DEAD CORES OF SINGULAR DIRICHLET BOUNDARY VALUE PROBLEMS WITH ϕ-LAPLACIAN*

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Abstract. The paper discusses the existence of positive solutions, dead core solutions and pseudodead core solutions of the singular Dirichlet problem $(\phi(u'))' = \lambda f(t,u,u')$, $u(0) = u(T) = A$. Here λ is the positive parameter, $A > 0$, f is singular at the value 0 of its first phase variable and may be singular at the value A of its first and at the value 0 of its second phase variable.

Keywords: singular Dirichlet boundary value problem, dead core, positive solution, dead core solution, pseudodead core solution, existence, ϕ-Laplacian

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1. Introduction

Let A and T be positive numbers. Throughout the paper $\|x\| = \max\{|x(t)|\colon 0 \leqslant t \leqslant T\}$ denotes the norm in $C^0[0,T]$, $L_1[0,T]$ is the set of Lebesgue integrable functions on $[0,T]$ and $\mathrm{AC}[0,T]$ is the set of absolutely continuous functions on $[0,T]$. Assume that $G \subset \mathbb{R}^2$. Now $\mathrm{Car}([0,T]\times G)$ stands for the set of functions $f\colon [0,T]\times G \to \mathbb{R}$ satisfying the local Carathéodory conditions on $[0,T]\times G$, that is:

(i) for each $(x,y) \in G$, the function $f(\cdot,x,y)\colon [0,T] \to \mathbb{R}$ is measurable,

(ii) for a.e. $t \in [0,T]$, the function $f(t,\cdot,\cdot)\colon G \to \mathbb{R}$ is continuous,

(iii) for each compact set $K \subset G$ there exists $h_K \in L_1[0,T]$ such that $|f(t,x,y)| \leqslant h_K(t)$ for a.e. $t \in [0,T]$ and all $(x,y) \in K$.

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We consider the singular Dirichlet boundary value problem

$$(\phi(u'(t)))' = \lambda f(t, u(t), u'(t)), \quad \lambda > 0, \tag{1.1}$$

$$u(0) = A, \quad u(T) = A \tag{1.2}$$

depending on the positive parameter λ. Here $\phi \in C^0(\mathbb{R})$ is increasing, $f \in \mathrm{Car}([0,T] \times D)$, $D = (0, A) \times (\mathbb{R} \setminus \{0\})$, is singular at the values 0 of its first phase variable and f admits singularities at the value A of its first phase variable and at the value 0 of its second phase variable.

We say that $f \in \mathrm{Car}([0,T] \times D)$ is singular at the values 0 and A of its first and at the value 0 of its second phase variable if

$$\lim_{x \to 0^+} f(t, x, y) = \infty, \quad \lim_{x \to A^-} f(t, x, y) = \infty$$

for a.e. $t \in [0,T]$ and all $y \in \mathbb{R} \setminus \{0\}$, and

$$\lim_{y \to 0} f(t, x, y) = \infty$$

for a.e. $t \in [0,T]$ and all $x \in (0, A)$.

A function $u \in C^1[0,T]$ is called a *positive solution of the problem* (1.1), (1.2) if $\phi(u') \in \mathrm{AC}[0,T]$, $u > 0$ on $[0,T]$, u satisfies (1.2) and (1.1) holds for a.e. $t \in [0,T]$.

We say that $u \in C^1[0,T]$ is a *dead core solution of the problem* (1.1), (1.2) if there exist $0 < \alpha < \beta < T$ such that $u(t) = 0$ for $t \in [\alpha, \beta]$, $u > 0$ on $[0,T] \setminus [\alpha, \beta]$, $\phi(u') \in \mathrm{AC}[0,T]$, u satisfies (1.2) and (1.1) holds for a.e. $t \in (0,T) \setminus [\alpha, \beta]$. The interval $[\alpha, \beta]$ is called the *dead core of* u or the *dead core of the problem* (1.1), (1.2). If $\alpha = \beta$ then we say that u is a *pseudodead core solution of the problem* (1.1), (1.2).

We may say, roughly speaking, that dead core solutions are such solutions which 'stay' on singularities of considered differential equations for a time interval (equal to the dead core) in contrast to 'ordinary solutions' which only 'go over' singularities of the differential equation but do not stay there for some time interval.

Problem (1.1), (1.2) is a mathematical model for steady-state diffusion and reactions of several chemical species (see, e.g., [1], [4], [6], [7]).

The aim of this paper is to discuss the existence of positive solutions, pseudodead core solutions and dead core solutions of the problem (1.1), (1.2). Even though the problem (1.1), (1.2) is singular all types of solutions are considered in the space $C^1[0,T]$.

In the paper we will use the following conditions on the function ϕ and f in the differential equation (1.1).

(H_1) $\phi\colon \mathbb{R} \to \mathbb{R}$ is an increasing and odd homeomorphism such that $\phi(\mathbb{R}) = \mathbb{R}$,

$$(\mathrm{H}_2)\quad \begin{cases} f \in \mathrm{Car}([0,T]\times D) \text{ where } D=(0,A)\times(\mathbb{R}\setminus\{0\}),\ f \text{ is singular at the} \\ \text{value 0 of its first phase variable and } f \text{ admits singularities at the} \\ \text{value } A \text{ of its first and at the value 0 of its second phase variable,} \end{cases}$$

$$(\mathrm{H}_3)\quad \begin{cases} \text{for a.e. } t\in[0,T] \text{ and all } (x,y)\in D, \\ \qquad \varphi(t) \leqslant f(t,x,y) \leqslant p(x)\omega(|y|), \\ \text{where } \varphi\in L_1[0,T],\ p\in C^0(0,A)\cap L_1[0,A],\ \omega\in C^0(0,\infty) \text{ are positive} \\ \text{and there exists } \delta\in(0,\frac{1}{2}A) \text{ such that } p \text{ is nonincreasing on } (0,\delta] \\ \text{and nondecreasing on } [A-\delta,A),\ \omega \text{ is nonincreasing on } (0,\delta] \text{ and} \\ \qquad \displaystyle\int_0^\infty \frac{\phi^{-1}(s)}{\omega(\phi^{-1}(s))}\,\mathrm{d}s=\infty. \end{cases}$$

R e m a r k 1.1. We observe that the condition in (H_3) on the monotonicity of the functions p and ω on some intervals can be omitted. It is used only to obtain 'nicer' growth conditions for approximating functions (see the inequality (1.5)). On the other hand, since f is singular at the value 0 of its first phase variable and f admits singularities at the value A of its first and at the value 0 of its second phase variable, the condition in (H_3) on the monotonicity of p and ω gives no restrictions on f.

Put $\mathbb{N}'=\{n\in\mathbb{N}\colon\ 1/n\leqslant\delta\}$ where δ is taken from (H_3) and $D_*=(0,A)\times\mathbb{R}$. For each $n\in\mathbb{N}'$ define $f_n^*\in\mathrm{Car}([0,T]\times D_*)$ and $f_n\in\mathrm{Car}([0,T]\times\mathbb{R}^2)$ by the formulas

$$f_n^*(t,x,y)=\begin{cases} f(t,x,y) \\ \qquad \text{for } t\in[0,T],\ (x,y)\in(0,A)\times(\mathbb{R}\setminus[-1/n,1/n]), \\ \frac{1}{2}n[f(t,x,1/n)(y+1/n)-f(t,x,-1/n)(y-1/n)] \\ \qquad \text{for } t\in[0,T],\ (x,y)\in(0,A)\times[-1/n,1/n], \end{cases}$$

$$f_n(t,x,y)=\begin{cases} f_n^*(t,A-1/n,y) & \text{for } t\in[0,T],\ (x,y)\in(A-1/n,\infty)\times\mathbb{R}, \\ f_n^*(t,x,y) & \text{for } t\in[0,T],\ (x,y)\in[1/n,A-1/n]\times\mathbb{R}, \\ [\phi(1/n)]^{-1}\phi(x)f_n^*(t,1/n,y) & \text{for } t\in[0,T],\ (x,y)\in[0,1/n]\times\mathbb{R}, \\ x & \text{for } t\in[0,T],\ (x,y)\in(-\infty,0)\times\mathbb{R}. \end{cases}$$

We have due to (H_3),

$$\varphi(t)\leqslant f_n(t,x,y) \quad \text{for a.e. } t\in[0,T] \text{ and all } (x,y)\in[1/n,\infty)\times\mathbb{R}, \tag{1.3}$$

$$0<f_n(t,x,y) \quad \text{for a.e. } t\in[0,T] \text{ and all } (x,y)\in(0,\infty)\times\mathbb{R} \tag{1.4}$$

and

$$\begin{cases} f_n(t,x,y)\leqslant p(x)\omega(|y|) \\ \text{for a.e. } t\in[0,T] \text{ and all } (x,y)\in(0,A)\times(\mathbb{R}\setminus\{0\}). \end{cases} \tag{1.5}$$

Since $f_n(t,0,y)=0$ for a.e. $t\in[0,T]$ and each $y\in\mathbb{R}$, and $\lim\limits_{n\to\infty} f_n(t,x,y)=f(t,x,y)$ for a.e. $t\in[0,T]$ and all $(x,y)\in(0,A)\times(\mathbb{R}\setminus\{0\})$, we discuss the existence of positive solutions, pseudodead core solutions and dead core solutions of the problem (1.1), (1.2) by considering solutions of the sequence of auxiliary regular Dirichlet problems

$$(\phi(u'(t)))'=\lambda f_n(t,u(t),u'(t)),\quad \lambda>0, \tag{1.6}$$

$$u(0)=A-\frac{1}{n},\quad u(T)=A-\frac{1}{n} \tag{1.7}$$

with $n\to\infty$. We note that this technique is related to that presented in [3] and [4]. In [4] the authors discuss positive and dead core solutions to the problem

$$\begin{cases} u''+q(t,u')=\lambda h(t,u), \quad \lambda>0,\\ u'(a)=0,\ \ \beta u'(b)+\alpha u(b)=A,\quad \beta\geqslant 0,\ \alpha,A>0,\end{cases} \tag{1.8}$$

where $q\in C^0((a,b]\times[0,\infty))$ is nonnegative, $q(t,0)=0$ for $t\in(a,b]$ and $h\in C^0([a,b]\times(0,A/\alpha])$ is positive. We note that the motivation for treating the problem (1.8) was the paper by Bobisud [5] dealing with the Robin problem

$$\begin{cases} u''=\lambda g_2(u), \quad \lambda>0,\\ -u'(-1)+\alpha u(-1)=A,\ \ u'(1)+\alpha u(1)=A,\quad \alpha,A>0,\end{cases} \tag{1.9}$$

where $g_2\in C^1(0,A/\alpha]$ is positive. Bobisud proved that if $g_2\in L_1[0,A/\alpha]$ then for λ sufficiently large the problem (1.9) has a dead core solution. Here u is called a dead core solution of (1.9) if there exists $\tau\in[0,1)$ such that $u\in C^1[-1,1]\cap C^2([-1,1]\setminus[-\tau,\tau])$, $u=0$ on $[-\tau,\tau]$, u satisfies the boundary condition in (1.9) and $u''(t)=\lambda g_2(u(t))$ for $t\in[-1,1]\setminus[-\tau,\tau]$.

In [3] we considered the existence of positive solutions, pseudodead core solutions and dead core solutions of the singular differential equation

$$(\phi(u'))'=\lambda(f_1(t,u,u')+f_2(t,u,u')),\quad \lambda>0,$$

satisfying the Dirichlet conditions (1.2) where $f_1\in C^0([0,T]\times((0,A]\times\mathbb{R}))$ is positive and $f_2\in C^0([0,T]\times([0,A]\times(\mathbb{R}\setminus\{0\})))$ is nonnegative. Here the existence results are proved by a regularization and sequential technique and the solvability of regular problems is proved by an existence principle presented in [2].

In this paper differential equations with Carathéodory nonlinearities are considered. Our existence results are proved by a combination of a regularization and sequential technique and the method of lower and upper functions (see e.g. [8], [9]). The theory presented in this paper improves and extends the corresponding results in [2].

Applying a combination of the method of lower and upper functions and the regularization and sequential techniques, we obtain among others a generalization of the results presented in [3].

By a solution of the problem (1.6), (1.7) we mean a function $u \in C^1[0,T]$ such that $\phi(u') \in \mathrm{AC}[0,T]$, u satisfies (1.7) and (1.6) holds for a.e. $t \in [0,T]$.

It is useful to introduce also the notion of a solution of the problem (1.1), (1.2). We say that u is a *solution of the problem* (1.1), (1.2) if there exists a subsequence $\{k_n\}$ of $\{n\}_{n\in\mathbb{N}'}$ such that $\lim\limits_{n\to\infty} u_{k_n} = u$ in $C^1[0,T]$ where u_{k_n} is a solution of the problem (1.6), (1.7) with k_n instead of n. In Section 3 (see Theorem 3.1) we will prove that any solution of the problem (1.1), (1.2), is either a positive solution or a pseudodead solution or a dead core solution of this problem.

The rest of the paper is organized as follows. Section 2 is devoted to the regular problem (1.6), (1.7). Using Lemmas 2.1 and 2.2 and Proposition 2.3, the solvability of the problem (1.6), (1.7) is proved (Lemma 2.4). Lemmas 2.5–2.8 present properties of solutions to the problem (1.6), (1.7) which are used in the next section. The main results are given in Section 3. Under the assumptions (H_1)–(H_3), for each $\lambda > 0$ the problem (1.1), (1.2) has a solution and this solution is either a positive solution or a pseudodead core solution or a dead core solution of the problem (1.1), (1.2) (Theorem 3.1). For sufficiently small positive values of λ the problem (1.1), (1.2) has only positive solutions (Corollary 3.2) and if values of λ are sufficiently large then the problem (1.1), (1.2) has only dead core solutions (Corollary 3.3). Finally, Corollary 3.4 states a relation between solutions of the problem (1.1), (1.2) with distinct values of the parameter λ in (1.1). An example demonstrates the application of our existence results.

2. Auxiliary regular problems

Lemma 2.1. *Let* (H_1)–(H_3) *hold and let* u_n *be a solution of the problem* (1.6), (1.7)*. Then*

$$0 < u_n(t) \leqslant A - \frac{1}{n} \quad \text{for } t \in [0,T], \tag{2.1}$$

u_n' *is increasing on* $[0,T]$ *and there exists a unique* $\alpha_n \in (0,T)$ *such that*

$$u_n' < 0 \ \text{ on } [0,\alpha_n), \quad u_n'(\alpha_n) = 0, \quad u_n' > 0 \ \text{ on } (\alpha_n, T]. \tag{2.2}$$

P r o o f. Suppose that $\min\{u_n(t)\colon 0 \leqslant t \leqslant T\} = u_n(\xi) < 0$. Then $\xi \in (0,T)$ and there exist $0 < a < b < T$ such that $u_n(a) = u_n(b) = 0$ and $u_n < 0$ on

(a,b). Hence $u_n'(a) \leqslant 0$, $u_n'(b) \geqslant 0$ and $(\phi(u_n'(t)))' = \lambda u_n(t) < 0$ for a.e. $t \in [a,b]$. Integrating the last inequality over $[a,b]$ gives $\phi(u_n'(b)) - \phi(u_n'(a)) < 0$. Consequently $u_n'(b) < u_n'(a)$, contrary to $u_n'(a) \leqslant 0$ and $u_n'(b) \geqslant 0$. Thus $u_n \geqslant 0$ on $[0,T]$. From (1.4) and $f_n(t,0,y) = 0$ for a.e. $t \in [0,T]$ and all $y \in \mathbb{R}$ it follows that $(\phi(u_n'))' \geqslant 0$ a.e. on $[0,T]$. Therefore $\phi(u_n')$ is nondecreasing on $[0,T]$ and so is u_n' since ϕ is increasing by (H_1). Moreover (1.7) implies $u_n'(\alpha_n) = 0$ for some $\alpha_n \in (0,T)$. We now show that $\min\{u_n(t)\colon 0 \leqslant t \leqslant T\} > 0$. To obtain a contradiction assume that $\min\{u_n(t)\colon 0 \leqslant t \leqslant T\} = u_n(\xi) = 0$. Then $\xi \in (0,T)$ and $u_n'(\xi) = 0$. Put

$$\eta_n = \min\{t \in [0,T]\colon u_n'(t) = 0\}, \quad \tau_n = \max\{t \in [0,T]\colon u_n'(t) = 0\}.$$

Then $\eta_n \leqslant \xi \leqslant \tau_n$ and since u_n' is nondecreasing on $[0,T]$ we have $u_n' < 0$ on $[0,\eta_n)$, $u_n' = 0$ on $[\eta_n,\tau_n]$ and $u_n' > 0$ on $(\tau_n,T]$. Consequently $u_n \leqslant A - 1/n$ on $[0,T]$, $u_n > 0$ on $[0,T] \setminus [\eta_n,\tau_n]$ and $u_n = 0$ on $[\eta_n,\tau_n]$. As a result $(\phi(u_n'))' > 0$ a.e. on $[\tau_n,T]$ and therefore u_n' is increasing on $[\tau_n,T]$ and positive on $(\tau_n,T]$. Thus there exists $t_1 \in (\tau_n,\tau_n+1]$ such that $0 < u_n \leqslant 1/n$ on $(\tau_n,t_1]$ and u_n, u_n' are positive and increasing on this interval. Moreover from the definition of the function f_n it follows that

$$(\phi(u_n'(t)))' = B\phi(u_n(t))q(t) \quad \text{for a.e. } t \in [\tau_n,t_1] \tag{2.3}$$

where $B = \lambda[\phi(1/n)]^{-1}$, $q(t) = f_n^*(t,1/n,u_n'(t)) \in L_1[\tau_n,t_1]$ and $q > 0$ a.e. on $[\tau_n,t_1]$. Integrating (2.3) over $[\tau_n,t] \subset [\tau_n,t_1]$ yields $\phi(u_n'(t)) = B\int_{\tau_n}^t \phi(u_n(s))q(s)\,\mathrm{d}s$ and using the properties of u_n and ϕ we obtain

$$\begin{aligned}\phi(u_n'(t)) \leqslant B\phi(u_n(t))\int_{\tau_n}^t q(s)\,\mathrm{d}s &\leqslant B\phi(u_n'(t)(t-\tau_n))\int_{\tau_n}^t q(s)\,\mathrm{d}s\\ &\leqslant B\phi(u_n'(t))\int_{\tau_n}^t q(s)\,\mathrm{d}s\end{aligned}$$

for $t \in [\tau_n,t_1]$. Since $\phi(u_n') > 0$ on $(\tau_n,t_1]$, the last inequality gives $1 \leqslant B\int_{\tau_n}^t q(s)\,\mathrm{d}s$ for $t \in (\tau_n,t_1]$, which is impossible. We have proved that $u_n > 0$ on $[0,T]$. Hence $(\phi(u_n'))' > 0$ a.e. on $[0,T]$ and consequently u_n' is increasing on $[0,T]$, α_n is the unique zero of u_n' and (2.1) and (2.2) hold. $\square$

We now state *a priori* bounds for the derivative of solutions to problem (1.6), (1.7).

Lemma 2.2. *Let* (H_1)–(H_3) *hold. Then there exists a positive constant* S *independent of* $n \in \mathbb{N}'$ *(and depending on* λ*) such that*

$$\|u_n'\| < S \tag{2.4}$$

for any solution u_n *of the problem* (1.6), (1.7).

P r o o f. Let u_n be a solution of the problem (1.6), (1.7). Lemma 2.1 shows that (2.1) and (2.2) are true (where $0 < \alpha_n < T$) and u_n' is increasing on $[0, T]$. Hence

$$\|u_n'\| = \max\{|u_n'(0)|, u_n'(T)\}. \tag{2.5}$$

In view of (1.5),

$$(\phi(u_n'(t)))' \leqslant \lambda p(u_n(t))\omega(-u_n'(t)) \quad \text{for a.e. } t \in [0, \alpha_n],$$
$$(\phi(u_n'(t)))' \leqslant \lambda p(u_n(t))\omega(u_n'(t)) \quad \text{for a.e. } t \in [\alpha_n, T].$$

Now integrating

$$\frac{(\phi(u_n'(t)))' u_n'(t)}{\omega(-u_n'(t))} \geqslant \lambda p(u_n(t)) u_n'(t) \tag{2.6}$$

over $[0, \alpha_n]$ and

$$\frac{(\phi(u_n'(t)))' u_n'(t)}{\omega(u_n'(t))} \leqslant \lambda p(u_n(t)) u_n'(t) \tag{2.7}$$

over $[\alpha_n, T]$, we have

$$\int_0^{\phi(|u_n'(0)|)} \frac{\phi^{-1}(s)}{\omega(\phi^{-1}(s))}\,\mathrm{d}s \leqslant \lambda \int_{u_n(\alpha_n)}^{A-1/n} p(s)\,\mathrm{d}s < \lambda \int_0^A p(s)\,\mathrm{d}s \tag{2.8}$$

and

$$\int_0^{\phi(u_n'(T))} \frac{\phi^{-1}(s)}{\omega(\phi^{-1}(s))}\,\mathrm{d}s \leqslant \lambda \int_{u_n(\alpha_n)}^{A-1/n} p(s)\,\mathrm{d}s < \lambda \int_0^A p(s)\,\mathrm{d}s, \tag{2.9}$$

respectively. By (H_3), there exists a positive constant K (depending on λ) such that

$$\int_0^u \frac{\phi^{-1}(s)}{\omega(\phi^{-1}(s))}\,\mathrm{d}s > \lambda \int_0^A p(s)\,\mathrm{d}s$$

whenever $u \geqslant K$ and therefore (2.8) and (2.9) give $\max\{\phi(|u_n'(0)|), \phi(u_n'(T))\} < K$. Consequently $\max\{|u_n'(0)|, u_n'(T)\} < \phi^{-1}(K)$ and (2.5) now shows that (2.4) is true with $S = \phi^{-1}(K)$. □

In order to prove the existence of a solution of the problem (1.6), (1.7) we apply the method of lower and upper functions. Let $h \in \mathrm{Car}([0,T]\times\mathbb{R}^2)$ and let ϕ satisfy (H_1). Consider the Dirichlet problem

$$(\phi(u'(t)))' = h(t,u(t),u'(t)), \tag{2.10}$$

$$u(0)=a, \quad u(T)=b, \tag{2.11}$$

where $a,b \in \mathbb{R}$. We say that $v \in C^1[0,T]$ is a *lower function of the problem* (2.10), (2.11) if $\phi(v') \in \mathrm{AC}[0,T]$, $(\phi(v'(t)))' \geqslant h(t,v(t),v'(t))$ for a.e. $t \in [0,T]$ and $v(0) \leqslant a$, $v(T) \leqslant b$. If the reverse inequalities hold, we say that v is an *upper function of the problem* (2.10), (2.11).

For the solvability of the problem (2.10), (2.11) the following result holds (see e.g. [8], [9]).

Proposition 2.3. *If there exists a lower function v and an upper function z of the problem* (2.10), (2.11), *$v(t) \leqslant z(t)$ for $t \in [0,T]$ and there exists $q \in L_1[0,T]$ such that*

$$|h(t,x,y)| \leqslant q(t) \quad \textit{for a.e. } t \in [0,T] \textit{ and all } v(t) \leqslant x \leqslant z(t),\ y \in \mathbb{R},$$

then the problem (2.10), (2.11) *has a solution u and $v(t) \leqslant u(t) \leqslant z(t)$ for $t \in [0,T]$.*

Lemma 2.4. *Let* (H_1)–(H_3) *hold. Then the problem* (1.6), (1.7) *has a solution u_n satisfying* (2.1) *and* (2.2) *for some $0 < \alpha_n < T$.*

P r o o f. Let S be a positive constant in Lemma 2.2. Put

$$\tilde h(t,x,y) = \begin{cases} f_n(t,x,y) & \text{for } t \in [0,T],\ (x,y) \in [0,\infty)\times\mathbb{R}, \\ 0 & \text{for } t \in [0,T],\ (x,y) \in (-\infty,0)\times\mathbb{R}, \end{cases} \tag{2.12}$$

and

$$h(t,x,y) = \chi(y)\tilde h(t,x,y) \quad \text{for } t \in [0,T],\ (x,y) \in \mathbb{R}^2 \tag{2.13}$$

where

$$\chi(y) = \begin{cases} 1 & \text{for } |y| \leqslant S, \\ 2 - \dfrac{|y|}{S} & \text{for } S < |y| \leqslant 2S, \\ 0 & \text{for } |y| > 2S. \end{cases}$$

Then (see (1.5))

$$(2.14)\qquad \begin{cases} h(t,x,y) \leqslant p(x)\omega(|y|) \\ \text{for a.e. } t \in [0,T] \text{ and all } (x,y) \in \Big(0, A - \dfrac{1}{n}\Big] \times (\mathbb{R} \setminus \{0\}) \end{cases}$$

and there exists $q \in L_1[0,T]$ such that

$$0 \leqslant h(t,x,y) \leqslant q(t) \quad \text{for a.e. } t \in [0,T] \text{ and all } (x,y) \in \mathbb{R}^2.$$

Since $h(t,0,0) = 0$ and $h(t, A - 1/n, 0) \geqslant 0$ for a.e. $t \in [0,T]$, we see that $v = 0$ and $z = A - 1/n$ is a lower and an upper function of the problem (2.10), (1.7). Hence Proposition 2.3 guarantees that this problem has a solution u such that $0 \leqslant u_n(t) \leqslant A - 1/n$ for $t \in [0,T]$. From (2.14) and from the proof of Lemma 2.2 it follows that $\|u_n'\| < S$. Hence $h(t, u_n(t), u_n'(t)) = f_n(t, u_n(t), u_n'(t))$ for $t \in [0,T]$ and so u_n is a solution of the problem (1.6), (1.7). By Lemma 2.1, u_n satisfies (2.1) and (2.2) for some $0 < \alpha_n < T$. □

Lemma 2.5. *Let* (H_1)–(H_3) *hold and let* u_n *be a solution of the problem* (1.6), (1.7). *Then* $\{u_n'\}_{n\in\mathbb{N}'}$ *is equicontinuous on* $[0,T]$.

Proof. By Lemmas 2.1 and 2.2, there exist a positive constant S and some $0 < \alpha_n < T$ such that (2.1), (2.2) and (2.4) hold. Set

$$H(v) = \int_0^{\phi(v)} \frac{\phi^{-1}(s)}{\omega(\phi^{-1}(s))}\,\mathrm{d}s \quad \text{for } v \in [0,\infty),$$

$$H^*(v) = \begin{cases} H(v) & \text{for } v \in [0,\infty), \\ -H(-v) & \text{for } v \in (-\infty,0), \end{cases}$$

and

$$P(v) = \int_0^v p(s)\,\mathrm{d}s \quad \text{for } v \in [0,A].$$

Then $H^* \in C^0(\mathbb{R})$ is an increasing and odd function, $H^*(\mathbb{R}) = \mathbb{R}$ by (H_3) and $P \in \mathrm{AC}[0,T]$ is increasing. Since $\{u_n'\}_{n\in\mathbb{N}'}$ is bounded in $C^0[0,T]$, $\{u_n\}_{n\in\mathbb{N}'}$ is equicontinuous on $[0,T]$ and consequently $\{P(u_n)\}_{n\in\mathbb{N}'}$ is equicontinuous on $[0,T]$ as well. Choose $\varepsilon > 0$. Then there exists $\nu > 0$ such that

$$|P(u_n(t_1)) - P(u_n(t_2))| < \varepsilon \quad \text{for } n \in \mathbb{N}'$$

whenever $t_1, t_2 \in [0,T]$ and $|t_1 - t_2| < \nu$. In order to prove that $\{u_n'\}_{n\in\mathbb{N}'}$ is equicontinuous on $[0,T]$, let $0 \leqslant t_1 < t_2 \leqslant T$ and $t_2 - t_1 < \nu$. If $t_2 \leqslant \alpha_n$ integrating (2.6) from t_1 to t_2 yields

$$0 < H^*(u_n'(t_2)) - H^*(u_n'(t_1)) \leqslant \lambda[P(u_n(t_1)) - P(u_n(t_2))] < \lambda\varepsilon,$$

and if $t_2 \geqslant \alpha_n$ integrating (2.7) over $[t_1, t_2]$ gives

$$0 < H^*(u_n'(t_2)) - H^*(u_n'(t_1)) \leqslant \lambda[P(u_n(t_2)) - P(u_n(t_1))] < \lambda\varepsilon.$$

Finally, if $t_1 < \alpha_n < t_2$ one can check that

$$0 < H^*(u_n'(t_2)) - H^*(u_n'(t_1)) < 2\lambda\varepsilon.$$

Summarizing, we have

$$0 \leqslant H^*(u_n'(t_2)) - H^*(u_n'(t_1)) < 2\lambda\varepsilon \quad \text{for } n \in \mathbb{N}'$$

whenever $0 \leqslant t_1 < t_2 \leqslant T$ and $t_2 - t_1 < \nu$. Hence $\{H^*(u_n')\}_{n\in\mathbb{N}'}$ is equicontinuous on $[0,T]$ and, since $\{u_n'\}_{n\in\mathbb{N}'}$ is bounded in $C^0[0,T]$ and H^* is continuous and increasing on $\mathbb{R}$, we see that $\{u_n'\}_{n\in\mathbb{N}'}$ is equicontinuous on $[0,T]$. □

The next result will be used for the existence of positive solutions of the problem (1.1), (1.2).

Lemma 2.6. *Let* (H$_1$)–(H$_3$) *hold. Then there exist* $\lambda_0 > 0$ *and* $c > 0$ *such that*

$$u_n(t) > c \quad \text{for } t \in [0,T] \text{ and } n \in \mathbb{N}', \tag{2.15}$$

where u_n *is a solution of the problem* (1.6), (1.7) *with* $\lambda \in (0, \lambda_0]$ *in* (1.6).

P r o o f. Let $\lambda_0 > 0$ satisfy the inequality

$$\lambda_0 < \left(\int_0^A p(s)\,\mathrm{d}s\right)^{-1} \int_0^{\phi(A/T)} \frac{\phi^{-1}(s)}{\omega(\phi^{-1}(s))}\,\mathrm{d}s.$$

Then there is an $\varepsilon \in (0, A/T)$ such that

$$\lambda_0 = \left(\int_0^A p(s)\,\mathrm{d}s\right)^{-1} \int_0^{\phi(A/T-\varepsilon)} \frac{\phi^{-1}(s)}{\omega(\phi^{-1}(s))}\,\mathrm{d}s.$$

Choose $\lambda \in (0, \lambda_0]$ and let u_n be a solution of the problem (1.6), (1.7). From (2.5) and (see (2.8) and (2.9))

$$\int_0^{\phi(\|u_n'\|)} \frac{\phi^{-1}(s)}{\omega(\phi^{-1}(s))}\,\mathrm{d}s < \lambda \int_0^A p(s)\,\mathrm{d}s \leqslant \lambda_0 \int_0^A p(s)\,\mathrm{d}s,$$

we have

$$\int_0^{\phi(\|u_n'\|)} \frac{\phi^{-1}(s)}{\omega(\phi^{-1}(s))}\,\mathrm{d}s < \int_0^{\phi(A/T-\varepsilon)} \frac{\phi^{-1}(s)}{\omega(\phi^{-1}(s))}\,\mathrm{d}s.$$

Hence $\|u_n'\| < A/T - \varepsilon$. Let $\min\{u_n(t)\colon\ 0 \leqslant t \leqslant T\} = u_n(\xi)$. Then

$$A - \frac{1}{n} - u_n(\xi) = u_n(0) - u_n(\xi) = -\int_0^{\xi} u_n'(t)\,\mathrm{d}t < \Big(\frac{A}{T} - \varepsilon\Big)\xi,$$
$$A - \frac{1}{n} - u_n(\xi) = u_n(T) - u_n(\xi) = \int_{\xi}^{T} u_n'(t)\,\mathrm{d}t < \Big(\frac{A}{T} - \varepsilon\Big)(T - \xi),$$

and consequently $u_n(\xi) > A - 1/n - \frac{1}{2}T(A/T - \varepsilon) = \frac{1}{2}A - 1/n + \frac{1}{2}\varepsilon T > \frac{1}{2}\varepsilon T$. Therefore $u_n > \frac{1}{2}\varepsilon T$ on $[0, T]$ and (2.15) is true with $c = \frac{1}{2}\varepsilon T$. □

The following result will be needed for the existence of dead core solutions of the problem (1.1), (1.2).

Lemma 2.7. *Let* (H$_1$)–(H$_3$) *hold. Then for each $c \in (0, T)$, there exists $\lambda_c > 0$ such that*

$$\lim_{n\to\infty} u_n(c) = 0 \tag{2.16}$$

where u_n is a solution of the problem (1.6), (1.7) *with $\lambda > \lambda_c$ in* (1.6).

P r o o f. Fix $c \in (0, T)$. Let φ and δ be taken from (H$_3$). Set $\varrho = \min\{c, T - c\}$,

$$\Lambda = \min\Big\{\int_{c/2}^{c} \varphi(t)\,\mathrm{d}t, \int_{c}^{(T-c)/2} \varphi(t)\,\mathrm{d}t\Big\} > 0, \quad \lambda_c = \frac{1}{\Lambda}\phi\Big(\frac{2A}{\varrho}\Big).$$

Let $\lambda \in (\lambda_c, \infty)$ and choose $\varepsilon \in (0, \delta)$. If we prove that

$$u_n(c) < \varepsilon \quad \text{for all } n > \frac{1}{\varepsilon}, \tag{2.17}$$

then (2.16) is true since $u_n \geqslant 0$ on $[0, T]$. In order to prove (2.17), we argue by contradiction and assume that there exists $n_0 > 1/\varepsilon$ such that $u_{n_0}(c) \geqslant \varepsilon$. The next part of the proof is divided into two cases, namely $u_{n_0}'(c) \leqslant 0$ and $u_{n_0}'(c) > 0$.

Case 1. Suppose $u_{n_0}'(c) \leqslant 0$. By Lemma 2.1, u_{n_0}' is increasing on $[0, T]$ and therefore if $u_{n_0}'(\frac{1}{2}c) < -2A/c$ then $u_{n_0}' < -2A/c$ on $[0, \frac{1}{2}c]$ which implies

$$u_{n_0}(0) = u_{n_0}\Big(\frac{1}{2}c\Big) - \int_0^{c/2} u_{n_0}'(t)\,\mathrm{d}t > u_{n_0}\Big(\frac{1}{2}c\Big) + A > A,$$

contrary to $u_{n_0}(0) = A - 1/n_0$. Hence

$$u_{n_0}'\Big(\frac{1}{2}c\Big) \geqslant -2\frac{A}{c} \quad \text{and} \quad 0 \geqslant u_{n_0}'(t) \geqslant -2\frac{A}{c} \quad \text{for } t \in \Big[\frac{1}{2}c, c\Big]. \tag{2.18}$$

Since $n_0 u_{n_0}(t) \geqslant n_0 \varepsilon > 1$ for $t \in [0, c]$, we have (see (1.3))

$$f_{n_0}(t, u_{n_0}(t), u'_{n_0}(t)) \geqslant \varphi(t) \quad \text{for a.e. } t \in \Big[\frac{1}{2}c, c\Big] \tag{2.19}$$

and therefore

$$(\phi(u'_{n_0}(t)))' \geqslant \lambda\varphi(t) > \lambda_c\varphi(t) \quad \text{for a.e. } t \in \Big[\frac{1}{2}c, c\Big]. \tag{2.20}$$

Thus $\phi(u'_{n_0}(c)) - \phi(u'_{n_0}(\frac{1}{2}c)) > \lambda_c \int_{c/2}^{c} \varphi(t)\,\mathrm{d}t \geqslant \lambda_c\Lambda$ and so

$$\phi\Big(-u'_{n_0}\Big(\frac{1}{2}c\Big)\Big) > -\phi(u'_{n_0}(c)) + \lambda_c\Lambda \geqslant \lambda_c\Lambda = \phi\Big(\frac{2A}{\varrho}\Big) \geqslant \phi\Big(\frac{2A}{c}\Big).$$

Consequently $-u'_{n_0}(\frac{1}{2}c) > 2A/c$, contrary to (2.18).

Case 2. Suppose $u'_{n_0}(c) > 0$. Then u'_{n_0} is positive and increasing on $[c, T]$ by Lemma 2.1. If $u'_{n_0}(\frac{1}{2}(T+c)) \geqslant 2A/(T-c)$ then $u'_{n_0} \geqslant 2A/(T-c)$ on $[\frac{1}{2}(T+c), T]$ and

$$u_{n_0}(T) = u_{n_0}\Big(\frac{1}{2}(T+c)\Big) + \int_{(T+c)/2}^{T} u'_{n_0}(t)\,\mathrm{d}t \geqslant u_{n_0}\Big(\frac{1}{2}(T+c)\Big) + A > A,$$

contrary to $u_{n_0}(T) = A - 1/n$. Hence

$$0 < u'_{n_0}(t) < \frac{2A}{T-c} \quad \text{for } t \in \Big[c, \frac{1}{2}(T+c)\Big]. \tag{2.21}$$

Since $n_0 u_{n_0}(t) \geqslant n_0\varepsilon > 1$ for $t \in [c, T]$, it follows that the inequality in (2.19) is satisfied a.e. on $[c, \frac{1}{2}(T+c)]$ and therefore on this interval also the inequality in (2.20) is true. Integrating $(\phi(u'_{n_0}))' > \lambda_c\varphi(t)$ over $[c, \frac{1}{2}(T+c)]$ yields $\phi(u'_{n_0}(\frac{1}{2}(T+c))) > \phi(u'_{n_0}(c)) + \lambda_c \int_c^{(T+c)/2} \varphi(t)\,\mathrm{d}t$. Thus

$$\phi\Big(u'_{n_0}\Big(\frac{1}{2}(T+c)\Big)\Big) > \lambda_c \int_c^{(T+c)/2} \varphi(t)\,\mathrm{d}t \geqslant \lambda_c\Lambda = \phi\Big(\frac{2A}{\varrho}\Big) \geqslant \phi\Big(\frac{2A}{T-c}\Big),$$

and so $u'_{n_0}((T+c)/c) > 2A/(T-c)$, contrary to (2.21). □

We now state a relation between solutions of the problem (1.6), (1.7) with distinct values of the parameter λ in (1.1).

Lemma 2.8. *Let* (H_1)–(H_3) *hold and let* $0 < \lambda_1 < \lambda_2$. *If* u_n *is a solution of the problem* (1.6), (1.7) *with* $\lambda = \lambda_1$ *in* (1.6) *then there exists a solution* v_n *of the problem* (1.6), (1.7) *with* $\lambda = \lambda_2$ *in* (1.6) *such that*

$$v_n(t) \leqslant u_n(t) \quad \text{for } t \in [0,T].$$

P r o o f. Let $j = 1, 2$ and S_j be a positive constant in Lemma 2.2 for a priori bounds for the derivative of solutions to the problem (1.6), (1.7) with $\lambda = \lambda_j$ in (1.6). Put $S = \max\{S_1, S_2\}$ and let the function h be defined as in (2.13). Consider the differential equation

$$(\phi(u'(t)))' = \lambda h(t, u(t), u'(t)). \tag{2.22}$$

Let u_n be a solution of the problem (1.6), (1.7) with $\lambda = \lambda_1$ in (1.6). Then u_n is also a solution of the problem (2.22), (1.7) with $j = 1$ since $\|u_n\| < S_1$ and $u_n > 0$ on $[0,T]$. The function $v = 0$ is a lower function of the problem (2.22), (1.7) with $\lambda = \lambda_2$ and from

$$(\phi(u_n'(t)))' = \lambda_1 f_n(t, u_n(t), u_n'(t)) \leqslant \lambda_2 f_n(t, u_n(t), u_n'(t)) \quad \text{for a.e. } t \in [0,T]$$

and $u_n(0) = u_n(T) = A - 1/n$, we see that u_n is an upper function of the problem (2.22), (1.7) with $\lambda = \lambda_2$. Thus the last problem has a solution v_n satisfying $0 \leqslant v_n(t) \leqslant u_n(t)$ for $t \in [0,T]$ by Proposition 2.3. Now arguing as in the proof of Lemma 2.4 we show that $\|v_n'\| < S_2$. Hence $h(t, v_n(t), v_n'(t)) = f_n(t, v_n(t), v_n'(t))$ for $t \in [0,T]$ and consequently v_n is a solution of the problem (1.6), (1.7) with $\lambda = \lambda_2$ in (1.6) which completes the proof. $\square$

3. Main results and an example

Theorem 3.1. *Let* (H_1)–(H_3) *hold. Then for all* $\lambda > 0$, *the problem* (1.1), (1.2) *has a solution. Moreover, any solution of the problem* (1.1), (1.2) *is either a positive solution or a pseudodead core solution or a dead core solution.*

P r o o f. Fix $\lambda > 0$. For all $n \in \mathbb{N}'$, there exists a solution u_n of the problem (1.6), (1.7) by Lemma 2.4. Lemmas 2.1 and 2.2 show that the relations (2.1), (2.2) and (2.4) hold where S is a positive constant, $0 < \alpha_n < T$ and u_n' is increasing on $[0,T]$. In addition $\{u_n'\}_{n\in\mathbb{N}'}$ is equicontinuous on $[0,T]$ by Lemma 2.5. Hence we can assume without loss of generality that $\{u_n\}_{n\in\mathbb{N}'}$ is convergent in $C^1[0,T]$ and let $\lim\limits_{n\to\infty} u_n = u$. We have proved that u is a solution of the problem (1.1), (1.2).

In order to prove the second part of the assertion of our theorem, let u be a solution of the problem (1.1), (1.2). Then there exists a subsequence of $\{n\}_{n\in\mathbb{N}'}$, denoting for simplicity again by $\{n\}_{n\in\mathbb{N}'}$, such that $u = \lim_{n\to\infty} u_n$ in $C^1[0,T]$ where u_n is a solution of the problem (1.6), (1.7). Then $u \in C^1[0,T]$ and $u(0) = u(T) = A$. From the properties of u_n it follows that u' is nondecreasing on $[0,T]$, $\|u'\| \leqslant S$ and $0 \leqslant u(t) \leqslant A$ for $t \in [0,T]$. The next part of the proof is broken into two cases if $\min\{u(t)\colon 0 \leqslant t \leqslant T\}$ is positive or equals zero.

Case 1. Suppose $\min\{u(t)\colon 0 \leqslant t \leqslant T\} > 0$. Then there exist $\varepsilon > 0$ and $n_0 \in \mathbb{N}'$ such that

$$u_n(t) \geqslant \varepsilon \quad \text{for } t \in [0,T],\ n \geqslant n_0. \tag{3.1}$$

Without loss of generality we can assume that $n_0 > 1/\varepsilon$. Then $u_n > 1/n_0 \geqslant 1/n$ on $[0,T]$ for $n \geqslant n_0$ and (see (1.3))

$$(\phi(u_n'(t)))' = \lambda f_n(t, u_n(t), u_n'(t)) \geqslant \lambda\varphi(t) \tag{3.2}$$

for a.e. $t \in [0,T]$ and all $n \geqslant n_0$. Hence

$$-\phi(u_n'(t)) = \phi(u_n'(\alpha_n)) - \phi(u_n'(t)) \geqslant \lambda \int_t^{\alpha_n} \varphi(s)\,\mathrm{d}s, \tag{3.3}$$

and therefore

$$u_n'(t) \leqslant -\phi^{-1}\Bigl(\lambda \int_t^{\alpha_n} \varphi(s)\,\mathrm{d}s\Bigr) \quad \text{for } t \in [0,\alpha_n],\ n \geqslant n_0. \tag{3.4}$$

Analogous reasoning shows that

$$u_n'(t) \geqslant \phi^{-1}\Bigl(\lambda \int_{\alpha_n}^t \varphi(s)\,\mathrm{d}s\Bigr) \quad \text{for } t \in [\alpha_n,T],\ n \geqslant n_0. \tag{3.5}$$

Passing if necessary to a subsequence we can assume that $\{\alpha_n\}$ is convergent, $\lim_{n\to\infty} \alpha_n = \alpha$. Letting $n \to \infty$ in (3.4) and (3.5) gives

$$u'(t) \leqslant -\phi^{-1}\Bigl(\lambda \int_t^{\alpha} \varphi(s)\,\mathrm{d}s\Bigr) \quad \text{for } t \in [0,\alpha], \tag{3.6}$$

$$u'(t) \geqslant \phi^{-1}\Bigl(\lambda \int_{\alpha}^t \varphi(s)\,\mathrm{d}s\Bigr) \quad \text{for } t \in [\alpha,T]. \tag{3.7}$$

Thus α is the unique zero of u' and (1.2) shows that $\alpha \in (0,T)$. Therefore

$$\lim_{n\to\infty} f_n(t, u_n(t), u_n'(t)) = f(t, u(t), u'(t)) \quad \text{for a.e. } t \in [0,T]$$

and since $\lambda\int_0^T f_n(t,u_n(t),u_n'(t))\,\mathrm{d}t = \phi(u_n'(T)) - \phi(u_n'(0)) < 2\phi(S)$ for $n \in \mathbb{N}'$, we have from Fatou's theorem that $\int_0^T f(t,u(t),u'(t))\,\mathrm{d}t < 2\phi(S)/\lambda$ and $f(t,u(t),u'(t)) \in L_1[0,T]$. Next, from (3.6) we obtain

$$A - u(t) = -\int_0^t u'(s)\,\mathrm{d}s \geqslant \int_0^t \phi^{-1}\Big(\lambda\int_s^\alpha \varphi(v)\,\mathrm{d}v\Big)\,\mathrm{d}s$$

and therefore

$$u(t) \leqslant A - \int_0^t \phi^{-1}\Big(\lambda\int_s^\alpha \varphi(v)\,\mathrm{d}v\Big)\,\mathrm{d}s \quad \text{for } t \in [0,\alpha]. \tag{3.8}$$

Similarly, using (3.7),

$$u(t) \leqslant A - \int_\alpha^t \phi^{-1}\Big(\lambda\int_\alpha^s \varphi(v)\,\mathrm{d}v\Big)\,\mathrm{d}s \quad \text{for } t \in [\alpha,T]. \tag{3.9}$$

Choose $0 < t_1 \leqslant \frac{1}{2}\alpha < t_2 < \alpha$. Then, noting (2.4), (3.1), (3.6) and (3.8), there exist $\tau > 0$, $\nu > 0$ and $n_1 \geqslant n_0$ such that

$$\varepsilon \leqslant u_n(t) < A - \tau, \quad -S < u_n'(t) \leqslant -\nu \quad \text{for } t \in [t_1,t_2] \text{ and } n \geqslant n_1.$$

Since

$$\lim_{n\to\infty} f_n(t,u_n(t),u_n'(t)) = f(t,u(t),u'(t)) \tag{3.10}$$

for a.e. $t \in [t_1,t_2]$ and (see (1.5))

$$f_n(t,u_n(t),u_n'(t)) \leqslant \max\{p(s)\colon\ \varepsilon \leqslant s \leqslant A-\tau\}\max\{\omega(s)\colon\ \nu \leqslant s \leqslant S\},$$

letting $n \to \infty$ in

$$\phi(u_n'(t)) = \phi\Big(u_n'\Big(\frac{1}{2}\alpha\Big)\Big) + \lambda\int_{\alpha/2}^t f_n(s,u_n(s),u_n'(s))\,\mathrm{d}s \tag{3.11}$$

yields

$$\phi(u'(t)) = \phi\Big(u'\Big(\frac{1}{2}\alpha\Big)\Big) + \lambda\int_{\alpha/2}^t f(s,u(s),u'(s))\,\mathrm{d}s \tag{3.12}$$

for $t \in [t_1,t_2]$ by the Lebesque Dominated Convergence Theorem. Since $0 < t_1 < \frac{1}{2}\alpha < t_2 < \alpha$ are arbitrary, (3.12) holds for $t \in (0,\alpha)$. Essentially the same reasoning applied now on $\alpha < t_1 < \frac{1}{2}(T-\alpha) < t_2 < T$ shows that

$$\phi(u'(t)) = \phi\Big(u'\Big(\frac{1}{2}(T-\alpha)\Big)\Big) + \lambda\int_{(T-\alpha)/2}^t f(s,u(s),u'(s))\,\mathrm{d}s \tag{3.13}$$

for $t \in (\alpha, T)$. Since $u' \in C^1[0,T]$ and $f(t,u(t),u'(t)) \in L_1[0,T]$, we deduce from (3.12) and (3.13) that $\phi(u') \in \mathrm{AC}[0,T]$ and (1.1) is satisfied for a.e. $t \in [0,T]$. Hence u is a positive solution of the problem (1.1), (1.2).

Case 2. Suppose that $\min\{u(t)\colon 0 \leqslant t \leqslant T\} = 0$, $u(\alpha) = u(\beta) = 0$ for some $0 < \alpha \leqslant \beta < T$ and $u > 0$ on $[0,T] \setminus [\alpha,\beta]$. As u' is nondecreasing on $[0,T]$, we have $u' < 0$ on $[0,\alpha)$, $u' = 0$ on $[\alpha,\beta]$ and $u' > 0$ on $(\beta,T]$. Hence $u = 0$ on $[\alpha,\beta]$,

$$\lim_{n\to\infty} f_n(t,u_n(t),u_n'(t)) = f(t,u(t),u'(t)) \quad \text{for a.e. } t \in [0,T] \setminus (\alpha,\beta)$$

and since

$$\lambda \int_0^\alpha f_n(t,u_n(t),u_n'(t))\,\mathrm{d}s = \phi(u_n'(\alpha)) - \phi(u_n'(0)) < 2\phi(S),$$
$$\lambda \int_\beta^T f_n(t,u_n(t),u_n'(t))\,\mathrm{d}s = \phi(u_n'(T)) - \phi(u_n'(\beta)) < 2\phi(S),$$

then Fatou's theorem gives that the function $f(t,u(t),u'(t))$ is integrable on the intervals $[0,\alpha]$ and $[\beta,T]$. Let $t_2 \in (\frac{1}{2}\alpha, \alpha)$. Then there exists $n_2 \in \mathbb{N}'$, $n_2 \geqslant 2/u(t_2)$ such that $u_n(t) \geqslant u_n(t_2) \geqslant \frac{1}{2}u(t_2)$ $(\geqslant 1/n)$, $-S < u_n'(t) \leqslant u_n'(t_2) \leqslant \frac{1}{2}u'(t_2)$ for $t \in [0,t_2]$ and $n \geqslant n_2$. Thus (3.2) is satisfied for a.e. $t \in [0,t_2]$ and integrating (3.2) over $[t,t_2]$ gives $-u_n'(t) > \phi^{-1}(\lambda \int_t^{t_2} \varphi(s)\,\mathrm{d}s)$ for $t \in [0,t_2]$ and $n \geqslant n_2$ since $\phi(u_n'(t_2)) < 0$. Hence

$$A - \frac{1}{n} - u_n(t) = u_n(0) - u_n(t) = -\int_0^t u_n'(s)\,\mathrm{d}s > \int_0^t \phi^{-1}\Big(\lambda \int_s^{t_2} \varphi(v)\,\mathrm{d}v\Big)\,\mathrm{d}s$$

and

$$u_n(t) < A - \int_0^t \phi^{-1}\Big(\lambda \int_s^{t_2} \varphi(v)\,\mathrm{d}v\Big)\,\mathrm{d}s$$

for $t \in [0,t_2]$ and $n \geqslant n_2$. Choose $t_1 \in (0, \frac{1}{2}\alpha)$. Then $A - c > u_n(t) \geqslant \frac{1}{2}u(t_2)$ for $t \in [t_1,t_2]$ and $n \geqslant n_2$ where $c = \int_0^{t_1} \phi^{-1}(\lambda \int_s^{t_2} \varphi(v)\,\mathrm{d}v)\,\mathrm{d}s$ and consequently

$$f_n(t,u_n(t),u_n'(t)) \\ \leqslant \max\Big\{p(s)\colon \frac{1}{2}u(t_2) \leqslant s \leqslant A - c\Big\} \max\Big\{\omega(s)\colon \frac{1}{2}|u'(t_2)| \leqslant s \leqslant S\Big\}$$

for $t \in [t_1,t_2]$ and $n \geqslant n_2$. Letting $n \to \infty$ in (3.11) gives (3.12) for $t \in [t_1,t_2]$ by the Lebesgue Dominated Convergence Theorem. Since $0 < t_1 < t_2 < \alpha$ are arbitrary, (3.12) is true for $t \in (0,\alpha)$ and from $u' \in C^1[0,T]$ and $f(t,u(t),u'(t)) \in L_1[0,\alpha]$ it follows that $\phi(u') \in \mathrm{AC}[0,\alpha]$ and u satisfies (1.1) a.e. on $[0,\alpha]$. A similar procedure can be applied to the interval $(\beta,T]$. Summarizing we have shown that $u = 0$ on $[\alpha,\beta]$, $\phi(u') \in \mathrm{AC}[0,T]$ and (1.1) holds for a.e. $t \in [0,T] \setminus [\alpha,\beta]$. Consequently, if $\alpha < \beta$ then u is a dead core solution of the problem (1.1), (1.2) and if $\alpha = \beta$ then u is a pseudodead core solution of this problem. □

Corollary 3.2. *Let* (H_1)–(H_3) *hold. Then there exists* $\lambda_0 > 0$ *such that the problem* (1.1), (1.2), *has only positive solutions for each* $\lambda \in (0, \lambda_0]$ *in* (1.1).

P r o o f. Let $\lambda_0 > 0$ be taken from Lemma 2.6. Choose $\lambda \in (0, \lambda_0]$ and let u be a solution of the problem (1.1), (1.2) whose existence is guaranteed by Theorem 3.1. Then $u = \lim\limits_{n\to\infty} u_{k_n}$ in $C^1[0,T]$ where $\{k_n\}$ is a subsequence of $\{n\}_{n\in\mathbb{N}'}$ and u_{k_n} is a solution of (1.6), (1.7) with k_n instead of n. Due to Lemma 2.6, $\inf\{u_{k_n}(t)\colon t \in [0,T],\ n \in \mathbb{N}\} > 0$ and consequently $u > 0$ on $[0,T]$. Hence all solutions of the problem (1.1), (1.2) are positive for each $\lambda \in (0, \lambda_0]$ in (1.1). □

Corollary 3.3. *Let* (H_1)–(H_3) *hold. Then there exists* $\lambda_* > 0$ *such that the problem* (1.1), (1.2) *has only dead core solutions for each* $\lambda \in (\lambda_*, \infty)$ *in* (1.1). *Moreover, for every* $0 < c_1 < c_2 < T$ *the problem* (1.1), (1.2) *has for sufficiently large values of* λ *only dead core solutions* u *and* $u(t) = 0$ *for* $t \in [c_1, c_2]$.

P r o o f. Let $0 < c_1 < c_2 < T$ be arbitrary but fixed numbers. Lemma 2.7 guarantees the existence of some $\lambda_* > 0$ such that if $\lambda > \lambda_*$ then

$$\lim_{n\to\infty} u_n(c_j) = 0 \quad \text{for } j = 1, 2 \tag{3.14}$$

where u_n is a solution of the problem (1.6), (1.7). Let u be a solution of the problem (1.1), (1.2) with $\lambda > \lambda_*$ in (1.1). Then $u = \lim\limits_{n\to\infty} u_{k_n}$ in $C^1[0,T]$ where $\{k_n\}$ is a subsequence of $\{n\}_{n\in\mathbb{N}'}$. From (3.14) it follows that $u(c_j) = 0$ for $j = 1, 2$ and since we know that u' is nondecreasing on $[0,T]$, $u(t) = 0$ for $t \in [c_1, c_2]$. Hence for $\lambda > \lambda_*$ the problem (1.1), (1.2) has only dead core solutions and moreover these solutions vanish on $[c_1, c_2]$. □

Corollary 3.4. *Let* (H_1)–(H_3) *hold and let* $0 < \lambda_1 < \lambda_2$. *If* u *is a solution of the problem* (1.1), (1.2) *with* $\lambda = \lambda_1$ *in* (1.1) *then there exists a solution* v *of the problem* (1.1), (1.2) *with* $\lambda = \lambda_2$ *in* (1.1) *such that*

$$v(t) \leqslant u(t) \quad \textit{for } t \in [0,T]. \tag{3.15}$$

P r o o f. Let u be a solution of the problem (1.1), (1.2) with $\lambda = \lambda_1$ in (1.1). Then there exists a subsequence $\{k_n\}$ of $\{n\}_{n\in\mathbb{N}'}$ such that $u = \lim\limits_{n\to\infty} u_{k_n}$ in $C^1[0,T]$ where u_{k_n} is a solution of the problem (1.6), (1.7) with $\lambda = \lambda_1$ and k_n instead of n. Due to Lemma 2.8, there exists a solution v_{k_n} of the problem (1.6), (1.7) with $\lambda = \lambda_2$ and k_n instead of n such that $v_{k_n}(t) \leqslant u_{k_n}(t)$ for $t \in [0,T]$. Lemmas 2.1, 2.2 and 2.5 show that $\{v_{k_n}\}$ is bounded in $C^1[0,T]$ and $\{v'_{k_n}\}$ is equicontinuous on $[0,T]$. Going if necessary to a subsequence, we can assume that $\{v_{k_n}\}$ is convergent in $C^1[0,T]$ and let $v = \lim_{n\to\infty} v_{k_n}$. Then v is a solution of the problem (1.1), (1.2) with $\lambda = \lambda_2$ in (1.1) and from the inequality $v_{k_n} \leqslant u_{k_n}$ on $[0,T]$ for $n \in \mathbb{N}$, we obtain the inequality (3.15). $\square$

Example 3.5. Consider the differential equation

$$(|u'|^{p-2}u')' = \lambda\Big(\frac{\mathrm{e}^t}{u^\alpha(A-u)^\beta} + \frac{1}{|u'|^\gamma} + u^\mu + |u'|^\nu\Big) \tag{3.16}$$

where $A > 0$, $p > 1$, $\alpha, \beta \in (0,1)$, $\gamma, \mu \in (0,\infty)$ and $\nu \in (0,p)$. Equation (3.16) is the special case of (1.1) with $\phi(u) = |u|^{p-2}u$ satisfying (H_1) and $f(t,x,y) = \mathrm{e}^t/(x^\alpha(A-x)^\beta) + 1/|y|^\gamma + x^\mu + |y|^\nu$ for $t \in [0,T]$ and $(x,y) \in (0,A) \times (\mathbb{R} \setminus \{0\})$. The function f satisfies (H_2) and from the estimate

$$f(t,x,y) \leqslant \Big(\frac{\mathrm{e}^T}{x^\alpha(A-x)^\beta} + x^\mu + 1\Big)\Big(\frac{1}{|y|^\gamma} + |y|^\nu + 1\Big)$$

we see that in (H_3) we can put $p(x) = \mathrm{e}^T/(x^\alpha(A-x)^\beta) + x^\mu + 1$ for $x \in (0,A)$ and $\omega(y) = 1/y^\gamma + y^\nu + 1$ for $y \in (0,\infty)$. Then $p \in C^0(0,A) \cap L_1[0,A]$ and $\omega \in C^0(0,\infty)$ are positive and the inequality (for $v > 1$)

$$\begin{aligned}\int_0^v \frac{\phi^{-1}(s)}{\omega(\phi^{-1}(s))}\,\mathrm{d}s &> \int_1^v \frac{s^{1/(p-1)}}{s^{-\gamma/(p-1)} + s^{\nu/(p-1)} + 1}\,\mathrm{d}s\\ &= \int_1^v \frac{s^{(1+\gamma)/(p-1)}}{1 + s^{(\nu+\gamma)/(p-1)} + s^{\gamma/(p-1)}}\,\mathrm{d}s\\ &> \frac{1}{3}\int_1^v s^{(1-\nu)/(p-1)}\,\mathrm{d}s = \frac{1}{3}\frac{p-1}{p-\nu}\big(v^{(p-\nu)/(p-1)} - 1\big)\end{aligned}$$

implies that $\int_0^\infty \phi^{-1}(s)/\omega(\phi^{-1}(s))\,\mathrm{d}s = \infty$. Thus the functions p and ω satisfy the conditions in (H_3) where δ is sufficiently small. Since $f(t,x,y) > 1/(x^\alpha(A-x)^\beta) \geqslant ((\alpha+\beta)/A)^{\alpha+\beta}\alpha^{-\alpha}\beta^{-\beta}$ for $t \in [0,T]$ and $(x,y) \in (0,A) \times (\mathbb{R} \setminus \{0\})$, we can set $\varphi(t) = ((\alpha+\beta)/A)^{\alpha+\beta}\alpha^{-\alpha}\beta^{-\beta}$ in (H_3). Applying Theorem 3.1, problem (3.16), (1.2) has a solution for each $\lambda > 0$ and any solution of this problem is either a positive solution or a pseudodead core solution or a dead core solution. In addition, if λ is sufficiently small then all solutions of problem (3.16), (1.2) are positive solutions by Corollary 3.2 and if λ is sufficiently large then all solutions are dead core solutions by Corollary 3.3.

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