THE DENSITY OF INFINITELY DIFFERENTIABLE FUNCTIONS IN SOBOLEV SPACES WITH MIXED BOUNDARY CONDITIONS*

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Abstract. We present a detailed proof of the density of the set $C^\infty(\overline{\Omega}) \cap V$ in the space of test functions $V \subset H^1(\Omega)$ that vanish on some part of the boundary $\partial\Omega$ of a bounded domain Ω.

Keywords: density theorems, finite element method

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Let Ω be a nonempty bounded domain in $\mathbb{R}^N$ $(N = 2$ or $3)$. The symbol $C^\infty(\overline{\Omega})$ denotes the set containing all restrictions to $\overline{\Omega}$ of infinitely smooth functions defined on $\mathbb{R}^N$ (see [5, 1.2.1, 1.2.3 and 5.2.1]). Further, the symbol V denotes the set of test functions belonging to $H^1(\Omega)$ (for detailed definition of V see Theorem 1), where $H^1(\Omega) \equiv H^{1,2}(\Omega)$ is the Sobolev space in the notation defined in [5, 5.4.1].

In this paper we present a detailed proof of the density of $C^\infty(\overline{\Omega}) \cap V$ in V (see Theorem 1) the use of which is necessary when proving the convergence of the finite element method without any regularity assumptions on the exact solution u of a given variational problem, i.e., when proving the relation (which we present in $(*)$ for the case of a variational problem corresponding to a second order elliptic boundary value problem)

$$(*) \qquad \lim_{h\to 0} \|\tilde{u} - u_h\|_{1,\Omega_h} = 0,$$

where $\tilde{u}$ is the Calderon extension (see [7, p. 77]) of the exact solution and u_h is the approximate solution by the finite element method. Many authors consider the density of $C^\infty(\overline{\Omega}) \cap V$ in V to be evident and using it they do not give any reference

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(see, for example, [3, p. 135]). The assertion of Theorem 1 was given (in a little more general form) in [4]. However, the proof presented in [4] is so concise that almost no reader will have patience to read and understand it. For this reason we present in this paper a sufficiently detailed proof of this result which is of basic importance in the theory of convergence of the finite element method.

We also restrict ourselves to the class of domains $\tilde{\mathcal{C}}^{0,1} \subset \mathcal{C}^{0,1}$, where $\mathcal{C}^{0,1}$ denotes the set of domains with Lipschitz continuous boundary, in the following sense: If $\Omega \in \tilde{\mathcal{C}}^{0,1}$ then $\Omega \in \mathcal{C}^{0,1}$ and the boundary $\partial\Omega$ of Ω consists of a finite number of smooth parts which have a finite number of relative maxima and minima and inflexions and in the three-dimensional case also a finite number of saddle points. To consider such a class of domains is sufficient for applications.

Further, we will consider parts γ_i of $\partial\Omega$, on which homogeneous Dirichlet boundary condition will be prescribed, which satisfy the following condition. Let $\gamma_i \subset \partial\Omega$ be a relatively open set (i.e., open in the metric space $\partial\Omega$). We say that γ_i has a Lipschitz relative boundary $\partial\gamma_i$ (i.e., the boundary in the metric space $\partial\Omega$) and write $\gamma_i \in \mathrm{LRB}$ if either $\dim\Omega = 2$, or if in the case $\dim\Omega = 3$ it has the following property:

Let X_0 be an arbitrary point of $\partial\gamma_i$ and let $N(X_0)$ be a neighbourhood of X_0 such that $N(X_0) \cap \partial\Omega$ is expressed as a graph $x_3 = a(x_1, x_2)$. Let further G_i be the image of $\gamma_i \cap N(X_0)$ in the projection to the plane x_1, x_2 with the boundary ∂G_i. Then G_i has the same property as Ω, i.e., ∂G_i is locally representable as a graph of a Lipschitz function in one variable (obviously this definition is independent of the description of $\partial\Omega$). (In this case we use the notation $\gamma_i \in \mathrm{LRB}$ as mentioned above).

Theorem 1 (on the density of $C^\infty(\overline{\Omega}) \cap V$ in V). *Let $\Omega \in \tilde{\mathcal{C}}^{0,1}$ and let*

$$V = \{v \in H^1(\Omega)\colon\ v = 0 \ \text{ on } \Gamma_1, \ \text{ where } \Gamma_1 \subset \partial\Omega, \\ \operatorname{meas}_{N-1}\Gamma_1 > 0 \ \text{ with } N = 2 \ \text{ or } N = 3\}, \tag{1}$$

where Γ_1 consists of a finite number of relatively open parts in $\partial\Omega$, say $\gamma_1, \ldots, \gamma_m$ such that $\gamma_i \in \mathrm{LRB}$ $(i = 1, \ldots, m)$. Then the set $C^\infty(\overline{\Omega}) \cap V$ is dense in V.

The proof of Theorem 1 will be divided into four parts:

1) formulation of Lemma 2;
2) the idea of the proof of Lemma 2;
3) the proof of Theorem 1 by means of Lemma 2;
4) the detailed proof of Lemma 2.

Lemma 2. *Let Θ be an N-dimensional parallelepiped*

$$\Theta = \{X \in \mathbb{R}^N : X' \in \Delta,\ x_N \in (-\beta, 0)\}, \tag{2}$$

where $\Delta = (-\alpha, \alpha)^{N-1}$ and α, β are positive numbers.

Let $G \subset \overline{G} \subset \Delta$ be a domain such that $G \in \tilde{\mathcal{C}}^{0,1}$ and let $\overline{G} \subset \mathcal{U}$, where $\mathcal{U} \subset \Delta$ is an open set.

Let us denote $\Theta_1 = (-\alpha, \alpha)^{N-1} \times (-\beta, \beta)$. Further, let us denote $\Gamma = G \times \{0\}$ and let $\mathcal{K} \subset \Theta_1$ be a compact set, $\Gamma \subset \mathcal{K}$. (See Fig. 1 in the case of $N = 2$.)

Then there exists a compact set $\mathcal{K}_1 \subset \Theta_1$, $\mathcal{K}_1 \supset \mathcal{K}$ (where $\mathcal{K}_1$ depends only on $\mathcal{K}$) with the following property ($\mathcal{K}_1$ will be defined at the end of the idea of the proof of Lemma 2 (see the text following relation (12))):

Let $u \in H^1(\Theta)$ be an arbitrary function which is equal to zero on Γ (in the sense of traces) and $\operatorname{supp} u \subset \mathcal{K}$.

Then there exists a sequence $\{u_n\} \subset C^\infty(\overline{\Theta}_1)$ such that $\operatorname{supp} u_n \subset \mathcal{K}_1 \setminus \overline{\Gamma}$, where $\Gamma = G \times \{0\}$, and $u_n \to u$ in the space $H^1(\Theta)$.

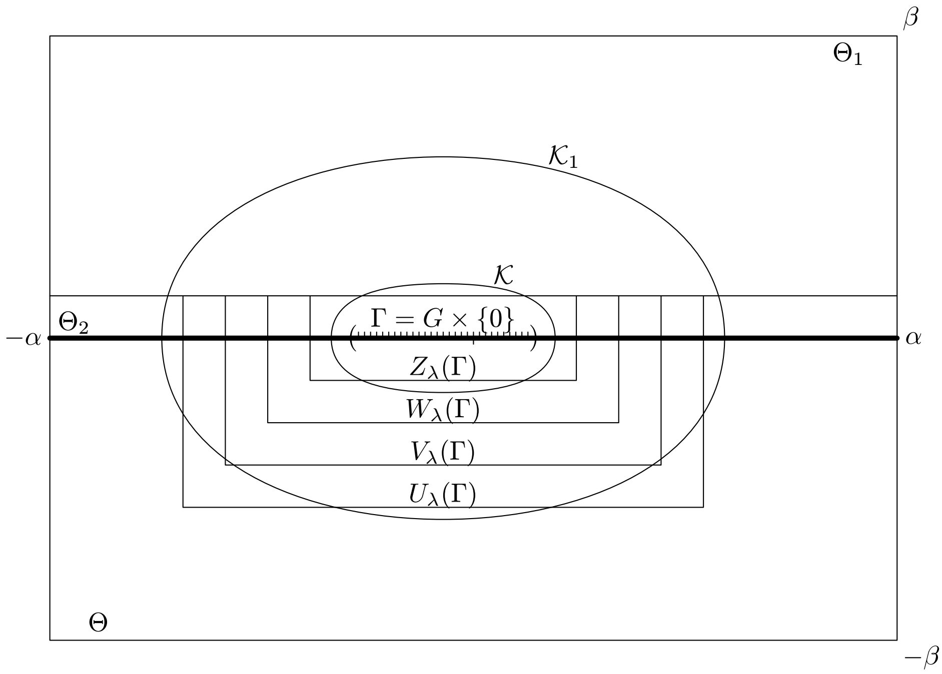

Figure 1. A two-dimensional example with G an interval.

The idea of the proof of Lemma 2. Due to the assumptions $\overline{G} \subset \Delta$, $\mathcal{K} \subset \Theta_1$ we have (see Fig. 1 for $N = 2$)

$$\operatorname{dist}(\mathcal{K}, \mathbb{R}^N \setminus \Theta_1) = \nu > 0. \tag{3}$$

Let us denote successively

$$\begin{aligned} U_\lambda(G) &= \{X' \in \Delta\colon \operatorname{dist}(X', G) < \lambda\},\\ V_\lambda(G) &= \Big\{X' \in \Delta\colon \operatorname{dist}(X', G) < \frac{3}{4}\lambda\Big\},\\ W_\lambda(G) &= \Big\{X' \in \Delta\colon \operatorname{dist}(X', G) < \frac{1}{2}\lambda\Big\},\\ Z_\lambda(G) &= \Big\{X' \in \Delta\colon \operatorname{dist}(X', G) < \frac{1}{4}\lambda\Big\} \end{aligned} \tag{4}$$

and, correspondingly (see Fig. 1 for $N = 2$),

$$\begin{aligned} U_\lambda(\Gamma) &= U_\lambda(G) \times \Big(-\lambda, \frac{1}{4}\lambda\Big),\\ V_\lambda(\Gamma) &= V_\lambda(G) \times \Big(-\frac{3}{4}\lambda, \frac{1}{4}\lambda\Big),\\ W_\lambda(\Gamma) &= W_\lambda(G) \times \Big(-\frac{1}{2}\lambda, \frac{1}{4}\lambda\Big),\\ Z_\lambda(\Gamma) &= Z_\lambda(G) \times \Big(-\frac{1}{4}\lambda, \frac{1}{4}\lambda\Big), \end{aligned} \tag{5}$$

where λ is supposed sufficiently small, thus satisfying

$$\lambda < \frac{1}{2}\nu. \tag{6}$$

Let us put $h = \frac{1}{4}\lambda$ and (see Fig. 2)

$$u_\lambda(X', x_N) := u(X', x_N - h), \quad [X', x_N] \in \Theta \tag{7}$$

with $u_\lambda \in H^1(\Theta_2)$, $\Theta_2 \equiv \Theta_2(h) = \Delta \times (-\beta, h)$,

$$v_\lambda(X) = \begin{cases} 0, & X \in W_\lambda(\Gamma),\\ u_\lambda(X), & X \in \Theta_2 \setminus \overline{W_\lambda(\Gamma)}, \end{cases} \tag{8}$$

$$w_\lambda(X) = (\omega_h * v_\lambda)(X). \tag{9}$$

In (9) we have used the brief notation for convolution

$$(\omega_h * u)(X) = \int_{\mathbb{R}^N} \omega_h(X - Y) u(Y) \,\mathrm{d}Y = \int_{\mathbb{R}^N} \omega_h(Y) u(X - Y) \,\mathrm{d}Y, \tag{10}$$

where the mollifier $\omega_h(Z)$ is defined by the relations

$$\omega_h(Z) = \begin{cases} \varkappa h^{-N} \exp\Big(\dfrac{\|Z\|^2}{\|Z\|^2 - h^2}\Big) & \text{for } \|Z\| < h, \\ 0 & \text{for } \|Z\| \geqslant h; \end{cases} \tag{11}$$

the symbol $\|\cdot\|$ denotes the Euclidean norm and the constant $\varkappa$ is defined by

$$\int_{\mathbb{R}^N} \omega_1(Z)\,\mathrm{d}Z = \int_{\mathbb{R}^N} \omega_h(Z)\,\mathrm{d}Z = 1, \tag{12}$$

from which we obtain

$$\varkappa = \Big(\int_{\mathbb{R}^N} \exp\Big(\frac{\|Z\|^2}{\|Z\|^2 - 1}\Big)\,\mathrm{d}Z\Big)^{-1}.$$

The first equality (12) follows from the fact that

$$\|Z\|^2/(\|Z\|^2 - h^2) = \|Z/h\|^2/(\|Z/h\|^2 - 1)$$

and from the substitution $Z = hX = [hx_1, \dots, hx_N]$ which implies $\mathrm{d}Z = h^N\,\mathrm{d}X$. Thus the convolution (10) is well-defined for $u \in L_2\Omega$ and its restriction onto $\overline{\Omega}$ belongs to the space $C^\infty(\overline{\Omega})$.

We see immediately that $u_\lambda \in H^1(\Theta)$. It follows from (4)–(6) that $\operatorname{supp} w_\lambda \subset \mathcal{K}_1$, where $\mathcal{K}_1 = \{X \in \mathbb{R}^N \colon \operatorname{dist}(X, \mathcal{K}) < \frac{1}{2}\nu\} \subset \Theta_1$ depends only on $\mathcal{K}$ (because, according to (3), ν depends on $\mathcal{K}$), and that $w_\lambda(X) = 0$ for $X \in Z_\lambda(\Gamma)$ (see Fig. 2). In what follows we show that $\|u - w_\lambda\|_{1,\Theta} \to 0$ for $\lambda \to 0$, which proves Lemma 2. $\square$

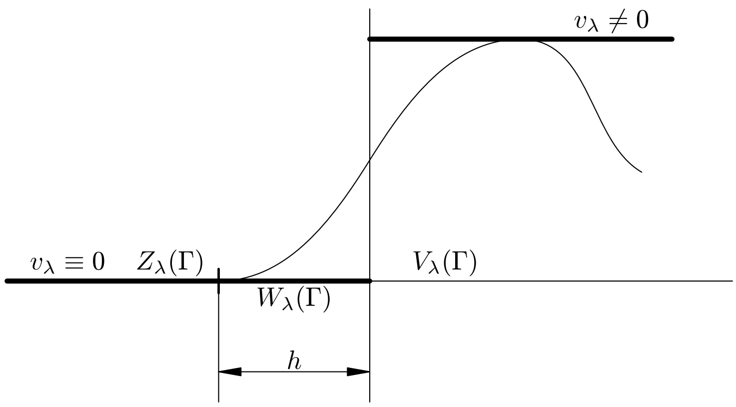

Figure 2. Concerning relations (7)–(12).

P r o o f of Theorem 1 by means of Lemma 2. Let $u \in V$ be an arbitrary but fixed function. We show that there exists such a sequence $\{u_n\} \subset C^\infty(\overline{\Omega})$ that

$$u_n = 0 \quad \text{on an } N\text{-dimensional neighbourhood of } \Gamma_1, \tag{13}$$

$$u_n \to u \quad \text{in the space } H^1(\Omega). \tag{14}$$

For better clarity of exposition we restrict ourselves to the two-dimensional case, i.e., $N = 2$.

The domain Ω has a Lipschitz continuous boundary and hence for any $X \in \partial\Omega$ there exists a (local) Cartesian coordinate system (x_1, x_2) and a Lipschitz function $a(x_1)$ with the domain of definition $\Delta = (-\alpha, \alpha) \subset \mathbb{R}^1$ such that[1]

$$\mathbb{U} = \{[x_1, x_2]\colon\ x_1 \in \Delta,\ a(x_1) - \beta < x_2 < a(x_1)\} \subset \Omega,$$
$$\mathbb{V} = \{[x_1, x_2]\colon\ x_1 \in \Delta,\ a(x_1) < x_2 < a(x_1) + \beta\} \subset \mathbb{R}^2 \setminus \overline{\Omega},$$

where $\alpha > 0$, $\beta > 0$ are suitable constants. Let us denote

$$\mathbb{Z} = \mathbb{U} \cup \mathbb{V} \cup \{[x_1, x_2]\colon\ x_1 \in \Delta,\ x_2 = a(x_1)\}.$$

Owing to the compactness of $\partial\Omega$ we can cover $\partial\Omega$ by a finite number of such domains $\mathbb{Z}_1, \ldots, \mathbb{Z}_m$. (The local Cartesian coordinate system x_1, x_2 and the function $a(x_1)$ corresponding to $\mathbb{Z}_r$ will be now denoted by x_1^r, x_2^r and $a_r(x_1^r)$, respectively.) Further, we can find a domain $\mathbb{Z}_0$ such that $\overline{\mathbb{Z}}_0 \subset \Omega$ ($\mathbb{Z}_0$ is considered in the global Cartesian coordinate system x_1, x_2) and

$$\overline{\Omega} \subset \bigcup_{r=0}^{m} \mathbb{Z}_r.$$

Owing to the compactness of $\overline{\Omega}$ we can construct a partition of unity, i.e., a system of functions $\varphi_r \in C_0^\infty(\mathbb{Z}_r)$ $(r = 0, 1, \ldots, m)$ which for $X \in \overline{\Omega}$ (the points X are considered in the global system x_1, x_2) satisfy

$$0 \leqslant \varphi_r(X) \leqslant 1, \qquad \sum_{r=0}^{m} \varphi_r(X) = 1.$$

We can transform $\mathbb{U}_r$ $(r = 1, \ldots, m)$ to the parallelepiped

$$\Theta = (-\alpha, \alpha)^{N-1} \times (-\beta, 0) \quad (\text{in our case } N = 2)$$

[1] If $N > 2$ then we substitute $[x_1, x_2]$ by $[X', x_N] = [x_1, \ldots, x_{N-1}, x_N]$ and the function $a(x_1)$ by $a(X')$ with the domain of definition $\Delta = (-\alpha, \alpha)^{N-1} \subset \mathbb{R}^{N-1}$.

by means of the lipschitzian mapping

$$\mathcal{T}_r\colon\ \xi_1 = x_1^r,\ \xi_2 = x_2^r - a_r(x_1^r).$$

This transformation maps continuously $H^1(\mathbb{U}_r)$ $(1 \leqslant r \leqslant m)$ onto $H^1(\Theta)$ (see [7, Lemma 2.3.2 on p. 66]) and $\operatorname{supp}\varphi_r$ onto a compact set $\mathcal{K}_r \subset \Theta_1 = \mathcal{T}_r(\mathbb{Z}_r)$, $\mathcal{K}_r = \mathcal{T}_r(\operatorname{supp}\varphi_r)$.

Let $G^r = \mathcal{T}_r(\Gamma_1 \cap \operatorname{supp}\varphi_r)$. We have $G^r \subset \overline{G^r} \subset \Delta$. Thus $\mathcal{K}_r \subset \Theta_1$ is the compact set $\mathcal{K}$ from Lemma 2. Let $\mathcal{K}_{r,1} \subset \Theta_1$ $(\mathcal{K}_{r,1} \supset \mathcal{K}_r)$ be the compact set $\mathcal{K}_1$ from Lemma 2.

Hence, according to Lemma 2, we can approach $\mathcal{T}_r(\varphi_r u)$ by a sequence $\{v_{n,r}\} \subset C^\infty(\overline{\Theta}_1)$, $\operatorname{supp} v_{n,r} \subset \mathcal{K}_{r,1} \setminus \overline{\Gamma^r}$ $(\Gamma^r = G^r \times \{0\})$.

The main step of the proof consists in the following argument: The functions $\tilde{u}_{n,r} = \mathcal{T}_r^{-1}(v_{n,r})$ belong to $H^1(C)$ ($C \subset \mathbb{R}^N$ is an N-dimensional cube which contains $\overline{\Omega}$),

$$\tilde{u}_{n,r} = 0 \quad \text{in a neighbourhood of } \overline{\Gamma}_1 \cap \operatorname{supp}\varphi_r \tag{15}$$

and $\tilde{u}_{n,r} \to u\varphi_r$ in $H^1(\Omega)$.

Applying the mollifier ω_h to $\tilde{u}_{n,r}$ we can replace $\tilde{u}_{n,r}$ by $u_{n,r} \subset C^\infty(\overline{\Omega})$ having the same property (15) and converging to $u\varphi_r$ in $H^1(\Omega)$. Finally, we approach $u\varphi_0$ by the sequence $\{u_{n,0}\} \subset C_0^\infty(\Omega)$ and write $u_n = \sum\limits_{r=0}^{m} u_{n,r}$, which proves the theorem.

□

The detailed p r o o f of Lemma 2. Let us denote $\sigma_\lambda = \Theta \setminus \overline{V_\lambda(\Gamma)}$, $P_\lambda = V_\lambda(\Gamma) \cap \Theta$ and recall that $\Theta_2 = \Delta \times (-\beta, h)$. The proof proceeds as follows: Due to the fact that $\overline{\Theta} = \overline{\sigma}_\lambda \cup \overline{P}_\lambda$, we can write

$$\begin{aligned} \|w_\lambda - u\|_{1,\Theta} &\leqslant \|u - u_\lambda\|_{1,\Theta} + \|u_\lambda - w_\lambda\|_{1,\Theta} \\ &\leqslant \|u - u_\lambda\|_{1,\Theta} + \|u_\lambda - w_\lambda\|_{1,\sigma_\lambda} + \|u_\lambda - w_\lambda\|_{1,P_\lambda} \\ &\leqslant \|u - u_\lambda\|_{1,\Theta} + \|u_\lambda - w_\lambda\|_{1,\sigma_\lambda} + \|u_\lambda\|_{1,P_\lambda} + \|w_\lambda\|_{1,P_\lambda} \end{aligned} \tag{16}$$

and prove successively (in parts A–D) that all terms on the right-hand side of (16) tend to zero with $\lambda \to 0$. The main difficulty is to prove that $\|w_\lambda\|_{1,P_\lambda} \to 0$ (if $\lambda \to 0$), in particular to prove

$$\Big\|\frac{\partial w_\lambda}{\partial x_i}\Big\|_{0,P_\lambda} \to 0 \quad \text{if } \lambda \to 0 \qquad (i = 1, \ldots, N).$$

A. First we prove

$$\lim_{\lambda \to 0} \|u - u_\lambda\|_{1,\Theta} = 0. \tag{17}$$

Let us denote

$$\overline{h} = (0, \ldots, 0, h).$$

As

$$\|u - u_\lambda\|_{0,\Theta} = \sqrt{\int_\Theta [u(X) - u(X - \overline{h})]^2 \,\mathrm{d}X},$$

we have, according to [5, Theorem 2.4.2] (the mean continuity theorem for L_2-functions),

$$\lim_{\lambda \to 0} \|u - u_\lambda\|_{0,\Theta} = 0. \tag{18}$$

As $D^\alpha u \in L_2(\Theta)$, $D^\alpha u_\lambda \in L_2(\Theta)$ ($|\alpha| = 1$) and as

$$\|D^\alpha u - D^\alpha u_\lambda\|_{0,\Theta} = \sqrt{\int_\Theta [(D^\alpha u)(X) - (D^\alpha u)(X - \overline{h})]^2 \,\mathrm{d}X},$$

we have again, according to [5, Theorem 2.4.2],

$$\lim_{\lambda \to 0} \|D^\alpha u - D^\alpha u_\lambda\|_{0,\Theta} = 0. \tag{19}$$

Relations (18) and (19) together give relation (17).

B. Now we prove

$$\lim_{\lambda \to 0} \|u_\lambda\|_{1,P_\lambda} = 0. \tag{20}$$

We have mentioned at the end of the *idea of the proof of Lemma* 2 that

$$u_\lambda \in H^1(\Theta). \tag{21}$$

As (see Fig. 3)

$$P_\lambda = V_\lambda(\Gamma) \cap \Theta, \tag{22}$$

we have, due to (21),

$$u_\lambda \in H^1(P_\lambda). \tag{23}$$

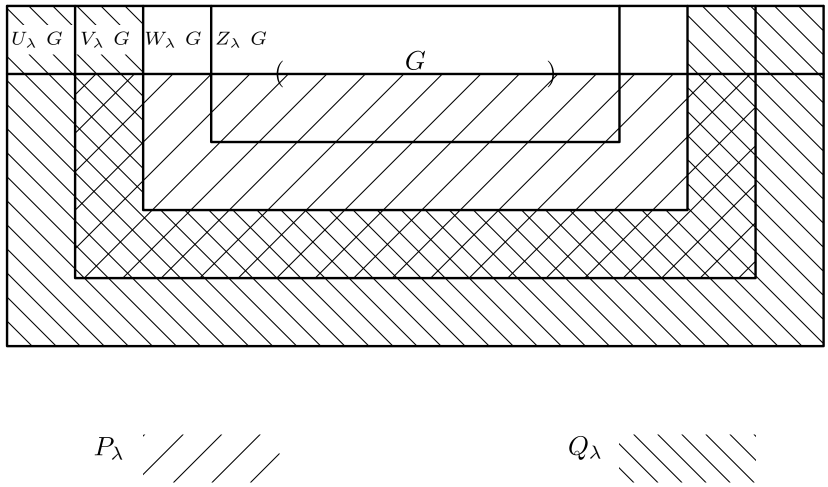

Figure 3. Domains P_λ and Q_λ.

By the definition of $V_\lambda(\Gamma)$ we have

$$\lim_{\lambda\to 0}(\operatorname{meas}_N V_\lambda(\Gamma)) = 0; \tag{24}$$

hence, taking into account (22), we obtain from (24)

$$\lim_{\lambda\to 0} \operatorname{meas}_N P_\lambda = 0. \tag{25}$$

Relation (23) yields

$$u_\lambda \in L_2(P_\lambda), \quad D^\alpha u_\lambda \in L_2(P_\lambda) \quad (|\alpha| = 1). \tag{26}$$

Relations (25) and (26) imply, according to the theorem on the absolute continuity of an integral, relation (20).

C. Our task in this subsection is to prove that

$$\lim_{\lambda\to 0} \|u_\lambda - w_\lambda\|_{1,\sigma_\lambda} \to 0, \tag{27}$$

where u_λ and w_λ are defined in (7)–(9). As

$$\sigma_\lambda = \Theta \setminus \overline{V_\lambda(\Gamma)} \tag{28}$$

we have

$$\Theta_2 \setminus \overline{W_\lambda(\Gamma)} \supset \sigma_\lambda. \tag{29}$$

As $u_\lambda \in H^1(\Theta)$ and $\operatorname{supp} u \subset \mathcal{K}$ we have (by (7)) $u_\lambda \in H^1(\Theta_2)$. Hence $v_\lambda \in H^1(\Theta_2 \setminus \overline{W_\lambda(\Gamma)})$ and (by (29)) $v_\lambda \in H^1(\sigma_\lambda)$. By (3), (7), (8) and the assumption $\operatorname{supp} u \subset \mathcal{K}$ we have $\operatorname{dist}(\operatorname{supp} v_\lambda, \mathbb{R}^N \setminus \Theta_1) \geqslant h$. Thus we can repeat with another notation the proof of [10, Lemma 3.6] and prove the implication

$$X \in \sigma_\lambda \cap \operatorname{supp} v_\lambda \Rightarrow \frac{\partial}{\partial x_i}(\omega_h * v_\lambda) = \omega_h * \frac{\partial v_\lambda}{\partial x_i}. \tag{30}$$

(Using the notation of [10, Chapter 3] we have $\omega_h * u = R_h u$.) By [10, Theorem 3.7] implication (30) yields (together with the preceding text)

$$\begin{aligned}\|u_\lambda - w_\lambda\|_{1,\sigma_\lambda} &= \sqrt{\int_{\sigma_\lambda \cap \operatorname{supp} v_\lambda} \sum_{|\alpha| \leqslant 1} [D^\alpha u_\lambda - D^\alpha(\omega_h * v_\lambda)]^2 \,\mathrm{d}X} \\ &= \sqrt{\int_{\sigma_\lambda \cap \operatorname{supp} v_\lambda} \sum_{|\alpha| \leqslant 1} [D^\alpha u_\lambda - \omega_h * D^\alpha u_\lambda]^2 \,\mathrm{d}X} \to 0 \quad \text{if } \lambda \to 0.\end{aligned}$$

This proves relation (27).

D. Our task is now to estimate the function w_λ as an element of $H^1(P_\lambda)$ (for the definition of P_λ see (22)). To this end, let us denote $Q_\lambda = U_\lambda(\Gamma) \setminus \overline{W_\lambda(\Gamma)}$. Owing to the choice of h we obtain for $X \in P_\lambda$ (cf. Fig. 3, properties of ω_h and definitions of u_λ and v_λ; we must realize that $h \to 0$—in Fig. 1 the variable quantity h is relatively large—and that $\mathcal{K}$ and $\mathcal{K}_1$ are fixed)

$$\begin{aligned}w_\lambda(X) &= \int_{\mathbb{R}^N} \omega_h(X - Y) v_\lambda(Y) \,\mathrm{d}Y \\ &= \int_{\mathbb{R}^N \setminus W_\lambda(\Gamma)} \omega_h(X - Y) v_\lambda(Y) \,\mathrm{d}Y \\ &= \int_{Q_\lambda} \omega_h(X - Y) u_\lambda(Y) \,\mathrm{d}Y \quad (X \in P_\lambda)\end{aligned} \tag{31}$$

(if $X \in P_\lambda$ then $\|X - Y\| < h$ only for the points $Y \in Q_\lambda$) and similarly (again for $X \in P_\lambda$)

$$\begin{aligned}\psi_i(X) = \frac{\partial w_\lambda}{\partial x_i}(X) &= \int_{Q_\lambda} \frac{\partial \omega_h}{\partial x_i}(X - Y) u_\lambda(Y) \,\mathrm{d}Y \\ &= -\int_{Q_\lambda} \frac{\partial \omega_h}{\partial y_i}(X - Y) u_\lambda(Y) \,\mathrm{d}Y \quad (X \in P_\lambda).\end{aligned} \tag{32}$$

Now we apply [5, Theorem 2.5.3] (see also [10, Theorem 3.7]) which asserts that

$$\lim_{\varepsilon \to 0+} \|D^\alpha(R_\varepsilon u) - D^\alpha u\|_{L_2(\Omega^*)} = 0 \quad (\overline{\Omega}^* \subset \Omega).$$

To this end let us denote by $\tilde{u}_\lambda$ the extension of the function u_λ by zero onto the domain Θ_1. In the notation of [5, Theorem 2.5.3] (or [10, Theorem 3.7]) we have

$$\Omega = \Theta_1, \quad \Omega^* = P_\lambda. \tag{33}$$

The extended function $\tilde{u}_\lambda$ satisfies

$$\tilde{u}_\lambda \in L_2(\Theta_1).$$

An application of [5, Theorem 2.5.3] (or [10, Theorem 3.7]) with $|\alpha| = 0$ to the functions

$$u = \tilde{u}_\lambda, \quad R_\varepsilon u = w_\lambda$$

and domains (33) yields

$$\lim_{\lambda \to 0} \|w_\lambda - u_\lambda\|_{0,P_\lambda} = 0. \tag{34}$$

Let us combine (34) with the inequality

$$\|w_\lambda\|_{0,P_\lambda} \leqslant \|u_\lambda\|_{0,P_\lambda} + \|w_\lambda - u_\lambda\|_{0,P_\lambda}$$

and relation (20). Then we obtain

$$\lim_{\lambda \to 0} \|w_\lambda\|_{0,P_\lambda} = 0. \tag{35}$$

The same device cannot be used in the case of the functions ψ_i (given by (32)) because $\tilde{u}_\lambda \notin H^1(\Theta_1)$ and hence

$$\frac{\partial \tilde{u}_\lambda}{\partial x_i} \notin L_2(\Theta_1).$$

Another approach must be used. Let us consider now the $L_2(P_\lambda)$-norm of ψ_i. We shall distinguish two cases: $i = N$ (considered in D1) and $i < N$ (see D2.1 and D2.2).

D1. Let $i = N$. As $Q_\lambda \in \tilde{\mathcal{C}}^{0,1}$ we can use the Green-Gauss-Ostrogradskij formula and obtain from (32)—see Fig. 3 ($X \in P_\lambda$) (note that $U_\lambda(G)$ and $W_\lambda(G)$ are $(N-1)$-dimensional sets)

$$\begin{aligned} \psi_N(X) = {} & \int_{Q_\lambda} \omega_h(X - Y) \frac{\partial u_\lambda}{\partial y_N}(Y)\,\mathrm{d}Y \\ & + \int_{U_\lambda(G)} \omega_h(X - [Y', -\lambda]) u_\lambda(Y', -\lambda)\,\mathrm{d}Y' \\ & - \int_{W_\lambda(G)} \omega_h\Big(X - \Big[Y', -\frac{1}{2}\lambda\Big]\Big) u_\lambda\Big(Y', -\frac{1}{2}\lambda\Big)\,\mathrm{d}Y' \\ & - \int_{U_\lambda(G) \setminus W_\lambda(G)} \omega_h\Big(X - \Big[Y', \frac{1}{4}\lambda\Big]\Big) u_\lambda\Big(Y', \frac{1}{4}\lambda\Big)\,\mathrm{d}Y'. \end{aligned} \tag{36}$$

Let us note that we have used the Green-Gauss-Ostrogradskij theorem in the form

$$\int_{Q_\lambda} \frac{\partial \omega_h}{\partial y_N}(X-Y)u_\lambda(Y)\,\mathrm{d}Y = \int_{\partial Q_\lambda} \omega_h(X-Y)u_\lambda(Y)n_N(Y)\,\mathrm{d}\sigma - \int_{Q_\lambda} \omega_h(X-Y)\frac{\partial u_\lambda}{\partial y_N}(Y)\,\mathrm{d}Y$$

where $n_N = -1$ on the base of Q_λ, $n_N = 1$ on the parts of ∂Q_λ parallel with the base of Q_λ and $n_N = 0$ otherwise (see Fig. 3).

First we shall prove that the first integral on the right-hand side of (36) tends to zero in $L_2(P_\lambda)$; this means that we shall prove

$$\lim_{\lambda\to 0}\int_{P_\lambda}\Big(\int_{Q_\lambda}\omega_h(X-Y)\frac{\partial u_\lambda}{\partial y_N}(Y)\,\mathrm{d}Y\Big)^2\,\mathrm{d}X = 0. \tag{37}$$

We have

$$\begin{aligned}\int_{P_\lambda}\Big(\int_{Q_\lambda}\omega_h(X-Y)\frac{\partial u_\lambda}{\partial y_N}(Y)\,\mathrm{d}Y\Big)^2\,\mathrm{d}X \\ = \varkappa\int_{P_\lambda}\Big\{\int_{Q_\lambda} h^{-N}\exp\Big(\frac{\|X-Y\|^2}{\|X-Y\|^2-h^2}\Big)\frac{\partial u_\lambda}{\partial y_N}(Y)\,\mathrm{d}Y\Big\}^2\,\mathrm{d}X.\end{aligned} \tag{38}$$

For the sake of brevity, let us denote

$$E_h(X,Y) := \exp\Big(\frac{\|X-Y\|^2}{\|X-Y\|^2-h^2}\Big), \quad F(Y) := \frac{\partial u_\lambda}{\partial y_N}(Y). \tag{39}$$

Using this notation we further denote

$$F_h(X) = \frac{1}{h^N}\int_{Q_\lambda} E_h(X,Y)F(Y)\,\mathrm{d}Y. \tag{40}$$

By (39), (40) we can write

$$|F_h(X)| \leqslant \frac{1}{h^N}\int_{Q_\lambda}\sqrt{E_h(X,Y)}\sqrt{E_h(X,Y)}|F(Y)|\,\mathrm{d}Y. \tag{41}$$

Let us square inequality (41) and to the resulting right-hand side let us apply the Schwarz inequality; we thus obtain

$$|F_h(X)|^2 \leqslant \frac{1}{h^N}\int_{Q_\lambda} E_h(X,Y)\,\mathrm{d}Y\cdot\frac{1}{h^N}\int_{Q_\lambda} E_h(X,Y)|F(Y)|^2\,\mathrm{d}Y. \tag{42}$$

We have (see [8, p. 218, relation (168)], where the case $N = 2$ is considered)

$$\int_{Q_\lambda} E_h(X, Y)\,\mathrm{d}Y = Ch^N.$$

Hence

$$\frac{1}{h^N}\int_{Q_\lambda} E_h(X, Y)\,\mathrm{d}Y = C. \tag{43}$$

Let us use (43) and let us integrate inequality (42) over P_λ with respect to X. We obtain

$$\int_{P_\lambda} |F_h(X)|^2\,\mathrm{d}X \leqslant \frac{C}{h^N}\int_{P_\lambda}\left\{\int_{Q_\lambda} E_h(X, Y)|F(Y)|^2\,\mathrm{d}Y\right\}\mathrm{d}X. \tag{44}$$

Now we use the Fubini theorem on the right-hand side of (44) and then relation (43) (where we write now P_λ instead of Q_λ). We conclude (if we use also notation $(39)_2$):

$$\begin{aligned}\int_{P_\lambda} |F_h(X)|^2\,\mathrm{d}X &\leqslant C\int_{Q_\lambda}\left\{|F(Y)|^2\frac{1}{h^N}\int_{P_\lambda} E_h(X, Y)\,\mathrm{d}X\right\}\mathrm{d}Y \\ &= C^2\int_{Q_\lambda}|F(Y)|^2\,\mathrm{d}Y = C^2\int_{Q_\lambda}\Big(\frac{\partial u_\lambda}{\partial y_N}(Y)\Big)^2\,\mathrm{d}Y.\end{aligned} \tag{45}$$

Using the relation

$$\lim_{\lambda\to 0}(\operatorname{meas}_N Q_\lambda) = 0$$

and the theorem on the absolute continuity of the Lebesgue integral we see that the right-hand side of (45) tends to zero if $\lambda \to 0$. This proves (37).

Now we prove that the second and the fourth integrals on the right-hand side of (36) are equal to zero for $X \in P_\lambda$. We recall that

$$h = \frac{1}{4}\lambda.$$

As to the second integral, for $X \in P_\lambda = V_\lambda(\Gamma)\cap\Theta$ we have

$$\begin{aligned}\omega_h(X - [Y', -\lambda]) &= \omega_h([X', x_N] - [Y', -\lambda]) \\ &= \omega_h(X' - Y', x_N + \lambda) \\ &= \omega_h(X' - Y', x_N + 4h) = \omega_h(Z),\end{aligned} \tag{46}$$

where we set

$$Z = [X' - Y', x_N + 4h].$$

We further have

$$\|Z\| = \sqrt{(X' - Y')^2 + (x_N + 4h)^2}, \tag{47}$$

where we set for the sake of brevity

$$(X' - Y')^2 = (x_1 - y_1)^2 + \ldots + (x_{N-1} - y_{N-1})^2.$$

The most inconvenient case is

$$(X' - Y')^2 = 0.$$

Since in the case $X \in P_\lambda = V_\lambda(\Gamma) \cap \Theta$ we have

$$x_N \in (-3h, 0) \tag{48}$$

we obtain from (47)

$$\|Z\| \geqslant h, \tag{49}$$

which implies by (46) (and by the fact that $\omega_h(Z) = 0$ for $\|Z\| \geqslant h$) that

$$\omega_h(X - [Y', -\lambda]) = 0.$$

This proves that the second integral on the right-hand side of (36) is equal to zero for $X \in P_\lambda$.

As to the fourth integral on the right-hand side of (36), we have

$$\omega_h\Big(X - \Big[Y', \frac{1}{4}\lambda\Big]\Big) = \omega_h(X' - Y', x_N - h).$$

As (48) holds we obtain again (49) with $Z = [X' - Y', x_N - h]$. This proves that the fourth integral on the right-hand side of (36) is equal to zero for $X \in P_\lambda$.

Let us consider the third integral on the right-hand side of (36) (in the second equality we use (7)):

$$\begin{aligned}\psi_{N,3}(X) &= \int_{W_\lambda(G)} \omega_h\Big(X - \Big[Y', -\frac{1}{2}\lambda\Big]\Big) u_\lambda\Big(Y', -\frac{1}{2}\lambda\Big)\, \mathrm{d}Y' \\ &= \int_{W_\lambda(G)} \omega_h\Big(X' - Y', x_N + \frac{1}{2}\lambda\Big) u\Big(Y', -\frac{3}{4}\lambda\Big)\, \mathrm{d}Y'.\end{aligned}$$

As

$$\omega_h\big(X' - Y', x_N + \frac{1}{2}\lambda\big) = 0 \quad \forall\, \|X' - Y'\| \geqslant h$$

we can write

$$\psi_{N,3}(X) = \int_{\|X'-Y'\|<h} \omega_h\Big(X'-Y', x_N + \frac{1}{2}\lambda\Big) u\Big(Y', -\frac{3}{4}\lambda\Big)\,\mathrm{d}Y'.$$

From this identity we obtain by the Schwarz inequality (and by extending the domain of integration in the case of the first integral)

$$\begin{aligned}
(50)\quad |\psi_{N,3}(X)|^2 &= \Big(\int_{\|X'-Y'\|<h} \omega_h\Big(X'-Y', x_N + \frac{1}{2}\lambda\Big) u\Big(Y', -\frac{3}{4}\lambda\Big)\,\mathrm{d}Y'\Big)^2 \\
&\leqslant \int_{\|X'-Y'\|<h} \omega_h^2\Big(X'-Y', x_N + \frac{1}{2}\lambda\Big)\,\mathrm{d}Y' \\
&\quad\times \int_{\|X'-Y'\|<h} u^2\Big(Y', -\frac{3}{4}\lambda\Big)\,\mathrm{d}Y' \\
&\leqslant \int_{\mathbb{R}^{N-1}} \omega_h^2\Big(X'-Y', x_N + \frac{1}{2}\lambda\Big)\,\mathrm{d}Y' \\
&\quad\times \int_{\|Y'-X'\|<h} u^2\Big(Y', -\frac{3}{4}\lambda\Big)\,\mathrm{d}Y' \\
&= \int_{\mathbb{R}^{N-1}} \omega_h^2\Big(Z', x_N + \frac{1}{2}\lambda\Big)\,\mathrm{d}Z' \int_{\|Y'-X'\|<h} u^2\Big(Y', -\frac{3}{4}\lambda\Big)\,\mathrm{d}Y'.
\end{aligned}$$

(We have used also the fact that $\omega_h(X'-Y', x_N+\frac{1}{2}\lambda) = \omega_h(Y'-X', x_N+\frac{1}{2}\lambda)$.) Integrating (50) over P_λ with respect to X (and extending the domain of integration in the case of the first integral) we obtain

$$\begin{aligned}
(51)\quad \|\psi_{N,3}\|^2_{0,P_\lambda} &= \int_{-3\lambda/4}^{0} \Big(\int_{\mathbb{R}^{N-1}} \omega_h^2\Big(Z', x_N + \frac{1}{2}\lambda\Big)\,\mathrm{d}Z'\Big)\,\mathrm{d}x_N \\
&\quad\times \int_{V_\lambda(G)} \Big(\int_{\|Y'-X'\|<h} u^2\Big(Y', -\frac{3}{4}\lambda\Big)\,\mathrm{d}Y'\Big)\,\mathrm{d}X' \\
&\leqslant \Big(\int_{\mathbb{R}^N} \omega_h^2(Z)\,\mathrm{d}Z\Big) \int_{V_\lambda(G)} \Big(\int_{\|Y'-X'\|<h} u^2\Big(Y', -\frac{3}{4}\lambda\Big)\,\mathrm{d}Y'\Big)\,\mathrm{d}X'.
\end{aligned}$$

Now we prove that

$$(52)\qquad \int_{\mathbb{R}^N} \omega_h^2(Z)\,\mathrm{d}Z = Ch^{-N}.$$

Let K_h be the disc (or the sphere) with its center at the origin and its radius equal to h. Then

$$\operatorname{meas}_N K_h = \begin{cases} \pi h^2 & \text{for } N = 2, \\ \frac{4}{3}\pi h^3 & \text{for } N = 3 \end{cases}$$

and the mean value theorem yields

$$\int_{\mathbb{R}^N} \omega_h^2(Z)\,\mathrm{d}Z = \int_{K_h} \omega_h^2(Z)\,\mathrm{d}Z = \varkappa h^{-2N} \cdot C_0 \cdot \operatorname{meas}_N K_h$$
$$= \varkappa h^{-2N} \cdot C_0 \cdot \tilde{C} h^N = C h^{-N},$$

where we set

$$C_0 = \exp\Bigl(\frac{\|Z_0\|^2}{\|Z_0\|^2 - h^2}\Bigr) \leqslant 1, \quad C = \varkappa C_0 \tilde{C}.$$

This proves relation (52).

From now on we shall assume that the functions considered are sufficiently *smooth* and we will extend our result by the density argument (i.e., by means of [5, Theorem 5.5.9]). For easier understanding we shall distinguish two cases: $N = 2$ and $N = 3$.

a) $N = 2$: In this case (considering for simplicity that $G = (a, b)$) we have by (4)

(53)
$$U_\lambda(G) = (a - \lambda, b + \lambda),$$
$$V_\lambda(G) = \Bigl(a - \frac{3}{4}\lambda, b + \frac{3}{4}\lambda\Bigr).$$

We recall that

$$h = \frac{1}{4}\lambda.$$

Let us consider the second integral on the right-hand side of (51). In the case of $N = 2$ we have

(54)
$$\int_{V_\lambda(G)} \Bigl(\int_{\|Y'-X'\|<h} u^2\Bigl(Y', -\frac{3}{4}\lambda\Bigr)\,\mathrm{d}Y'\Bigr)\,\mathrm{d}X'$$
$$= \int_{a-3\lambda/4}^{b+3\lambda/4} \Bigl(\int_{x_1-h}^{x_1+h} u^2(y_1, -\frac{3}{4}\lambda\Bigr)\,\mathrm{d}y_1\Bigr)\,\mathrm{d}x_1.$$

Using the mean value theorem we obtain (owing to the sufficient smoothness of functions considered)

$$\int_{a-3\lambda/4}^{b+3\lambda/4} \Bigl(\int_{x_1-h}^{x_1+h} u^2\Bigl(y_1, -\frac{3}{4}\lambda\Bigr)\,\mathrm{d}y_1\Bigr)\,\mathrm{d}x_1 = \int_{a-3\lambda/4}^{b+3\lambda/4} u^2\Bigl(\tilde{y}_1, -\frac{3}{4}\lambda\Bigr)\,\mathrm{d}x_1 \cdot \int_{x_1-h}^{x_1+h} \mathrm{d}y_1$$
$$= 2h \int_{a-3\lambda/4}^{b+3\lambda/4} u^2\Bigl(\tilde{y}_1, -\frac{3}{4}\lambda\Bigr)\,\mathrm{d}x_1,$$

where

(55)
$$\tilde{y}_1 = x_1 + \eta, \quad \eta \in (-h, h).$$

The transformation $x_1+\eta=t$ with (55) yields

$$2h\int_{a-3\lambda/4}^{b+3\lambda/4} u^2\Big(\tilde{y}_1,-\frac{3}{4}\lambda\Big)\,\mathrm{d}x_1 = 2h\int_{a-3\lambda/4+\eta}^{b+3\lambda/4+\eta} u^2\Big(t,-\frac{3}{4}\lambda\Big)\,\mathrm{d}t$$
$$\leqslant 2h\int_{a-\lambda}^{b+\lambda} u^2\Big(t,-\frac{3}{4}\lambda\Big)\,\mathrm{d}t.$$

Using (53) and combining the result just obtained with (54) we obtain (in the case $N=2$)

$$\int_{V_\lambda(G)}\Big(\int_{\|Y'-X'\|<h} u^2\Big(Y',-\frac{3}{4}\lambda\Big)\,\mathrm{d}Y'\Big)\,\mathrm{d}X' \leqslant 2h\int_{U_\lambda(G)} u^2\Big(Y',-\frac{3}{4}\lambda\Big)\,\mathrm{d}Y' \tag{56}$$

and thus we have (in the case of $N=2$), due to (51) and (52),

$$\|\psi_{N,3}\|^2_{0,P_\lambda} \leqslant C_2h^{-1}\int_{U_\lambda(G)} u^2(Y',-\frac{3}{4}\lambda)\,\mathrm{d}Y'. \tag{57}$$

b) $N=3$: Our task is now to prove relation (57) in the case $N=3$ (i.e., to prove relation (62)). Now the domains $U_\lambda(G)$ and $V_\lambda(G)$ are given by $(4)_1$ and $(4)_2$, respectively. Using the mean value theorem we obtain

$$\int_{V_\lambda(G)}\Big(\int_{\|Y'-X'\|<h} u^2\Big(Y',-\frac{3}{4}\lambda\Big)\,\mathrm{d}Y'\Big)\,\mathrm{d}X' \tag{58}$$
$$= \int_{V_\lambda(G)} u^2\Big(\tilde{Y}',-\frac{3}{4}\lambda\Big)\,\mathrm{d}X'\cdot\int_{\|Y'-X'\|<h}\mathrm{d}Y'$$
$$= \pi h^2\int_{V_\lambda(G)} u^2\Big(\tilde{Y}',-\frac{3}{4}\lambda\Big)\,\mathrm{d}X',$$

where

$$\tilde{Y}' = X' + [\eta_1,\eta_2],\quad [\eta_1,\eta_2]\in\{\|Y'-X'\|<h\}. \tag{59}$$

This means that

$$\eta_1\in(-h\cos\alpha,h\cos\alpha),\quad \eta_2\in(-h\sin\alpha,h\sin\alpha),\quad \alpha\in\langle 0,\pi\rangle. \tag{60}$$

The transformation

$$x_1+\eta_1=s_1,\quad x_2+\eta_2=s_2$$

with (59) and (60) yields

$$\pi h^2\int_{V_\lambda(G)} u^2\Big(\tilde{Y}',-\frac{3}{4}\lambda\Big)\,\mathrm{d}X' \leqslant \pi h^2\int_{U_\lambda(G)} u^2\Big(S',-\frac{3}{4}\lambda\Big)\,\mathrm{d}S'.$$

Using $(4)_1$, $(4)_2$ and combining the result just obtained with (58) we obtain (in the case of $N=3$)

$$\int_{V_\lambda(G)}\Big(\int_{\|Y'-X'\|<h} u^2\Big(Y',-\frac{3}{4}\lambda\Big)\,\mathrm{d}Y'\Big)\,\mathrm{d}X' \leqslant \pi h^2\int_{U_\lambda(G)} u^2\Big(Y',-\frac{3}{4}\lambda\Big)\,\mathrm{d}Y' \tag{61}$$

and thus we have (if $N=3$), due to (51) and (52),

$$\|\psi_{N,3}\|^2_{0,P_\lambda} \leqslant C_2h^{-1}\int_{U_\lambda(G)} u^2\Big(Y',-\frac{3}{4}\lambda\Big)\,\mathrm{d}Y'. \tag{62}$$

Let us consider the integral on the right-hand side of (62) or on the right-hand side of (57). (Note that relation (62) has the same form as relation (57) which is written in the case of $N=2$.) We obtain for $Y'\in U_\lambda(G)$ (let us point out that we assume u to be *smooth* enough and extend then the result by the density argument (see [5, Theorem 5.5.9])):

$$u\Big(Y',-\frac{3}{4}\lambda\Big) = u(Y',0) + \int_0^{-3\lambda/4}\frac{\partial u}{\partial y_N}(Y',\xi)\,\mathrm{d}\xi;$$

now we square this relation, use the inequality $(a+b)^2\leqslant 2(a^2+b^2)$ and then apply the Schwarz inequality; finally, we extend the interval of integration $(-\frac{3}{4}\lambda,0)$ to $(-\lambda,0)$ arriving at

$$\begin{aligned}
u^2\Big(Y',-\frac{3}{4}\lambda\Big) &\leqslant 2\Big[u^2(Y',0)+\Big(\int_0^{-3\lambda/4}\frac{\partial u}{\partial y_N}(Y',\xi)\,\mathrm{d}\xi\Big)^2\Big]\\
&= 2\Big[u^2(Y',0)+\Big(-\int_{-3\lambda/4}^0\frac{\partial u}{\partial y_N}(Y',\xi)\,\mathrm{d}\xi\Big)^2\Big]\\
&\leqslant 2\Big[u^2(Y',0)+\frac{3}{4}\lambda\int_{-3\lambda/4}^0\Big(\frac{\partial u}{\partial y_N}(Y',\xi)\Big)^2\,\mathrm{d}\xi\Big]\\
&\leqslant 2\Big[u^2(Y',0)+\frac{3}{4}\lambda\int_{-\lambda}^0\Big(\frac{\partial u}{\partial y_N}(Y',\xi)\Big)^2\,\mathrm{d}\xi\Big].
\end{aligned}$$

Hence ($h=\frac{1}{4}\lambda$)

$$\begin{aligned}
h^{-1}\int_{U_\lambda(G)} u^2\Big(Y',-\frac{3}{4}\lambda\Big)\,\mathrm{d}Y' \leqslant C_3\Big[h^{-1}&\int_{U_\lambda(G)} u^2(Y',0)\,\mathrm{d}Y'\\
&+\int_{U_\lambda(\Gamma)}\Big(\frac{\partial u}{\partial y_N}(Y',\xi)\Big)^2\,\mathrm{d}Y'\,\mathrm{d}\xi\Big].
\end{aligned} \tag{63}$$

The theorem on the absolute continuity of an integral implies

$$\int_{U_\lambda(\Gamma)}\Big(\frac{\partial u}{\partial y_N}(Y',\xi)\Big)^2\,\mathrm{d}Y'\,\mathrm{d}\xi \to 0 \quad (\text{with } \lambda\to 0) \tag{64}$$

and further we have (owing to the fact that $u\big|_{\Gamma} = 0$, $\Gamma = G \times \{0\}$)

$$\int_{U_\lambda(G)} u^2(Y', 0)\,\mathrm{d}Y' = \int_{U_\lambda(G)\setminus G} u^2(Y', 0)\,\mathrm{d}Y'. \tag{65}$$

When analyzing the integral on the right-hand side of (65) we shall distinguish between the cases $N = 2$ and $N \geqslant 3$ (in applications it is sufficient to consider the case $N = 3$).

1) The case $N = 2$. Again, let for simplicity $G = (a, b)$ (see Fig. 1). Then

$$\int_{U_\lambda(G)\setminus G} u^2(Y', 0)\,\mathrm{d}Y' = \int_{a-\lambda}^{a} u^2(y_1, 0)\,\mathrm{d}y_1 + \int_{b}^{b+\lambda} u^2(y_1, 0)\,\mathrm{d}y_1 = I_1 + I_2.$$

The integral I_1 will be written in the form

$$I_1 = -\int_{a}^{a-\lambda} u^2(y_1, 0)\,\mathrm{d}y_1$$

and the substitution $y_1 = a - t$, where $t \in (0, \lambda)$, will be used. We obtain, according to the theorem on substitution in a simple integral,

$$I_1 = \int_0^\lambda u^2(a - t, 0)\,\mathrm{d}t.$$

Similarly, using in the integral I_2 the substitution $y_1 = b + t$ where $t \in (0, \lambda)$, we obtain

$$I_2 = \int_0^\lambda u^2(b + t, 0)\,\mathrm{d}t.$$

Hence, according to the theorem on differentiation of an integral as a function of the upper limit and by the l'Hospital rule (and taking into account that the first integral on the right-hand side of (63) is multiplied by h^{-1}), we obtain

$$\begin{aligned} \lim_{\lambda\to0+} \frac{1}{\lambda} \int_{U_\lambda(G)\setminus G} & u^2(Y', 0)\,\mathrm{d}Y' \\ &= \lim_{\lambda\to0+} \frac{1}{\lambda}\Big(\int_0^\lambda u^2(a - t, 0)\,\mathrm{d}t + \int_0^\lambda u^2(b + t, 0)\,\mathrm{d}t\Big) \\ &= \lim_{\lambda\to0+} (u^2(a - \lambda, 0) + u^2(b + \lambda, 0)) \\ &= u^2(a, 0) + u^2(b, 0) = 0, \end{aligned} \tag{66}$$

because $u(a, 0) = 0$, $u(b, 0) = 0$ (u is, according to our assumption, a *smooth* function). All these results yield $\|\psi_N\|_{0,P_\lambda} \to 0$ with $\lambda \to 0$ (when $N = 2$).

2) In this part of the proof we use, beside the Trace Theorem in the form of [5, Theorem 6.8.13] (which concerns the mapping $\mathfrak{R}\colon H^{1,p} \to H^{1-\frac{1}{p},p}(\partial\Omega)$), the following theorem (we cite it from [1], where it is introduced by the following words: "We shall not attempt any proof" and references to the works of Besov [2], Uspenskij [9] and Lizorkin [6] are given. Adams further writes: "The theorem is stated for $\mathbb{R}^N$ but can obviously be extended to domains with sufficient regularity" (as, for example, domains $\Omega \in \tilde{\mathcal{C}}^{0,1}$).

Theorem A. *Let $s > 0$, $1 < p \leqslant q < \infty$, and $1 \leqslant k \leqslant n$. Let $\chi = s - n/p + k/q$. If*
(1) $\chi \geqslant 0$ and $p < q$, or
(2) $\chi > 0$ and χ is not an integer, or
(3) $\chi \geqslant 0$ and $1 < p \leqslant 2$,
then (direct imbedding theorem)

$$W^{s,p}(\mathbb{R}^n) \to W^{\chi,q}(\mathbb{R}^k). \tag{67}$$

Imbedding (67) *does not necessarily hold for $p = q > 2$ and χ a nonnegative integer.*

In this part of the proof we restrict ourselves to the case $N = 3$. Our considerations start now again from relation (65). Using the Trace Theorem in the form of [5, Theorem 6.8.13] we see that

$$u(Y', 0) \in H^{\frac{1}{2}}(\Delta). \tag{68}$$

Now we use Theorem A: We have (using the notation of Theorem A)

$$s = \frac{1}{2}, \quad p = q = 2, \quad n = N - 1, \quad k = n - 1 = N - 2.$$

Hence

$$\chi = s - \frac{n}{p} + \frac{k}{q} = \frac{1}{2} - \frac{N-1}{2} + \frac{N-2}{2} = 0.$$

Thus, according to the assertion of Theorem A,

$$H^{\frac{1}{2}}(\mathbb{R}^{N-1}) = W^{\frac{1}{2},2}(\mathbb{R}^{N-1}) \to H^0(\mathbb{R}^{N-2}) = L_2(\mathbb{R}^{N-2}),$$

and consequently

$$H^{\frac{1}{2}}(\Delta) \to L_2(\partial\Delta)$$

and also (again according to Theorem A, because $G \subset \Delta$)

$$H^{\frac{1}{2}}(\Delta) \to L_2(\partial G).$$

Thus (as $u(Y',0) = u\big|_G$)

$$u(Y',0)\big|_{\partial G} \in L_2(\partial G).$$

Then the convergence

$$\frac{1}{\lambda}\int_{U_\lambda(G)\setminus G} u^2(Y',0)\,\mathrm{d}Y' \to 0 \quad (\text{with } \lambda \to 0)$$

follows from the properties of traces: Of course, *locally* we have

$$\int_{\mathcal{R}} u^2(Y',0)\,\mathrm{d}Y' \leqslant \int_{\mathcal{R}'}\left(\int_{a(z_1)-K\lambda}^{a(z_1)} u^2(z_1,z_2,0)\,\mathrm{d}z_2\right)\mathrm{d}z_1,$$

where $\mathcal{R}$ is the intersection of $U_\lambda(G)\setminus G$ with some suitable neighbourhood

$$\mathcal{R}_1 = \{[z_1,z_2]\colon\ z_1 \in \mathcal{R}',\ z_2 \in (a(z_1)-\gamma, a(z_1)+\gamma)\}$$

of any fixed point of ∂G (see Fig. 4) and $a(z_1)$ is the function which represents ∂G

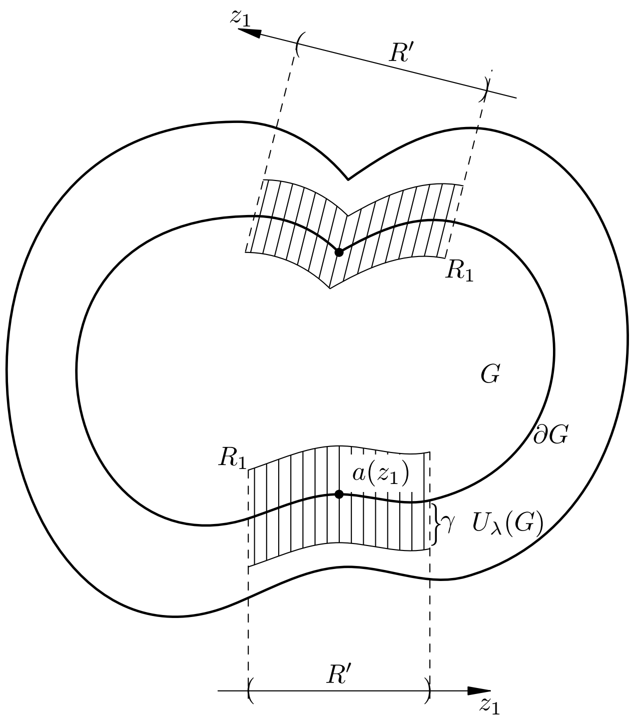

Figure 4. Concerning the case $N = 3$ (see 1 b) of part D1.)

with respect to a local Cartesian coordinate system of axes (z_1, z_2) with a Lipschitz constant L and $K \leqslant \sqrt{L^2+1}$. The function

$$\Phi(\eta) = \int_{\mathcal{R}'} u^2(z_1, a(z_1) - \eta, 0)\,\mathrm{d}z_1 \tag{69}$$

is a continuous function of $\eta \in \langle -\gamma, 0\rangle$: indeed, the function $u(z_1, z_2, 0)$ is assumed sufficiently smooth on $\mathcal{R}' \times (-\gamma, \gamma)$, hence it is continuous on $\overline{\mathcal{R}'} \times \langle -\gamma, \gamma\rangle$; by Cantor's theorem it is there uniformly continuous. As $a(z_1)$ is Lipschitz continuous on $\overline{\mathcal{R}'}$, we have $|a(z_1) - a(z_1')| < L|z_1 - z_1'|$. All these facts imply that (according to the definition of uniform continuity of a function of two variables)

$$|u(z_1, a(z_1) - \eta, 0) - u(z_1', a(z_1') - \eta', 0)| < \varepsilon \quad \text{for } |z_1 - z_1'| < \frac{\delta}{2L}, \quad |\eta - \eta'| < \frac{\delta}{2}.$$

It suffices now to use the theorem on continuity of an integral with respect to a parameter.

Thus, using the Fubini theorem and making the change of variables in the form $\eta = a(z_1) - z_2$, we obtain by means of the l'Hospital rule and the theorem on differentiation of the integral as a function of the upper limit

$$\lim_{\lambda \to 0+} \frac{1}{\lambda} \int_{\mathcal{R}} u^2(Y', 0)\,\mathrm{d}Y' \leqslant K \lim_{\lambda \to 0+} \frac{1}{K\lambda} \int_{-K\lambda}^{0} \Phi(\eta)\,\mathrm{d}\eta = -K\Phi(0) = 0,$$

because $u\big|_\Gamma = 0$ and the function $\Phi(\eta)$ is given by relation (69); we must keep in mind that $u^2(z_1, a(z_1), 0)\big|_{z_1 \in \mathcal{R}'} = u^2\big|_{\partial G} = u^2\big|_G = u^2\big|_\Gamma = 0$.

D2.1. $N = 2$. First we shall discuss the case $N = 2$. Let, for simplicity, the set G be an open interval on the axis x_1:

$$G = (a, b).$$

Then

$$U_\lambda(G) = (a - \lambda, b + \lambda), \quad V_\lambda(G) = \Big(a - \frac{3}{4}\lambda, b + \frac{3}{4}\lambda\Big),$$
$$W_\lambda(G) = \Big(a - \frac{1}{2}\lambda, b + \frac{1}{2}\lambda\Big), \quad Z_\lambda(G) = \Big(a - \frac{1}{4}\lambda, b + \frac{1}{4}\lambda\Big),$$

and

$$U_\lambda(\Gamma) = (a - \lambda, b + \lambda) \times \Big(-\lambda, \frac{1}{4}\lambda\Big),$$
$$V_\lambda(\Gamma) = \Big(a - \frac{3}{4}\lambda, b + \frac{3}{4}\lambda\Big) \times \Big(-\frac{3}{4}\lambda, \frac{1}{4}\lambda\Big),$$
$$W_\lambda(\Gamma) = \Big(a - \frac{1}{2}\lambda, b + \frac{1}{2}\lambda\Big) \times \Big(-\frac{1}{2}\lambda, \frac{1}{4}\lambda\Big),$$
$$Z_\lambda(\Gamma) = \Big(a - \frac{1}{4}\lambda, b + \frac{1}{4}\lambda\Big) \times \Big(-\frac{1}{4}\lambda, \frac{1}{4}\lambda\Big).$$

By (32), Fig. 3 and the Green-Gauss-Ostrogradskij theorem we have

$$
\begin{aligned}
(70)\quad \psi_1(X) = \frac{\partial w_\lambda}{\partial x_1}(X) &= \int_{Q_\lambda} \frac{\partial \omega_h}{\partial x_1}(X-Y)u_\lambda(Y)\,\mathrm{d}Y \\
&= -\int_{Q_\lambda} \frac{\partial \omega_h}{y_1}(X-Y)u_\lambda(Y)\,\mathrm{d}Y = \\
&= \int_{Q_\lambda} \omega_h(X-Y)\frac{\partial u_\lambda}{\partial y_1}(Y)\,\mathrm{d}Y \\
&\quad - \int_{-\lambda}^{\lambda/4} \omega_h(X-[b+\lambda,y_2])u_\lambda(b+\lambda,y_2)\,\mathrm{d}y_2 \\
&\quad + \int_{-\lambda/2}^{\lambda/4} \omega_h\Big(X-\Big[b+\frac{1}{2}\lambda,y_2\Big]\Big)u_\lambda\Big(b+\frac{1}{2}\lambda,y_2\Big)\,\mathrm{d}y_2 \\
&\quad - \int_{-\lambda/2}^{\lambda/4} \omega_h\Big(X-\Big[a-\frac{1}{2}\lambda,y_2\Big]\Big)u_\lambda\Big(a-\frac{1}{2}\lambda,y_2\Big)\,\mathrm{d}y_2 \\
&\quad + \int_{-\lambda}^{\lambda/4} \omega_h(X-[a-\lambda,y_2])u_\lambda(a-\lambda,y_2)\,\mathrm{d}y_2.
\end{aligned}
$$

Similarly to Section D1 the first integral on the right-hand side of (70) tends to zero in $L_2(P_\lambda)$; this means

$$
(71)\qquad \lim_{\lambda\to 0}\int_{P_\lambda}\Big(\int_{Q_\lambda}\omega_h(X-Y)\frac{\partial u_\lambda}{\partial y_1}(Y)\,\mathrm{d}Y\Big)^2\,\mathrm{d}X = 0,
$$

where (see Fig. 3)

$$
P_\lambda = V_\lambda(\Gamma)\cap\Theta, \quad Q_\lambda = U_\lambda(\Gamma)\setminus\overline{W_\lambda(\Gamma)}.
$$

Now we prove that the second and fifth integrals on the right-hand side of (70) are equal to zero. We have

$$
\begin{aligned}
\omega_h(X-[b+\lambda,y_2]) &= \omega_h([x_1,x_2]-[b+\lambda,y_2]) \\
&= \omega_h(x_1-b-\lambda,x_2-y_2) \\
&= \omega_h(x_1-b-4h,x_2-y_2) = \omega_h(Z),
\end{aligned}
$$

where we set

$$
Z = [x_1-b-4h,x_2-y_2].
$$

Thus

$$
\|Z\| = \sqrt{(x_1-b-4h)^2+(x_2-y_2)^2}.
$$

The most inconvenient situation occurs when

$$(x_2 - y_2)^2 = 0.$$

As $X \in P_\lambda = V_\lambda(\Gamma) \cap \Theta$ we have

$$x_1 \in \Big(a - \frac{3}{4}\lambda, b + \frac{3}{4}\lambda\Big) = (a - 3h, b + 3h). \tag{72}$$

In such a situation the inequality

$$\|Z\| \geqslant h \tag{73}$$

holds because for every x_1 satisfying (72) we have

$$(x_1 - b - 4h)^2 > [(b + 3h) - b - 4h]^2 = h^2.$$

Hence the *second* integral on the right-hand side of (70) is equal to zero.

A similar situation arises in the case of the *fifth* integral on the right-hand side of (70) because

$$\omega_h(X - [a - \lambda, y_2]) = \omega_h(x_1 - a + 4h, x_2 - y_2) = \omega_h(Z)$$

and again we have

$$(x_1 - a + 4h)^2 > [(a - 3h) - a + 4h]^2 = h^2$$

for every x_1 satisfying (72), hence relation (73) holds again.

As to the *third* and *fourth* integrals on the right-hand side of (70) they can be different from zero. We see it (in the case of the third integral) from the relation

$$\omega_h\Big(X - \Big[b + \frac{1}{2}\lambda, y_2\Big]\Big) = \omega_h(x_1 - b - 2h, x_2 - y_2) = \omega_h(Z)$$

where, due to (72), we can have $x_1 = b + 2h$, and thus it is possible that

$$\|Z\| < h.$$

Thus, let us consider

$$\psi_{1,3}(X) = \int_{-\lambda/2}^{\lambda/4} \omega_h\Big(X - \Big[b + \frac{1}{2}\lambda, y_2\Big]\Big) u_\lambda\Big(b + \frac{1}{2}\lambda, y_2\Big)\, \mathrm{d}y_2, \tag{74}$$

$$\psi_{1,4}(X) = \int_{-\lambda/2}^{\lambda/4} \omega_h\Big(X - \Big[a - \frac{1}{2}\lambda, y_2\Big]\Big) u_\lambda\Big(a - \frac{1}{2}\lambda, y_2\Big)\, \mathrm{d}y_2. \tag{75}$$

Taking into account the preceding text and relation (70) we can write

$$|\psi_1(X)|^2 = (\psi_{1,3}(X) + \psi_{1,4}(X))^2 \leqslant 2(|\psi_{1,3}(X)|^2 + |\psi_{1,4}(X)|^2); \tag{76}$$

thus it suffices to consider the case of relation (74); the considerations in the case of (75) follow the same lines.

Using the Schwarz inequality and relation (74) we obtain

$$|\psi_{1,3}(X)|^2 \leqslant \int_{-\lambda/2}^{\lambda/4} \omega_h^2\Big(X - \Big[b + \frac{1}{2}\lambda, y_2\Big]\Big)\,\mathrm{d}y_2 \int_{-\lambda/2}^{\lambda/4} u_\lambda^2\Big(b + \frac{1}{2}\lambda, y_2\Big)\,\mathrm{d}y_2. \tag{77}$$

Inequality (77) will be rewritten to the form

$$\begin{aligned} |\psi_{1,3}(X)|^2 &\leqslant \int_{-\lambda/2}^{\lambda/4} \omega_h^2\Big(x_1 - b - \frac{1}{2}\lambda, x_2 - y_2\Big)\,\mathrm{d}y_2 \int_{-\lambda/2}^{\lambda/4} u_\lambda^2\Big(b + \frac{1}{2}\lambda, y_2\Big)\,\mathrm{d}y_2 \\ &\leqslant \int_{-\infty}^{\infty} \omega_h^2\Big(x_1 - b - \frac{1}{2}\lambda, y_2 - x_2\Big)\,\mathrm{d}y_2 \int_{-\lambda/2}^{\lambda/4} u_\lambda^2\Big(b + \frac{1}{2}\lambda, y_2\Big)\,\mathrm{d}y_2. \end{aligned} \tag{78}$$

In the first integral on the right-hand side of (78) we transform the variables in the form

$$y_2 - x_2 = z_2 \Rightarrow \mathrm{d}y_2 = \mathrm{d}z_2$$

and in the second integral we change the notation of the integration variable. We obtain

$$|\psi_{1,3}(X)|^2 \leqslant \int_{-\infty}^{\infty} \omega_h^2\Big(x_1 - b - \frac{1}{2}\lambda, z_2\Big)\,\mathrm{d}z_2 \int_{-\lambda/2}^{\lambda/4} u_\lambda^2\Big(b + \frac{1}{2}\lambda, s\Big)\,\mathrm{d}s. \tag{79}$$

Let us integrate inequality (79) over P_λ:

$$\begin{aligned} \int_{P_\lambda} &|\psi_{1,3}(X)|^2\,\mathrm{d}X \\ &\leqslant \int_{P_\lambda} \Big\{\int_{-\infty}^{\infty} \omega_h^2\Big(x_1 - b - \frac{1}{2}\lambda, z_2\Big)\,\mathrm{d}z_2 \int_{-\lambda/2}^{\lambda/4} u_\lambda^2\Big(b + \frac{1}{2}\lambda, s\Big)\,\mathrm{d}s\Big\}\,\mathrm{d}X \\ &= \int_{a-3\lambda/4}^{b+3\lambda/4} \Big(\int_{-\infty}^{\infty} \omega_h^2\Big(x_1 - \frac{1}{2}\lambda - b, z_2\Big)\,\mathrm{d}z_2\Big)\,\mathrm{d}x_1 \\ &\quad \times \int_{-3\lambda/4}^{0} \mathrm{d}x_2 \int_{-\lambda/2}^{\lambda/4} u_\lambda\Big(b + \frac{1}{2}\lambda, s\Big)\,\mathrm{d}s \\ &\leqslant \int_{-\infty}^{\infty} \Big(\int_{-\infty}^{\infty} \omega_h^2\Big(x_1 - b - \frac{1}{2}\lambda, z_2\Big)\,\mathrm{d}z_2\Big)\,\mathrm{d}x_1 \cdot 3h \int_{-\lambda/2}^{\lambda/4} u_\lambda\Big(b + \frac{1}{2}\lambda, s\Big)\,\mathrm{d}s. \end{aligned} \tag{80}$$

In the first integral on the right-hand side of (80) we transform the variables in the form

$$x_1 - b - \frac{1}{2}\lambda = z_1 \Rightarrow \,\mathrm{d}x_1 = \,\mathrm{d}z_1;$$

hence we obtain

$$\|\psi_{1,3}\|^2_{0,P_\lambda} \leqslant \int_{\mathbb{R}^2} \omega_h^2(Z)\,\mathrm{d}Z \cdot 3h \int_{-\lambda/2}^{\lambda/4} u_\lambda\Bigl(b + \frac{1}{2}\lambda, s\Bigr)\,\mathrm{d}s. \tag{81}$$

Relations (81) and (52) (with $N = 2$) yield

$$\|\psi_{1,3}\|^2_{0,P_\lambda} \leqslant C_2 h^{-1} \int_{-\lambda/2}^{\lambda/4} u_\lambda^2\Bigl(b + \frac{1}{2}\lambda, s\Bigr)\,\mathrm{d}s. \tag{82}$$

Inequality (82) is a one-dimensional analogue of inequalities (57) and (62). In this case we proceed as follows (we use definition (7) of the function u_λ and the definition of h: $h = \frac{1}{4}\lambda$):

$$\begin{aligned} h^{-1} \int_{-\lambda/2}^{\lambda/4} u_\lambda^2\Bigl(b + \frac{1}{2}\lambda, s\Bigr)\,\mathrm{d}s &= h^{-1} \int_{-2h}^{h} u_\lambda^2\Bigl(b + \frac{1}{2}\lambda, s\Bigr)\,\mathrm{d}s \\ &= \frac{1}{h} \int_{-2h}^{0} u^2\Bigl(b + \frac{1}{2}\lambda, s - h\Bigr)\,\mathrm{d}s + \frac{1}{h} \int_{0}^{h} u^2\Bigl(b + \frac{1}{2}\lambda, s - h\Bigr)\,\mathrm{d}s. \end{aligned} \tag{83}$$

Let $h \to 0$ and let us evaluate each limit on the right-hand side of (83) separately. In both cases we have the limit of type $\frac{0}{0}$; thus we can use the l'Hospital rule and the theorem on differentiation of an integral as a function of the upper limit. Again, as in part D1, we assume the function u to be sufficiently *smooth* and then extend the result by a density argument (see [5, Theorem 5.5.9]). First we consider the limit of the second term on the right-hand side of (83):

$$\lim_{h\to 0} \frac{1}{h} \int_0^h u^2\Bigl(b + \frac{1}{2}\lambda, s - h\Bigr)\,\mathrm{d}s = \lim_{h\to 0} u^2(b + 2h, 0) = 0, \tag{84}$$

because $[b, 0] \in \overline{\Gamma} = \overline{G \times \{0\}}$ and $u\big|_\Gamma = 0$. (The function u is smooth.)

As to the first term on the right-hand side of (83), we proceed as follows:

$$\begin{aligned} -2 \lim_{h\to 0} \frac{1}{2h} \int_0^{-2h} u^2\Bigl(b + \frac{1}{2}\lambda, s - h\Bigr)\,\mathrm{d}s &= -2 \lim_{h\to 0} u^2(b + 2h, -3h) \\ &= -2u^2(b, 0) = 0. \end{aligned} \tag{85}$$

Relations (81)–(85) yield

$$\lim_{\lambda\to 0} \|\psi_{1,3}\|^2_{0,P_\lambda} = 0. \tag{86}$$

We can prove in the same way that

$$\lim_{\lambda\to 0}\|\psi_{1,4}\|^2_{0,P_\lambda}=0. \tag{87}$$

Relations (76) and (86), (87) imply

$$\lim_{\lambda\to 0}\|\psi_{1}\|^2_{0,P_\lambda}=0. \tag{88}$$

The proof of Lemma 2 in the case $N=2$ is complete.

D2.2. $N=3$. Now we shall consider the situation $i<N$ in the case $N=3$, this means the cases $i=1$ and $i=2$.

For greater simplicity we assume that $G\subset\overline{G}\subset\Delta$ is a simply connected domain. Further we assume that the boundary ∂G consists of a finite number of smooth arcs; this assumption is sufficient for applications. In the case $N=3$ the sets $U_\lambda(G),\dots,Z_\lambda(G)$ and $U_\lambda(\Gamma),\dots,Z_\lambda(\Gamma)$ are defined by relations (4) and (5), respectively. Domains (5) are three-dimensional cylinders with bases parallel to the coordinate plane (x_1,x_2) and with the lateral area of a cylinder formed by straight-lines parallel to the axis x_3. The projections of both bases of the cylinder $M_\lambda(\Gamma)$ $(M=U,\dots,Z)$ are identical with the two-dimensional domain $M_\lambda(G)$ which lies in the coordinate plane (x_1,x_2).

Let $M^1_\lambda(\Gamma)$ and $M^2_\lambda(\Gamma)$ be the lower and upper bases of the cylinder $M_\lambda(\Gamma)$, respectively, and let $M^3_\lambda(\Gamma)=\partial M_\lambda(\Gamma)\setminus(M^1_\lambda(\Gamma)\cup M^2_\lambda(\Gamma))$. Let $X\in\partial M_\lambda(\Gamma)$ be an arbitrary point except for the points at which $\partial M_\lambda(\Gamma)$ is not smooth. Let $n(X)$ be the unit outer normal to $\partial M_\lambda(\Gamma)$ at the point X. Then we have

$$X\in M^1_\lambda(\Gamma)\Rightarrow n(X)=(0,0,-1), \tag{89}$$

$$X\in M^2_\lambda(\Gamma)\Rightarrow n(X)=(0,0,1), \tag{90}$$

$$X\in M^3_\lambda(\Gamma)\Rightarrow n(X)=(n_1(X),n_2(X),0),\quad n_1^2(X)+n_2^2(X)=1. \tag{91}$$

Now we shall compute the functions $\psi_i(X)$ given by relation (32) for $i=1$ and $i=2$. By the Green-Gauss-Ostrogradskij theorem and relations (89)–(91) we have

$$\begin{aligned}\psi_i(X)=&\int_{Q_\lambda}\omega_h(X-Y)\frac{\partial u_\lambda}{\partial y_i}(Y)\,\mathrm{d}Y\\ &-\int_{U^3_\lambda(\Gamma)}\omega_h(X-Y)u_\lambda(Y)n_i(Y)\,\mathrm{d}\sigma_Y\\ &+\int_{W^3_\lambda(\Gamma)}\omega_h(X-Y)u_\lambda(Y)n_i(Y)\,\mathrm{d}\sigma_Y.\end{aligned} \tag{92}$$

The last two integrals appearing on the right-hand side of (92) are surface integrals of the first kind. Similarly to Section D1 the first integral on the right-hand side of (92) tends to zero in $L_2(P_\lambda)$; this means

$$\lim_{\lambda\to 0}\int_{P_\lambda}\Bigl(\int_{Q_\lambda}\omega_h(X-Y)\frac{\partial u_\lambda}{\partial y_1}(Y)\,\mathrm{d}Y\Bigr)^2\,\mathrm{d}X=0, \tag{93}$$

where (see Fig. 3)

$$P_\lambda = V_\lambda(\Gamma)\cap\Theta,\quad Q_\lambda = U_\lambda(\Gamma)\setminus\overline{W_\lambda(\Gamma)}.$$

Now we prove that the second integral on the right-hand side of (92) is equal to zero. We have

$$\|X-Y\|=\sqrt{(x_1-y_1)^2+(x_2-y_2)^2+(x_3-y_3)^2}.$$

The most inconvenient situation occurs when $x_3-y_3=0$ (in this case the points X, Y lie in a plane which is parallel to the coordinate plane (x_1,x_2)) and when the point X is very close to $V_\lambda^3(\Gamma)$ or lies on $V_\lambda^3(\Gamma)$. As $Y\in U_\lambda^3(\Gamma)$ and $X\in P_\lambda$ we see that in *every* case we have $\|X-Y\|\geqslant h$. Thus $\omega_h(X-Y)=0$ and the second integral on the right-hand side of (92) is equal to zero. Hence

$$\psi_i(X)=\int_{W_\lambda^3(\Gamma)}\omega_h(X-Y)u_\lambda(Y)n_i(Y)\,\mathrm{d}\sigma_Y.$$

Using the Schwarz inequality and the fact that $|n_i(Y)|\leqslant 1$ we obtain

$$\begin{aligned}|\psi_i(X)|^2&\leqslant\int_{W_\lambda^3(\Gamma)}\omega_h^2(X-Y)\,\mathrm{d}\sigma_Y\int_{W_\lambda^3(\Gamma)}u_\lambda^2(Y)\,\mathrm{d}\sigma_Y\\&=\int_{W_\lambda^3(\Gamma)}\omega_h^2(X-Y)\,\mathrm{d}\sigma_Y\int_{W_\lambda^3(\Gamma)}u_\lambda^2(Z)\,\mathrm{d}\sigma_Z.\end{aligned}\tag{94}$$

Integration of inequality (94) over P_λ yields

$$\|\psi_i\|_{0,P_\lambda}^2\leqslant\int_{P_\lambda}\Bigl\{\int_{W_\lambda^3(\Gamma)}\omega_h^2(X-Y)\,\mathrm{d}\sigma_Y\Bigr\}\,\mathrm{d}X\cdot\int_{W_\lambda^3(\Gamma)}u_\lambda^2(Z)\,\mathrm{d}\sigma_Z.\tag{95}$$

First let us consider the second surface integral appearing on the right-hand side of (95). As $W_\lambda^3(\Gamma)=\partial W_\lambda(G)\times(-\frac12\lambda,\frac14\lambda)$ we can write using definition (7) of the function u_λ and the properties of a line integral of the first kind:

$$\begin{aligned}\int_{W_\lambda^3(\Gamma)}u_\lambda^2(Z)\,\mathrm{d}\sigma_Z&=\int_{\partial W_\lambda(G)\times(-\frac12\lambda,\frac14\lambda)}u^2(z_1,z_2,z_3-h)\,\mathrm{d}\sigma_Z\\&=\sum_{k=1}^m\int_{-\lambda/2}^{\lambda/4}\Bigl\{\int_{R_k'}u^2(z_1,a_{k,\lambda}(z_1),z_3-h)\sqrt{1+[a_{k,\lambda}'(z_1)]^2}\,\mathrm{d}z_1\Bigr\}\,\mathrm{d}z_3,\end{aligned}\tag{96}$$

where $a_{k,\lambda}$ is a local representation of the curve $\partial W_\lambda(G)$. (The situation is similar to that which is sketched in Fig. 4.) We have $|a'_{k,\lambda}(z_1)|^2 < C_k$, where the constant C_k depends only on the Lipschitz constant of the function $\alpha_k(z_1)$ which represents locally the curve ∂G (which does not depend on λ; however, $\partial W_\lambda(G)$ depends, similarly as $W_\lambda(G)$, on λ—see $(4)_3$; attention: the symbols $a_{k,\lambda}$ and α_k represent two different parallel arcs). The symbol R'_k ($k = 1, \ldots, m$) denotes a segment on the z_1-axis of the kth local coordinate system (z_1, z_2). (It would be more precise to use the notation $(z_1^{(k)}, z_2^{(k)})$ instead of the notation (z_1, z_2); however, it would have been very cumbersome.) Thus

$$(97)\qquad \int_{W^3_\lambda(\Gamma)} u^2_\lambda(Z)\,\mathrm{d}\sigma_Z \leqslant C_0 \sum_{k=1}^{m} \int_{-\lambda/2}^{\lambda/4} \left\{ \int_{R'_k} u^2(z_1, a_{k,\lambda}(z_1), z_3 - h)\,\mathrm{d}z_1 \right\} \mathrm{d}z_3,$$

where $C_0 = \max(C_1, \ldots, C_m)$. Let us consider the expression

$$(98)\qquad \begin{aligned} \frac{1}{h}\int_{-\lambda/2}^{\lambda/4} &\left\{ \int_{R'_k} u^2(z_1, a_{k,\lambda}(z_1), z_3 - h)\,\mathrm{d}z_1 \right\} \mathrm{d}z_3 \\ &= \int_{R'_k} \left\{ \frac{1}{h}\int_{-2h}^{0} u^2(z_1, a_{k,\lambda}(z_1), z_3 - h)\,\mathrm{d}z_3 \right. \\ &\qquad \left. + \frac{1}{h}\int_0^h u^2(z_1, a_{k,\lambda}(z_1), z_3 - h)\,\mathrm{d}z_3 \right\} \mathrm{d}z_1, \end{aligned}$$

where we have used the Fubini theorem. Let $h \to 0$ in (98). First let us consider the second expression appearing on the right-hand side of (98). To this end, the following fact should be noted: if $h \to 0$ then $\partial W_\lambda(G)$ tends to ∂G. The corresponding kth part of ∂G is described in the local coordinate system (z_1, z_2) by the function $\alpha_k(z_1)$. Hence (as the limit is of the type $\frac{0}{0}$, we use the l'Hospital rule and the theorem on differentiation of an integral as a function of the upper limit)

$$\lim_{h\to 0} \frac{1}{h}\int_0^h u^2(z_1, a_{k,\lambda}(z_1), z_3 - h)\,\mathrm{d}z_3 = \lim_{h\to 0} u^2(z_1, a_{k,\lambda}(z_1), 0) = u^2(z_1, \alpha_k(z_1), 0).$$

As $[z_1, \alpha_k(z_1), 0] \in \partial G$ and $u\big|_\Gamma = 0$, where $\Gamma = G \times \{0\}$, we have by the assumed continuity of the function u

$$u^2(z_1, \alpha_k(z_1), 0) = 0,$$

which yields

$$(99)\qquad \lim_{h\to 0} \frac{1}{h}\int_0^h u^2(z_1, a_{k,\lambda}(z_1), z_3 - h)\,\mathrm{d}z_3 = 0.$$

As to the first expression appearing on the right-hand side of (98) we have by the same argument

$$\lim_{h\to 0}\frac{1}{h}\int_{-2h}^{0}u^2(z_1,a_{k,\lambda}(z_1),z_3-h)\,\mathrm{d}z_3=-2\lim_{h\to 0}u^2(z_1,a_{k,\lambda}(z_1),-3h)$$
$$=-2u^2(z_1,\alpha_k(z_1),0)=0. \tag{100}$$

We see from relations (95)–(100) that to complete the proof of Lemma 2 in the case $N=3$ means to prove

$$\int_{P_\lambda}\left\{\int_{W_\lambda^3(\Gamma)}\omega_h^2(X-Y)\,\mathrm{d}\sigma_Y\right\}\mathrm{d}X=Ch^{-1}. \tag{101}$$

The Fubini theorem and the mean value theorem yield

$$\begin{aligned}\int_{P_\lambda}\left\{\int_{W_\lambda^3(\Gamma)}\omega_h^2(X-Y)\,\mathrm{d}\sigma_Y\right\}\mathrm{d}X&=\int_{W_\lambda^3(\Gamma)}\left\{\int_{P_\lambda}\omega_h^2(X-Y)\,\mathrm{d}X\right\}\mathrm{d}\sigma_Y\\&=\int_{W_\lambda^3(\Gamma)}\left\{\int_{\substack{\|X-Y\|<h\\Y\in W_\lambda^3(\Gamma)}}\omega_h^2(X-Y)\,\mathrm{d}X\right\}\mathrm{d}\sigma_Y\\&=\int_{W_\lambda^3(\Gamma)}\omega_h^2(X_0-Y)\,\mathrm{d}\sigma_Y\int_{\substack{\|X-Y\|<h\\Y\in W_\lambda^3(\Gamma)}}\mathrm{d}X.\end{aligned} \tag{102}$$

We have

$$\int_{\substack{\|X-Y\|<h\\Y\in W_\lambda^3(\Gamma)}}\mathrm{d}X=\frac{4}{3}\pi h^3. \tag{103}$$

As to the first integral on the right-hand side of (102), the case $X_0\in W_\lambda^3(\Gamma)$ is most inconvenient. As the inequality $\|X_0-Y\|<h$ must hold for $\omega_h^2(X_0-Y)>0$ we find

$$\int_{W_\lambda^3(\Gamma)}\omega_h^2(X_0-Y)\,\mathrm{d}\sigma_Y=\int_{\sigma(X_0)}\omega_h^2(X_0-Y)\,\mathrm{d}\sigma_Y,$$

where

$$\sigma(X_0)\subset W_\lambda^3(\Gamma),\quad \operatorname{meas}_2\sigma(X_0)\leqslant Ch^2.$$

Hence, taking into account (11) with $N=3$, we conclude that

$$\int_{W_\lambda^3(\Gamma)}\omega_h^2(X_0-Y)\,\mathrm{d}\sigma_Y=\omega_h^2(X_0-Y_0)\int_{\sigma(X_0)}\mathrm{d}\sigma_Y\leqslant Ch^{-4}. \tag{104}$$

Relations (102)–(104) imply the desired result (101). Lemma 2 is completely proved. □

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