issue of Maximin models in disguise.

We conclude with a reminder of the preeminent role this paradigm plays in the definition of *robustness* in fields as diverse as optimization, control, economics, engineering, and statistics.

2. Worst-Case Analysis

Taking a worst-case approach to uncertainty seems to be something that is second nature to us. The basic characteristic of this position is summed up in the widely held adage: *When in doubt, assume the worst*!

So, a "worst case" search of Amazon's books database generates a huge number of books and articles with the terms "worst case" and "worst scenario" in their titles; and a search of the web further reveals how widespread this concept is in common parlance. But, as noted by Rustem and Howe [19, p. v], the idea goes further back in time:

The gods to-day stand friendly, that we may,
Lovers of peace, lead on our days to age!
But, since the affairs of men rests still incertain,
Let's reason with the worst that may befall.

Julius Caesar, Act 5, Scene 1
William Shakespeare (1564–1616)

This approach is also known as the *Worst-Case Scenario Method* and can of course be formulated in various ways.

In this discussion we shall refer specifically to the abstract mathematical formulation that this idea is given by Hlaváček [12] for the treatment of input data problems that are subject to uncertainty. The model consists of three ingredients:

- A state variable u.
- A set $\mathcal{U}_{\text{ad}}$ of admissible input data.
- A criterion function $\Phi = \Phi(A; u)$, where $A \in \mathcal{U}_{\text{ad}}$.

For simplicity we assume that for any input data $A \in \mathcal{U}_{\text{ad}}$ there exists a unique solution, call it $u(A)$, to the given state problem.

In this setup the uncertainty is with regard to the input data A: It is unknown which element of the admissible set $\mathcal{U}_{\mathrm{ad}}$ will be observed, hence it is unknown what state will be observed.

Assuming that we prefer $\Phi(A; u)$ to be small, the *Worst Scenario Problem* is as follows:

$$A^0 := \arg \max_{A \in \mathcal{U}_{\mathrm{ad}}} \Phi(A; u(A)). \tag{2.1}$$

In other words, we invoke the maxim *"When in doubt, assume the worst!"* to resolve the uncertainty in the true value of A.

With no loss of generality then, assuming that "more is better", the worst-case scenario problem can be formulated thus:

$$z^* := \min_{u \in U} f(u), \tag{2.2}$$

where

- U denotes the *uncertainty set.*
- f denotes the *objective function.*

Formally, U is an arbitrary set and f is an arbitrary real-valued function on U such that f attains a (global) minimum on U. Obviously, in situations where "less is better", the min in (2.2) is replaced by max.

Note that the term "worst" in this context implies the existence of some implicit external *preference structure.* Elishakoff [9, p. 6884] points out that this type of worst-case analysis, which he calls "anti-optimization", can be combined effectively with optimization techniques, citing Adali et al [1], [2] as examples of such schemes in the area of buckling of structures. Other examples can be found in [6], [17].

But the fact is that for well over eighty years now, beginning with von Neumann's [27], [28] pioneering work on classical *game theory*, this has been done routinely not only in the area of Decision Theory, but in statistics, operations research, economics, engineering and so on. To demonstrate this point we shall first recall how the worst-case analysis is encapsulated in Wald's [29], [30], [31] famous Maximin paradigm.

3. Maximin paradigm

Maximin is the classic mathematical formulation of the application of the worst-case approach in *decision-making under uncertainty*. An instructive verbal formulation of the paradigm is given by the philosopher John Rawls [18, p. 152] in his discussion of his *theory of justice*:

> The maximin rule tells us to rank alternatives by their worst possible outcomes: we are to adopt the alternative the worst outcome of which is superior to the worst outcome of the others.

In classical Decision Theory [10], [20] this paradigm has become the standard non-probabilistic model for dealing with uncertainty. This has been the case ever since Wald [29], [30], [31] adapted von Neumann's [27], [28] Maximin paradigm for game theory by casting "uncertainty", or "Nature", as one of the two players. The assumption is that the decision maker (DM) plays first, and then Nature selects the least favorable state associated with the decision selected by DM. The total reward to DM is determined by the decision selected by DM and the state selected by Nature.

The appeal of this simple paradigm is in its apparent ability to dissolve the uncertainty associated with Nature's selection of its states. This is due to the underlying assumption that Nature is a consistent *adversary* and as such her decisions are predictable: Nature consistently selects the least favorable *state* associated with the decision selected by the decision maker. This, in turn, eliminates the uncertainty from the analysis.

The price tag attached to this convenience is, however, significant. By eliminating the uncertainty through a single-minded focus on the worst outcome, the Maximin may yield highly "conservative" outcomes [26]. It is not surprising, therefore, that over the years a number of attempts have been made to modify this paradigm with a view to mitigate its extremely "pessimistic" stance. The most famous variation is no doubt Savage's *Minimax Regret* model [10], [20], [22]. But, the fact remains that, for all this effort, the Maximin paradigm provides no easy remedy for handling decision problems subject to severe uncertainty/variability [11].

4. Math formulations

The first point to note is that the Maximin paradigm can be given more than one mathematical formulation. For our purposes, however, it will suffice to consider the two most commonly encountered (equivalent) formulations. These are: the *classical formulation* and the *mathematical programming* formulation. As we shall see, these formulations can often be simplified by exploiting specific features of the problem under consideration.

Both formulations employ the following three basic, simple, intuitive, abstract constructs:

- A *decision space*, D.
  A set consisting of all the decisions available to the decision maker.
- *State spaces* $S(d) \subseteq \mathbb{S}$, $d \in D$.
  $S(d)$ denotes the set of states associated with decision $d \in D$. We refer to $\mathbb{S}$ as the *state space*.
- A real-valued function f on $D \times \mathbb{S}$.
  $f(d, s)$ denotes the value of the outcome generated by the decision-state pair (d, s). We refer to f as the *objective function*.

The decision situation represented by this model is as follows: the decision maker (DM) is intent on selecting a decision that will optimize the value generated by the objective function f. However, this value depends not only on the decision d selected by the DM, but also on the state s selected by Nature.

Since Nature is a consistent adversary, it will always select a state $s \in S(d)$ that is least favorable to the DM. Thus, if the DM is maximizing, Nature will minimize $f(d, s)$ with respect to s over $S(d)$. And if DM is minimizing, Nature will maximize $f(d, s)$ with respect to s over $S(d)$.

4.1. Classical formulation. This formulation has two forms, depending on whether the DM seeks to maximize or minimize the objective function:

$$\text{Maximin Model:} \quad z^* = \max_{d \in D} \min_{s \in S(d)} f(d, s), \tag{4.1}$$

$$\text{Minimax Model:} \quad z^\circ = \min_{d \in D} \max_{s \in S(d)} f(d, s). \tag{4.2}$$

Note that in these formulations the "outer" optimization represents the DM and the "inner" optimization—what Elishakoff [9] calls "anti-optimization"—represents Nature. This means that the DM "plays" first and Nature's response is contingent on the decision selected by the DM.

In short, in this framework the *worst-case analysis* is conducted by Nature, namely by the "inner" optimization of the Maximin/Minimax formats. The terms "Maximin" and "Minimax" thus convey in the most vividly descriptive manner the essence of the conflict between the inner and outer optimization operations.

Since the Minimax model and the Maximin model are equivalent (via the multiplication of the objective function by -1), we shall henceforth concentrate on the Maximin model.

4.2. Mathematical programming formulation. Often it proves more convenient to express the above models as "conventional" optimization models by eliminating the "inner" optimization altogether. Here is the equivalent Maximin model resulting from such a re-formulation of the respective classical model:

$$\text{(4.3)} \qquad \text{Maximin Model:} \quad z^* := \max_{d \in D,\, v \in \mathbb{R}} \{v\colon\ v \leqslant f(d,s),\ \forall\, s \in S(d)\}.$$

Note that in this formulation v is a decision variable and that the clause "$\forall\, s \in S(d)$" in the functional constraint entails that in cases where the state spaces are "continuous" rather than discrete, the Maximin model represents a *semi-infinite* optimization problem [19].

5. Modeling issues

The preceding discussion on the mathematical formulation of the Maximin model may have given the impression that modeling it is a straightforward affair that is carried out almost effortlessly. Yet, the fact of the matter is that the opposite is true. As indicated by Sniedovich [23], [24], formulating the components that form part of the model is often a tricky business that requires of the modeler/analyst considerable insight and ingenuity. To illustrate this point, consider the following optimization model:

$$\text{(5.1)} \qquad w^* := \max_{y \in Y} \{f(y)\colon\ g(y,u) \in C,\ \forall\, u \in \mathcal{U}(y)\},$$

where

- Y and C are arbitrary sets.
- f is a real-valued function on Y.
- g is a function on $Y \times U$.
- For each $y \in Y$ the set $\mathcal{U}(y)$ is a non-empty subset of U.
- It is assumed that $g(y,u) \in C,\ \forall\, u \in \mathcal{U}(y)$ for at least one $y \in Y$.

Here U represents the *uncertainty set*, namely u represents a parameter whose "true" value is unknown. All that is known about the true value of u is that it is an element of a given set U.

It should be noted that this model includes, as special cases, Lombardi's [17] *anti-optimization model* and Ben-Haim's [3], [4] *information-gap robustness model.*

So, the following question is of interest to us: Is the model stipulated in (5.1) a Maximin model?

And the answer is: yes it is!

Theorem 5.1.

$$\max_{y\in Y}\{f(y)\colon\ g(y,u)\in C,\ \forall\, u\in\mathcal{U}(y)\} = \max_{y\in Y}\ \min_{u\in\mathcal{U}(y)} h(y,u), \tag{5.2}$$

where

$$h(y,u) := \begin{cases} f(y), & g(y,u)\in C,\\ -\infty, & g(y,u)\notin C,\end{cases} \quad y\in Y,\ u\in\mathcal{U}(y). \tag{5.3}$$

P r o o f.

$$\begin{aligned} \max_{y\in Y}\ \min_{u\in\mathcal{U}(y)} h(y,u) &= \max_{y\in Y,\, v\in\mathbb{R}}\{v\colon\ v\leqslant h(y,u),\ \forall\, u\in\mathcal{U}(y)\}\\ &= \max_{y\in Y,\, v\in\mathbb{R}}\{v\colon\ v\leqslant f(y),\ g(y,u)\in C,\ \forall\, u\in\mathcal{U}(y)\}\\ &= \max_{y\in Y}\{f(y)\colon\ g(y,u)\in C,\ \forall\, u\in\mathcal{U}(y)\}. \end{aligned} \tag{5.4}$$

□

For obvious reasons, Sniedovich [24] labels models such as (5.1) *Maximin models in disguise*. To illustrate this point, consider Lombardi's [17, p. 100] *anti-optimization* oriented model:

$$w^\circ := \min_{x\in X}\{f(x)\colon\ 0\leqslant \min_{p\in P}\ g_j(x,p),\ j=1,\dots,N\}, \tag{5.5}$$

where P represents the uncertainty space. The equivalent Minimax formulation is as follows:

$$w^\circ := \min_{x\in X}\max_{p\in P}\varphi(x,p), \tag{5.6}$$

where

$$\varphi(x,p) := \begin{cases} f(x), & 0\leqslant g_j(x,p),\ j=1,\dots,N,\\ \infty, & \text{otherwise},\end{cases} \quad x\in X,\ p\in P. \tag{5.7}$$

By the same token, de Faria and de Almeida [6, p. 3960] formulate their optimization/antioptimization model explicitly as a Maximin model.

Similarly, consider Ben-Haim's [4, p. 40] information-gap robustness model

$$\hat{\alpha}(q,r_c) := \max\{\alpha\geqslant 0\colon\ r_c\leqslant \min_{u\in U(\alpha,\tilde{u})} r(q,u)\},\quad q\in Q, \tag{5.8}$$

where $U(\alpha, \tilde{u})$ represents a region of uncertainty of size α centered at the estimate $\tilde{u}$ of the true value of the parameter u.

The equivalent Maximin formulation is as follows:

$$\hat{\alpha}(q, r_c) := \max_{\alpha \geqslant 0} \min_{u \in U(\alpha, \tilde{u})} \sigma(q, \alpha, u), \tag{5.9}$$

where

$$\sigma(q, \alpha, u) := \begin{cases} \alpha, & r_c \leqslant r(q, u), \\ -\infty, & r_c > r(q, u), \end{cases} \quad q \in Q, \ \alpha \geqslant 0, \ u \in U(\alpha, \tilde{u}). \tag{5.10}$$

A fuller account of this simple illustration in [23], [24] shows that Ben-Haim's [4, p. 101] assertion that information-gap's robustness model is not a Maximin model is demonstrably erroneous. This reinforces Hlaváček et al's [13, p. xix] assessment that "... The worst scenario method represents a substantial part of the information-gap theory..."

Specifically, the worst-case analysis deployed by information-gap decision theory is represented by the "inner" optimization of the Maximin model (5.9). The "outer" optimization represents the decision maker's choice of the "best" (largest) safe region of uncertainty around the estimate $\tilde{u}$.

More details on the relationship between information-gap decision theory and Wald's Maximin paradigm can be found in [24].

For the record we point out that classical Decision Theory also recognizes the "optimistic" approach to uncertainty, captured by the Maximax model:

$$\text{Maximax Model:} \quad z^* = \max_{d \in D} \max_{s \in S(d)} f(d, s). \tag{5.11}$$

Here Nature cooperates with the decision-maker, hence it is as if there is in effect only one "player":

$$\max_{d \in D} \max_{s \in S(d)} f(d, s) = \max_{d \in D,\, s \in S(d)} f(d, s). \tag{5.12}$$

Hurwicz [15] combined this "optimistic" approac h to uncertainty with Wald's "pessimistic" approach to produce the famous *optimism-pessimism index* (see [10], [20]).

6. Discussion

In classical decision theory, robust optimization, statistics, economics, control theory, engineering, and so on, the quest for "robustness" is almost synonymous with an application of Wald's "maximin/minimax" paradigm. For instance, Huber [14, p. 17] observes:

> But as we defined robustness to mean insensitivity with regard to small deviations from assumptions, any quantitative measure of robustness must somehow be concerned with the maximum degradation of performance possible for an ε-deviation from the assumptions. An *optimally robust* procedure then minimizes this degradation and hence will be a minimax procedure of some kind.

The following quote is the abstract of the entry **Robust Control** in the on line New Palgrave Dictionary of Economics, Second Edition (2008), by *Noah Williams*[1]:

> Robust control is an approach for confronting model uncertainty in decision making, aiming at finding decision rules which perform well across a range of alternative models. This typically leads to a minimax approach, where the robust decision rule minimizes the worst-case outcome from the possible set. This article discusses the rationale for robust decisions, the background literature in control theory, and different approaches which have been used in economics, including the most prominent approach due to Hansen and Sargent.

More details on the central role played by Maximin in robust optimization can be found in [5], [16]. Other aspects of this important and well established paradigm are discussed in [7], [8], [21].

7. Conclusions

From the viewpoint of classical Decision Theory, worst-case analysis in the face of severe uncertainty is a game between the decision-maker and an antagonistic Nature. The difference between the worst-case of classical Game Theory and Wald's paradigm is that in the latter case the decision maker plays first, so that Nature's decision may depend on the decision selected by the decision maker.

This setup is represented by the "inner" optimization, namely the "min", of the abstract classical Maximin model

$$\overset{\text{DM}}{\max_{d\in D}} \ \overset{\text{Nature}}{\min_{s\in S(d)}} f(d,s). \tag{7.1}$$

[1] http://www.dictionaryofeconomics.com

As we have seen, the abstract nature of its three mathematical constructs: the decision space (D), the collection of state spaces ($S(d),\ d \in D$) and the objective function f, gives this simple model great expressive power.

But the other side of the coin is that precisely for this reason, modeling this paradigm requires an imaginative treatment of its components.

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