Consensus models with aggregation operators for minimum quadratic cost in group decision making

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Abstract

In group decision making (GDM), to facilitate an acceptable consensus among the experts from different fields, time and resources are paid for persuading experts to modify their opinions. Thus, consensus costs are important for the GDM process. Notwithstanding, the unit costs in the common linear cost functions are always fixed, yet experts will generally express more resistance if they have to make more compromises. In this study, we use the quadratic cost functions, the marginal costs of which increase with the opinion changes. Aggregation operators are also considered to expand the applications of the consensus methods. Moreover, this paper further analyzes the minimum cost consensus models under the weighted average (WA) operator and the ordered weighted average (OWA) operators, respectively. Corresponding approaches are developed based on strictly convex quadratic programming and some desirable properties are also provided. Finally, some examples and comparative analyses are furnished to illustrate the validity of the proposed models.

Keywords Aggregation operator · Group decision making · Minimum cost consensus model

1 Introduction

Group decision making (GDM) requires the opinions provided by a group of experts (or individuals) to address unstructured problems, such as negotiations and conflict resolutions [1]. In GDM, experts often represent different interest groups, and their opinions may differ substantially even with the same interests and knowledge backgrounds [2–5]. However, in the real-life decision environments, the consent of all or most of experts to the collective option is very crucial [6–14].

Generally, solving GDM problems comprises two processes, the consensus process and the selection process. The former is dedicated to maximize the degree of consensus or agreement among experts [2, 3, 15–20]. Usually, the consensus process is coordinated by a moderator who has strong team leadership qualities and communication skills [21]. The selection process yields the final alternative according to the experts' knowledge, which includes two different steps: aggregation of individual opinions and exploitation of group opinion [22, 23].

In general, the consensus process is complex and resource-consuming, during which moderator needs to convince experts to modify and change their opinions by spending time and resources [2]. Thus, cost is a significant issue in GDM [15, 24, 25]. For a moderator, what he/she concerns is not only the final consensus result, but also the cost in compensation for experts' opinion change. To this end, this paper intends to develop a consensus approach, though which moderator can assist experts in reaching consensus at the minimum cost. The concept of minimum cost consensus was initiated by Ben-Arieh and Easton [24] and Ben-Arieh, Easton and Evans [15], based on which different minimum cost consensus models have been proposed for various GDM situations.

Although notable progress has been made in the research of minimum-cost consensus, there are some issues worth noting. (1) In real life, an expert generally expresses more resistance

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when the required opinion change rises, and the utility function of his/her opinion could be nonlinear [26, 27]. However, most existing methods are constructed under the linear cost functions, in which the unit adjustment cost of an expert is always fixed no matter how much the opinion is adjusted. Apparently, these studies did not consider the psychological changes of experts during the negotiation process. (2) Furthermore, in GDM problems, different aggregation operators are frequently used to aggregate individual opinions into a group collective opinion, according to their specific meanings and inherent relationships in a particular context. In order to expand the applications of minimum cost consensus theory, aggregation operators are necessary to be taken into account. (3) Finally, for the GDM problem, a unique optimal solution is more explicit and efficient than multiple solutions, which would lead to conflicts among experts once again. Therefore, another issue is how to design a minimum cost consensus approach that can obtain a unique optimal solution directly.

To the best of our knowledge, no study has addressed the above three issues simultaneously, thereby, this paper develops a kind of minimum quadratic-cost consensus models with aggregation operators. The proposed models have the following features: (1) Consensus costs are measured by quadratic cost functions, the unit costs of which are monotonically increasing with the magnitude of opinion change, showing the rising resistance of experts when more modifications are required. Compared with the linear costs, quadratic costs are closer to the real-life consensus costs of a moderator. Furthermore, under quadratic costs, the unique optimal solutions to the proposed consensus models are obtained by using strictly convex quadratic programming, avoiding the case of multiple solutions. (2) Aggregation operators are used to build the relationship between individual opinions and the group opinion. The importance of each expert's opinion can also be intuitively reflected through the allocation of weights. (3) The order positions of the experts' original opinions will not be changed by the proposed method, when the unit costs are uniform and the ordered weighted average (OWA) operators are used. This feature is called the property of order preservation in this paper.

The remainder of this paper is organized as follows. Section 2 introduces the related works and the basic knowledge with respect to the aggregation operators and the minimum cost consensus models. Section 3 constructs the minimum quadratic cost consensus model under aggregation operators, which will reduce to the model of Ben-Arieh, Easton and Evans [15], when the adjusted individual opinions are equal to each other. Further, this section focuses on the consensus models based on the weighted average (WA) operator and OWA operators. The corresponding properties of the proposed models under the different aggregation operators are

explored in Sect. 4. In Sect. 5, three numerical examples are employed to manifest the validity of the proposed models, and a comparative analysis is also provided. Finally, concluding remarks are presented in Sect. 6.

2 Preliminaries and related works

This section introduces the basic knowledge of aggregation operators and related works on minimum cost consensus models.

2.1 Aggregation operators

To obtain a collective opinion from individual ones, aggregation operators are substantially used in the GDM process. They reveal the inherent relationships between individual opinions and the group collective opinion in specific GDM problems. Let $\{o_1, o_2, ..., o_n\}$ be a set of opinions to aggregate. An aggregation operator is a function that derives the collective opinion o grounded on the individuals' one

$$\overline{o} = F(o_1, o_2, ..., o_n).$$

Here we mainly introduce two typical aggregation operators: the WA operator and the OWA operator [28]. The WA operator is a simple additive weighting approach considering the importance degree of each expert's opinion. Let $\{o_1, o_2, ..., o_n\}$ be as above. The WA operator is defined as

$$WA_w(o_1, o_2, ..., o_n) = \sum_{i=1}^{n} w_i o_i,$$

where $w = (w_1, w_2, ..., w_n)^T$ is an associated weight vector, such that $w_i \in [0, 1]$ and $\sum_{i=1}^{n} w_i = 1$.

The OWA operator aggregates the individuals' opinions by assigning appropriate weights to the reordered elements. Let $\{o_1, o_2, ..., o_n\}$ and $w = (w_1, w_2, ..., w_n)^T$ be as above. The OWA operator is defined as [28]

$$OWA_w(o_1, o_2, ..., o_n) = \sum_{i=1}^{n} w_i o_{\sigma(i)},$$

where $\{\sigma(1), \sigma(2), ..., \sigma(n)\}$ is a permutation of $\{1, 2, ..., n\}$ such that $o_{\sigma(i-1)} \geq o_{\sigma(i)}$ for $i = 2, 3, ..., n$. Since Yager [28] introduced the OWA operator in 1988, notable progress has been made in developing OWA operators, especially the corresponding weight generation methods (see reference [29]). Recently, Chen, Yu, Chin and Martinez [30] proposed an enhanced approach to OWA weight generation based on the interweaving method.

2.2 Minimum cost consensus models

Suppose that there are n experts $E = \{e_1, e_2, ..., e_n\}$ and $o_i \in R$ ($i = 1, 2, ..., n$) denotes expert i's initial opinion. Let $\bar{o}_i$ and $\bar{o}$ be the adjusted individual opinion and collective opinion, respectively. Besides, c_i denotes the cost of moving expert i's opinion one unit, then the linear cost paid to expert i is formulated as $c_i|\bar{o}_i - o_i|$. Ben-Arieh and Easton [24] first proposed the minimum cost consensus model as follows

$$\min_{\bar{o}} \sum_{i=1}^n c_i |\bar{o} - o_i|$$
$$\text{s.t.} \bar{o}_i = \bar{o}, i = 1, 2, ..., n$$
$$(M-1)$$

where $O = \{\bar{o} \in R | \bar{o} > 0\}$ is a set of all possible consensus opinions. Particularly, when the adjusted opinion $\bar{o}_i$ is identical to the collective opinion o, expert i is recognized as reaching consensus. This full and unanimous consensus is generally called as "hard" consensus.

Besides, Ben-Arieh and Easton [24] introduced the concept of ε consensus, which allows individuals' opinions to approach the group opinion adequately. Compared with hard consensus, this kind of consensus is less strict and can be achieved in a more flexible way, which we call "soft" consensus [13]. The tolerated deviation threshold is denoted as ε. Thus, all adjusted opinions fitting into consensus should be within the interval $[\bar{o} - \varepsilon, \bar{o} + \varepsilon]$. Obviously, only the opinions outside the interval need to be modified. Thus, the minimum cost consensus model with ε threshold is constructed as:

$$\min_{\bar{o}} \left[\sum_{i:o_i < \bar{o} - \varepsilon} c_i \left(\bar{o} - \varepsilon - o_i \right) + \sum_{i:o_i > \bar{o} + \varepsilon} c_i \left(o_i - \bar{o} - \varepsilon \right) \right]$$
$$\text{s.t.} \left| \bar{o}_i - \bar{o} \right| \le \varepsilon, i = 1, 2, ..., n$$

$$(M-2)$$

Under the nonlinear opinion elasticity, Ben-Arieh, Easton and Evans [15] formulated quadratic cost function $f_i = c_i(\bar{o} - o_i)^2$, then the unanimous consensus at the minimum quadratic cost (CMQC) is obtained by the following model [15]:

$$\min_{\bar{o}} \sum_{i=1}^n c_i \left(\bar{o} - o_i \right)^2$$
$$\text{s.t.} \bar{o}_i = \bar{o}, i = 1, 2, ..., n$$
$$(M-3)$$

A natural extension to CMQC is the ε consensus allowing all experts' opinions within the given distance of group opinion (ε CMQC). The consensus model for ε CMQC is constructed as:

$$\min_{\bar{o}} \left[\sum_{i:o_i < \bar{o} - \varepsilon} c_i \left(\bar{o} - \varepsilon - o_i \right)^2 + \sum_{i:o_i > \bar{o} + \varepsilon} c_i \left(\bar{o} + \varepsilon - o_i \right)^2 \right]$$
$$\text{s.t.} \left| \bar{o}_i - \bar{o} \right| \le \varepsilon, i = 1, 2, ..., n$$

$$(M-4)$$

In addition, Ben-Arieh, Easton and Evans [15] also proposed the algorithm to yield the maximum number of experts that satisfy the consensus under a given budget B. The method to find the quadratic maximum expert consensus (QMEC) is presented as below:

$$\max_{\bar{o}} \sum_{i=1}^n z_i$$
$$\text{s.t.} \begin{cases} \sum_{i=1}^n c_i \left(\bar{o}_i - o_i \right)^2 \le B \\ z_i = \begin{cases} 1, & \text{if } \bar{o}_i = \bar{o} \\ 0, & \text{else} \end{cases}, i = 1, 2, ..., n \end{cases}$$
$$(M-5)$$

where only when the opinion of an expert is the same as the group opinion (i.e., $\bar{o}_i = \bar{o}$), is the expert confirmed as a member of consensus group ($z_i = 1$).

It is worth noting that, although methods in references [15, 24] are groundbreaking and effective, but neither of them considered aggregation operators, and ignored the internal relationships between the individual opinions and the group one. Thus, only in some special cases are the above methods able to be used. Motivated by this, Zhang, Dong, Xu and Li [31] incorporated aggregation operators into the minimum-cost consensus methods and furnished corresponding linear-programming approaches. By using aggregation operator $F(\bar{o}_1, \bar{o}_2, ..., \bar{o}_n)$ is derived from the adjusted experts' opinions. The minimum cost consensus model under aggregation operators is constructed as:

$$\min_{\bar{o}_i} \sum_{i=1}^n c_i |\bar{o}_i - o_i|$$
$$\text{s.t.} \begin{cases} \left| \bar{o}_i - \bar{o} \right| \le \varepsilon, i = 1, 2, ..., n \\ \bar{o} = F \left(\bar{o}_1, \bar{o}_2, ..., \bar{o}_n \right) \end{cases}$$
$$(M-6)$$

where $|\bar{o}_i - \bar{o}|$ measures the deviation of expert i's opinion from the consensus opinion and $\varepsilon \ge$ is the consensus threshold. When ε (i.e., $\bar{o}_1 = \bar{o}_2 = ... = \bar{o}_n = \bar{o}$), model (M-6) reduces to (M-1). It should be noted that, (M-6) can be solved based on linear-programming models, so there may be multiple solutions in some cases (please see Example 2) and the uniqueness of optimal solutions can't be guaranteed.

Besides, Cheng, Zhou, Cheng, Zhou and Xie [32] investigated the minimum cost consensus model with directional constraints in the asymmetric cost context. Gong, Zhang, Forrest, Li and Xu [2] and Gong, Xu, Zhang, Ozturk, Herrera-Viedma and Xu [33] constructed minimum cost consensus models and maximum return consensus models to balance the interests of both the moderator and the experts. Further, Zhang, Kou and Peng [23] proposed the soft minimum cost consensus models with a consensus level function and a generalized aggregation operator. Zhang, Liang, Gao and Zhang [34] developed a minimum cost consensus model with incomplete linguistic distribution assessments. Recently, Lu, Xu, Herrera-Viedma and Han [35] furnished a minimum cost consensus model based on robust optimization for the large-scale group decision making in social

network. Wu, Dai, Chiclana, Fujita and Herrera-Viedma [36] proposed a minimum adjustment cost based feedback mechanism for network social GDM with distributed linguistic trust. Wu, Yang, Tu and Chen [37] devised multi-stage optimization models with interval additive preference relations. A comparison between the above related works in the existing literatures and the proposed method in this paper is presented in Table 1.

3 Minimum quadratic cost consensus models under aggregation operators

3.1 The proposed model

In various scenarios, different aggregation operators are employed to aggregate experts' opinions into a collective one. Furthermore, utilizing different aggregation operators also impacts the consensus level. Thus, aggregation operators are pivotal to consensus methods.

For an expert, every number denoting his/her opinion has a specific utility value [27]. In the practical decision-making contexts, the utility functions do not just monotonically increase or decrease as experts' opinions change. Instead, the utility functions generally show nonlinear trends, such as parabolic and S-shaped utility functions [26, 27]. In order to convince experts to adjust their original opinions, a moderator compensates experts for the utility they sacrifice by paying consensus costs. Generally, individuals are willing to cooperate with the moderator if the required opinion adjustments are relatively small, while they would show more resistance if the adjustments grow. Considering this psychological factor, this paper herein uses the quadratic cost functions [15], which embody the experts' emotion changes. Although existing literatures mainly adopt the linear consensus cost functions, but they ignored the psychological changes of the experts, which leads to some limitations in practical application. Additionally, the quadratic cost function $f_i = c_i(o_i - \overline{o}_i)^2$ also results in distinct approaches and model properties. Thus, this paper proposes a generalized consensus model

with aggregation operators which aims at minimizing the quadratic costs, i.e.,

$$\min_{o_i} \sum_{i=1}^{n} c_i \left(\overline{o}_i - o_i \right)^2.$$

At the same time, the group consensus is fulfilled under the constraint that each expert's opinion is close enough to the collective opinion, that is

$$|\overline{o}_i - \overline{o}| \leq \varepsilon, i = 1, 2, \ldots, n$$

where $\overline{o}$ is the adjusted group opinion in (M-6).

In this way, a generalized consensus model for minimum quadratic cost is constructed as follows:

$$\min_{\overline{o}_i} \sum_{i=1}^{n} c_i \left(\overline{o}_i - o_i \right)^2 \qquad (M-7)$$

$$\text{s.t.} \begin{cases} \left| \overline{o}_i - \overline{o} \right| \leq \varepsilon, i = 1, 2, \ldots, n \\ \overline{o} = F\left(\overline{o}_1, \overline{o}_2, \ldots, \overline{o}_n \right) \end{cases}$$

Obviously, model (M-7) reduces to (M-3) when $\overline{o}_i = \overline{o}$ for $i = 1,2,\ldots,n$. In addition, it's proved that (M-7) can degrade into (M-4), when utilizing the OWA operator with weight vector $(1/2, 0, \ldots, 0, 1/2)^T$ [31]. Thus, model (M-7) is a generalized version of the consensus method of Ben-Arieh, Easton and Evans [15]. The flowchart of the proposed method is described in Fig. 1.

### 3.2 Minimum cost consensus models under some common operators

It's clear that the proposed consensus model varies according to the aggregation operator. Since the WA operator and the OWA operators [28] are the most common and basic aggregation operators, this section further analyzes (M-7) under these two kinds of operators with quadratic programming models.

(1) WA operator

Table 1 Comparisons between different minimum cost consensus models

References	Cost function	Aggregation operators	Considers unique solution	Consensus measured
This paper	Quadratic	Used	Yes	Soft consensus
[23, 31, 35]	Linear	Used	No	Soft consensus
[36]	Linear	Used	No	Soft consensus
[2, 24, 32, 33]	Linear	Not used	No	Hard consensus
[34]	Linear	Not used	No	Soft consensus
[15]	Quadratic	Not used	Yes	Hard consensus
[37]	Linear	Not used	Yes	Soft consensus
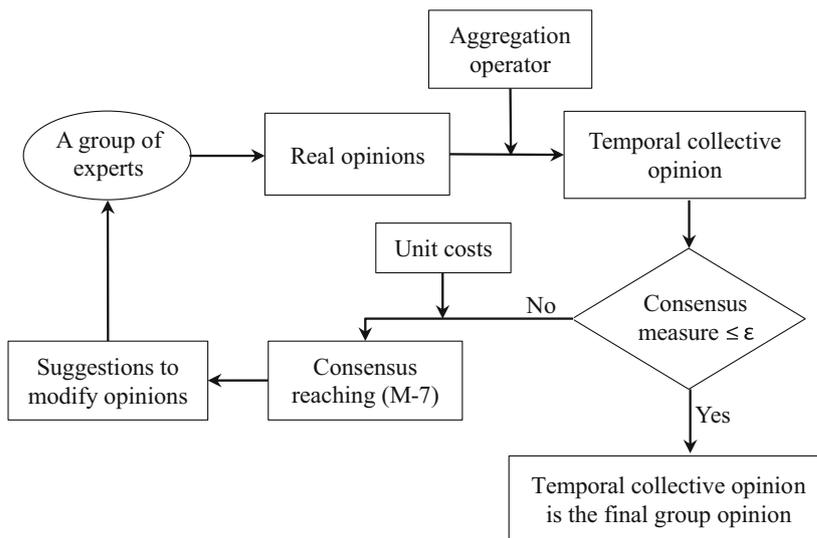
Using WA_w operator to derive the collective opinion from individual opinions, model (M-7) can be described as:

Using WA_w operator to derive the collective opinion from individual opinions,

$$\min_{\overline{o}_i} \sum_{i=1}^{n} c_i \left(\overline{o}_i - o_i\right)^2 \qquad (M-8)$$

$$s.t. \begin{cases} \left|\overline{o}_i - \overline{o}\right| \leq \varepsilon, i = 1, 2, \ldots, n \\ \overline{o} = \sum_{i=1}^{n} w_i \overline{o}_i \end{cases}$$

Denote Ω_8 as the feasible set to (M-8).

Theorem 1. *Let* $x_i = \overline{o}_i$, *then* (M-8) *can be described as the following strictly convex quadratic programming model:*

$$\min_{x_i} \sum_{i=1}^{n} c_i (x_i - o_i)^2 \qquad (M-9)$$

$$s.t. \begin{cases} x_i - \sum_{i=1}^{n} w_i x_i \leq \varepsilon, i = 1, 2, \ldots, n \\ \sum_{i=1}^{n} w_i x_i - x_i \leq \varepsilon, i = 1, 2, \ldots, n \end{cases}$$

Proof. The constraints in (M-9) guarantee that $\overline{o} = \sum_{i=1}^{n} w_i \overline{o}_i$ and $\left|\overline{o}_i - \overline{o}\right| \leq \varepsilon$. Thus, (M-8) can be transformed into (M-9). Let $f(x) = \min_{x_i} \sum_{i=1}^{n} c_i (x_i - o_i)^2$. Since $f(x)$ is the sum of strictly convex functions, it is itself strictly convex. Moreover, the constraints of (M-9) are linear. Hence, (M-8) can be transformed into the strictly convex quadratic programming model. This completes the proof of Theorem 1.

(2) OWA operator

When OWA_w operator is used to aggregate the individual opinions, (M-7) is presented as follows:

$$\min_{\overline{o}_i} \sum_{i=1}^{n} c_i \left(\overline{o}_i - o_i\right)^2$$

$$s.t. \begin{cases} \left|\overline{o}_i - \overline{o}\right| \leq \varepsilon, i = 1, 2, \ldots, n \\ \overline{o} = OWA_w \left(\overline{o}_1, \overline{o}_2, \ldots, \overline{o}_n\right) \end{cases} \qquad (M-10)$$

Let Ω_{10} be the feasible set to (M-10). Given the nonlinear constraint condition $\overline{o} = OWA_w(\overline{o}_1, \overline{o}_2, \ldots, \overline{o}_n)$, model (M-10) can't be straightly transformed into a quadratic programming model. Thus, two specific cases of (M-10) are introduced below.

Case A: The unit adjustment costs of all expert are identical to each other, which means that $c_1 = c_2 = \ldots = c_n = c$. In order to minimize the consensus cost, we hope the modifications to the individual opinions are as small as possible. The consensus model in this case is constructed as model (M-11).

Case B: In many competitions, a frequently used scoring rule is to remove the highest and lowest scores of participants first, and then calculate the average of the remaining scores to obtain the collective evaluation. It's clear that this practice utilizes the OWA_{w_1} operator with weight vector $w_1 = (0, 1/(n-2), \ldots, 1/(n-2), 0)^T$. Considering that some judges may deliberately express extreme opinions for their own or alliance benefits [38], this scoring format can prevent them from undermining the equity of the competition. The corresponding consensus models are shown below.

1) Case A: With the uniform unit cost of each expert, model (M-10) is equivalent to:

$$\min_{x_i} \sum_{i=1}^{n} (x_i - o_i)^2$$
$$s.t. \begin{cases} \left| \overline{o}_i - \overline{o} \right| \leq \varepsilon, i = 1, 2, \ldots, n \\ \overline{o} = OWA_w \left(\overline{o}_1, \overline{o}_2, \ldots, \overline{o}_n \right) \end{cases} \quad (M-11)$$

Denote Ω_{11} the feasible set corresponding to (M-11). Let $x_i = \overline{o}_i$ and $\sigma(1), \sigma(2), \ldots, \sigma(n)$ be a permutation of $\{1, 2, \ldots, n\}$ such that $o_{\sigma(i-1)} \geq o_{\sigma(i)}$ for all $i = 2, 3, \ldots, n$ (i.e., $o_{\sigma(i)}$ is the ith largest value in $\{o_1, o_2, \ldots, o_n\}$). Let $\{\tau(1), \tau(2), \ldots, \tau(n)\}$ be another permutation of $\{1, 2, \ldots, n\}$ such that $x_{\tau(i-1)} \geq x_{\tau(i)}$ for all $i = 2, 3, \ldots, n$. (M-11) can be transformed into the following model:

$$\min_{x_i} \sum_{i=1}^{n} (x_i - o_i)^2 \quad (M-12)$$
$$s.t. \begin{cases} x_i - \sum_{i=1}^{n} w_i x_{\tau(i)} \leq \varepsilon, i = 1, 2, \ldots, n \\ \sum_{i=1}^{n} w_i x_{\tau(i)} - x_i \leq \varepsilon, i = 1, 2, \ldots, n \end{cases}$$

Denote Ω_{12} as the feasible set to (M-12). Before presenting the approach to solve (M-12), we introduce a new model:

$$\min_{x_i} \sum_{i=1}^{n} (x_i - o_i)^2$$
$$s.t. \begin{cases} x_i - \sum_{i=1}^{n} w_i x_{\tau(i)} \leq \varepsilon, i = 1, 2, \ldots, n \\ \sum_{i=1}^{n} w_i x_{\tau(i)} - x_i \leq \varepsilon, i = 1, 2, \ldots, n \\ x_{\sigma(i)} - x_{\sigma(i-1)} \leq 0, i = 2, 3, \ldots, n \end{cases} \quad (M-13)$$

The constraint conditions $x_{\sigma(i)} - x_{\sigma(i-1)} \leq 0$ ($i = 2, 3, \ldots, n$) guarantee that $\tau(i) = \sigma(i)$. Consequently,

$$OWA_w(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} w_i x_{\tau(i)} = \sum_{i=1}^{n} w_i x_{\sigma(i)}.$$

Based on this, model (M-13) is equivalent to the following model:

$$\min_{x_i} \sum_{i=1}^{n} (x_i - o_i)^2$$
$$s.t. \begin{cases} x_i - \sum_{i=1}^{n} w_i x_{\sigma(i)} \leq \varepsilon, i = 1, 2, \ldots, n \\ \sum_{i=1}^{n} w_i x_{\sigma(i)} - x_i \leq \varepsilon, i = 1, 2, \ldots, n \\ x_{\sigma(i)} - x_{\sigma(i-1)} \leq 0, i = 2, 3, \ldots, n \end{cases} \quad (M-14)$$

Let $f_1(x) = \min_{x_i} \sum_{i=1}^{n} (x_i - o_i)^2$. We prove that $f_1(x)$ is a strictly convex function. Besides, $\sigma(i)$ is definite according to $\{o_1, o_2, \ldots, o_n\}$, so the constraints in (M-14) are

Dong, Xu, Li and Feng [39] proposed an OWA-based consensus operator under the continuous linguistic model, finding that the initial opinions of experts can be optimally preserved by their consensus model. The optimal solution to Dong

et al.'s [39] model is obtained by strictly convex quadratic programming. Based on the idea of Dong, Xu, Li and Feng [39], this paper introduces Lemma 1 and Lemma 2 to study the relationship between (M-12) and (M-14).

Lemma 1. Let $o_p, o_q \in \{o_1, o_2, \ldots, o_n\}$ and $\{\overline{o}_1^*, \overline{o}_2^*, \ldots, \overline{o}_n^*, \overline{o}^*\}$ be the optimal solution to (M-12). Then, $\overline{o}_p^* \geq \overline{o}_q^*$ for $o_p > o_q$

Proof. Using reduction to absurdity, suppose that $\overline{o}_q^* > \overline{o}_p^*$. Let a set of values be $\{\ddot{o}_1, \ddot{o}_2, \ldots, \ddot{o}_n, \ddot{o}\}$, where.

$$\ddot{o}_i = \begin{cases} \overline{o}_q^*, i = p \\ \overline{o}_p^*, i = q \\ \overline{o}_i^*, i \neq p, q \end{cases}$$

Since $\ddot{o} = OWA_w(\ddot{o}_1, \ddot{o}_2, \ldots, \ddot{o}_n) = OWA_w(\overline{o}_1^*, \overline{o}_2^*, \ldots, \overline{o}_n^*) = \overline{o}^*$ we have that $\max_i |\ddot{o}_i - \ddot{o}| = \max_i |\overline{o}_i^* - \overline{o}^*| \leq \varepsilon$ holds for all $i = 1, 2, \ldots, n$, so $\{\ddot{o}_1, \ddot{o}_2, \ldots, \ddot{o}_n, \ddot{o}\} \in \Omega_{12}$. Furthermore,

$$\sum_{i=1}^{n} \left(\overline{o}_i^* - o_i \right)^2 - \sum_{i=1}^{n} \left(\ddot{o}_i - o_i \right)^2$$
$$= \left(\overline{o}_p^* - o_p \right)^2 + \left(\overline{o}_q^* - o_q \right)^2 - \left(\ddot{o}_p - o_p \right)^2 - \left(\ddot{o}_q - o_q \right)^2$$
$$= \left(\overline{o}_p^* - o_p \right)^2 + \left(\overline{o}_q^* - o_q \right)^2 - \left(\overline{o}_q^* - o_p \right)^2 - \left(\overline{o}_p^* - o_q \right)^2$$
$$= 2 \left(\overline{o}_p^* - \overline{o}_q^* \right) (o_q - o_p) > 0,$$

which contradicts the fact that $\{\overline{o}_1^*, \overline{o}_2^*, \ldots, \overline{o}_n^*, \overline{o}^*\}$ is the optimal solution to (M-12). This completes the proof of Lemma 1.

Lemma 2. Let $o_p, o_q \in \{o_1, o_2, \ldots, o_n\}$ and $o_p = o_q$. If $\{\overline{o}_1^*, \overline{o}_2^*, \ldots, \overline{o}_n^*, \overline{o}^*\}$ is the optimal solution to (M-12), then $\{\ddot{o}_1, \ddot{o}_2, \ldots, \ddot{o}_n, \ddot{o}\}$ is also the optimal solution to (M-12), where

$$\ddot{o}_i = \begin{cases} \overline{o}_q^*, i = p \\ \overline{o}_p^*, i = q \\ \overline{o}_i^*, i \neq p, q \end{cases}$$

Proof. Based on the proof of Lemma 1, we get that $\{\ddot{o}_1, \ddot{o}_2, \ldots, \ddot{o}_n, \ddot{o}\} \in \Omega_{12}$ and $\sum_{i=1}^{n} (\ddot{o}_i - o_i)^2 = \sum_{i=1}^{n} (\overline{o}_i^* - o_i)^2$. Consequently, $\{\ddot{o}_1, \ddot{o}_2, \ldots, \ddot{o}_n, \ddot{o}\}$ is also the optimal solution to (M-12). This completes the proof of Lemma 2.

Remark 1. Lemma 1 shows the property of order preservation, i.e., if a value is large in the original set, then its adjusted value is still large in the optimal solution. Lemma 2 shows that if a set of values is the solution of (M-12), when two values of the set are interchanged, the interchanged set is still the solution of (M-12).

Based on Lemmas 1 and 2, the following Theorem 2 is obtained.

Theorem 2. *If $\left\{\overline{o}_1^*, \overline{o}_2^*, \ldots, \overline{o}_n^*, \overline{o}^*\right\}$ is the optimal solution to* (M-14), *then* $\left\{\overline{o}_1^*, \overline{o}_2^*, \ldots, \overline{o}_n^*, \overline{o}^*\right\}$ *is the optimal solution to* (M-12).

Proof. Let $\left\{\overline{\overline{o}}_1^*, \overline{\overline{o}}_2^*, \ldots, \overline{\overline{o}}_n^*, \overline{\overline{o}}^*\right\}$ be an optimal solution to (M-12) and $\Omega_{14} \subseteq \Omega_{12}$ denotes the feasible set to (M-14). Since Ω_{14}, $\sum_{i=1}^n\left(\overline{o}_i^* - o_i\right)^2 \geq \sum_{i=1}^n\left(\overline{\overline{o}}_i^* - o_i\right)^2$ is obtained.

Here, we consider the following two cases:

a) $o_i \neq o_j$ for any $o_i, o_j \in \{o_1, o_2, \ldots, o_n\}$. Based on Lemma 1, we have that $\overline{\overline{o}}_{\sigma(i-1)}^* \geq \overline{\overline{o}}_{\sigma(i)}^*$, so $\left\{\overline{\overline{o}}_1^*, \overline{\overline{o}}_2^*, \ldots, \overline{\overline{o}}_n^*, \overline{\overline{o}}^*\right\} \in \Omega_{14}$. Consequently,

$$\sum_{i=1}^n\left(\overline{o}_i^* - o_i\right)^2 \leq \sum_{i=1}^n\left(\overline{\overline{o}}_i^* - o_i\right)^2$$

which means that $\sum_{i=1}^n\left(\overline{o}_i^* - o_i\right)^2 = \sum_{i=1}^n\left(\overline{\overline{o}}_i^* - o_i\right)^2$. Given that the optimal solution to (M-14) exists and is unique, $\left\{\overline{\overline{o}}_1^*, \overline{\overline{o}}_2^*, \ldots, \overline{\overline{o}}_n^*, \overline{\overline{o}}^*\right\} = \left\{\overline{o}_1^*, \overline{o}_2^*, \ldots, \overline{o}_n^*, \overline{o}^*\right\}$.

b) Let $o_p, o_q \in \{o_1, o_2, \ldots, o_n\}$ and $o_p = o_q$. Without loss of generality, we suppose that $o_i \neq o_j$ for any $o_i, o_j \in \{o_1, o_2, \ldots, o_n\}/\{o_p\}$. Lemma 2 guarantees that $\left\{\overset{..*}{o}_1, \overset{..*}{o}_2, \ldots, \overset{..*}{o}_n, \overset{..*}{o}\right\}$ is also the optimal solution to (M-12), where

$$\overset{..*}{o}_i = \begin{cases} \overline{\overline{o}}_q^*, & i = p \\ \overline{\overline{o}}_p^*, & i = q \\ \overline{\overline{o}}_i^*, & i \neq p, q \end{cases}.$$

Based on Lemma 1, we can know that if $o_i \in \{o_1, o_2, \ldots, o_n\}/\{o_p, o_q\}$ is the kth largest variable in $\{o_1, o_2, \ldots, o_n\}$, $\overline{\overline{o}}_i^*/\overset{..*}{o}_i$ is still the kth largest variable in $\left\{\overline{\overline{o}}_1^*, \overline{\overline{o}}_2^*, \ldots, \overline{\overline{o}}_n^*\right\}/\left\{\overset{..*}{o}_1, \overset{..*}{o}_2, \ldots, \overset{..*}{o}_n\right\}$. Therefore, either $\left\{\overline{\overline{o}}_1^*, \overline{\overline{o}}_2^*, \ldots, \overline{\overline{o}}_n^*, \overline{\overline{o}}^*\right\}$ or $\left\{\overset{..*}{o}_1, \overset{..*}{o}_2, \ldots, \overset{..*}{o}_n, \overset{..*}{o}\right\}$ is the feasible solution to (M-14). Without loss of generality, assume that $\left\{\overset{..*}{o}_1, \overset{..*}{o}_2, \ldots, \overset{..*}{o}_n, \overset{..*}{o}\right\} \in \Omega_{14}$. Similar to case a), $\left\{\overset{..*}{o}_1, \overset{..*}{o}_2, \ldots, \overset{..*}{o}_n, \overset{..*}{o}\right\} = \left\{\overline{o}_1^*, \overline{o}_2^*, \ldots, \overline{o}_n^*, \overline{o}^*\right\}$ can be obtained. This completes the proof of Theorem 2.

Theorem 2 guarantees the optimal solution to (M-12) can be obtained through a strictly convex quadratic programming model.

2) Case B: This case utilizes an OWA operator with weight vector $w_1 = (0, 1/(n-2), \ldots, 1/(n-2), 0)^T$. The corresponding model under OWA_{w_1} is presented as follows:

$$\min_{\overline{o}_i} \sum_{i=1}^n c_i\left(\overline{o}_i - o_i\right)^2 \qquad (M-15)$$

$$s.t. \begin{cases} \left|\overline{o}_i - \overline{o}\right| \leq \varepsilon, i = 1, 2, \ldots, n \\ \overline{o} = \frac{1}{n-2}\left(\sum_{i=1}^n \overline{o}_i - \max_i\left\{\overline{o}_i\right\} - \min_i\left\{\overline{o}_i\right\}\right) \end{cases}$$

Denote Ω_{15} as the feasible set corresponding to (M-15). Before solving (M-15), Lemma 3 is introduced.

Lemma 3. *If $\overline{o}_m = \min_i\{\overline{o}_i\}$ and $\overline{o}_h = \max_i\{\overline{o}_i\}$ ($m, h \in \{1, 2, \ldots, n\}$), (M-15) can be described as:*

$$\min_{\overline{o}_i} \sum_{i=1}^n c_i\left(\overline{o}_i - o_i\right)^2$$

$$s.t. \begin{cases} \left|\overline{o}_i - \overline{o}\right| \leq \varepsilon, i = 1, 2, \cdots, n \\ \overline{o} = \frac{1}{n-2}\left(\sum_{i=1}^n \overline{o}_i - \overline{o}_m - \overline{o}_h\right) \\ \overline{o}_m \leq \overline{o}_i, i = 1, 2, \cdots, n \\ \overline{o}_i \leq \overline{o}_h, i = 1, 2, \cdots, n \end{cases} \quad \mathbf{M}\left(\overline{o}_m, \overline{o}_h\right)$$

Proof. In model $M(\overline{o}_m, \overline{o}_h)$, the constraints $\overline{o}_m \leq \overline{o}_i$ ($i = 1, 2, \ldots, n$) and $\overline{o}_h \geq \overline{o}_i$ ($i = 1, 2, \ldots, n$) guarantee that $\overline{o}_m = \min_i\{\overline{o}_i\}$ and $\overline{o}_h = \max_i\{\overline{o}_i\}$, respectively. Hence, $\overline{o} = \frac{1}{n-2}\left(\sum_{i=1}^n \overline{o}_i - \max_i\{\overline{o}_i\} - \min_i\{\overline{o}_i\}\right)$ is identical to $\overline{o} = \frac{1}{n-2}\left(\sum_{i=1}^n \overline{o}_i - \overline{o}_h - \overline{o}_m\right)$. Consequently, (M-15) can be equivalently described as $M(\overline{o}_m, \overline{o}_h)$, when $\overline{o}_m = \min_i\{\overline{o}_i\}$ and $\overline{o}_h = \max_i\{\overline{o}_i\}$. This completes the proof of Lemma 3.

Let $x_i = \overline{o}_i$, then $M(\overline{o}_m, \overline{o}_h)$ ($m = 1, 2, \ldots, n$ and $h = 1, 2, \ldots, n$) can be similarly transformed into the strictly convex quadratic programming models. The relationship between (M-15) and $M(\overline{o}_m, \overline{o}_h)$ is embodied in Theorem 3. Let Ω_{15} and Ω_{mh} be the feasible sets corresponding to (M-15) and $M(\overline{o}_m, \overline{o}_h)$, respectively. Naturally, $\Omega_{mh} \subseteq \Omega_{15}$ holds for all $m, h \in \{1, 2, \ldots, n\}$.

Theorem 3. *Let $\left\{\overline{o}_1^{mh}, \overline{o}_2^{mh}, \ldots, \overline{o}_n^{mh}, \overline{o}^{mh}\right\}$ be the optimal solution to $M(\overline{o}_m, \overline{o}_h)$ and $P_{mh} = \sum_{i=1}^n c_i\left(o_i - \overline{o}_i^{mh}\right)^2$. If $P_{jk} = \min\{P_{mh}|m, h = 1, 2, \ldots, n\}$ ($j, k \in \{1, 2, \ldots, n\}$), the corresponding optimal solution $\left\{\overline{o}_1^{jk}, \overline{o}_2^{jk}, \ldots, \overline{o}_n^{jk}, \overline{o}^{jk}\right\}$ to $M(\overline{o}_j, \overline{o}_k)$ is the optimal solution to (M-15).*

Proof. Let $\left\{\overline{\overline{o}}_1^e, \overline{\overline{o}}_2^e, \ldots, \overline{\overline{o}}_n^e, \overline{\overline{o}}^e\right\}$ be an optimal solution to (M-15). Since $\Omega_{mh} \subseteq \Omega_{15}$ holds for any $m, h \in \{1, 2, \ldots, n\}$, we have that $\sum_{i=1}^n c_i\left(o_i - \overline{o}_i^{jk}\right)^2 \geq \sum_{i=1}^n c_i\left(o_i - \overline{\overline{o}}_i^e\right)^2$.

Simultaneously, the minimum and maximum values ($\overline{o}_m$ and $\overline{o}_h$) of $\left\{\overline{\overline{o}}_1^e, \overline{\overline{o}}_2^e, \ldots, \overline{\overline{o}}_n^e\right\}$ are known, so $\left\{\overline{\overline{o}}_1^e, \overline{\overline{o}}_2^e, \ldots, \overline{\overline{o}}_n^e, \overline{\overline{o}}^e\right\}$ is also a feasible solution to $M(\overline{o}_m, \overline{o}_h)$. Thus, $\sum_{i=1}^n c_i\left(o_i - \overline{o}_i^{jk}\right)^2 \leq \sum_{i=1}^n c_i\left(o_i - \overline{\overline{o}}_i^e\right)^2$.

Consequently, $P_{jk} = \sum_{i=1}^{n} c_i \left(o_i - \overline{o}_i^{jk}\right)^2 = \sum_{i=1}^{n} c_i \left(o_i - \overline{\overline{o}}_i^e\right)^2$.

Therefore, the corresponding optimal solution $\left\{\overline{o}_1^{jk}, \overline{o}_2^{jk}, \ldots, \overline{o}_n^{jk}, \overline{o}^{jk}\right\}$ to $M(\overline{o}_j, \overline{o}_k)$ is also an optimal solution to (M-15). This completes the proof of Theorem 3.

Remark 2. Theorem 3 shows that (M-15) can be solved through $n(n-1)/2$ quadratic programming models (i.e., $M(\overline{o}_m, \overline{o}_h$), where m, $h \in \{1, 2, \ldots, n\}$ and $m \neq h$). Similarly, $n(n-1)/2$ linear programming models can be employed to solve the linear cost version of (M-15) in Zhang, Dong, Xu and Li [31]. It should be noted that, when all c_i ($i = 1, 2, \ldots, n$) are equal, only one quadratic programming model needs to be solved. In this case, (M-15) is classified as Case A of (M-10) where the order preservation property mentioned in Remark 1 holds. Furthermore, we speculate that this property still holds for (M-15) when the differences among c_i ($i = 1, 2, \ldots, n$) are relatively small, yet this conjecture hasn't been proved.

Remark 3. As known to all, there are many classical methods such as Lemke algorithm to solve strictly convex quadratic programming. In this paper, we employ the function "quadprog" of software MATLAB to solve the proposed consensus models so that the optimal solutions can be efficiently obtained.

4 Properties of the models

Section 3 constructs the consensus models for minimum quadratic cost under the WA and OWA operators and reveals the property of order preservation in (M-12). More interesting and desired properties will be shown in this section, which embody the characteristics of the proposed models under the different aggregation operators.

Lemma 4. Let $\{o_1, o_2, \ldots, o_n\}$ and $\left\{\overline{o}_1^*, \overline{o}_2^*, \ldots, \overline{o}_n^*\right\}$ be the original opinions and the modified opinions corresponding to (M-8) or (M-10), respectively. Then, we get that $\min_i \{o_i^*\} \geq \min_i \{o_i\}$ and $\max_i \{o_i^*\} \leq \max_i \{o_i\}$.

Proof. The statement $\min_i \{\overline{o}_i^*\} \geq \min_i \{o_i\}$ is first proved through reduction to absurdity. Assuming that $\exists Q \subseteq \{1, 2, \ldots, n\}$ and $Q = \left\{q | \overline{o}_q^* < \min\{o_i\}\right\}$. Let $\{\ddot{o}_1, \ddot{o}_2, \ldots, \ddot{o}_n, \ddot{o}\}$ be a set of values, where

$$\ddot{o}_i = \begin{cases} \min_i\{o_i\}, i \in Q \\ \overline{o}_i^*, i \notin Q \end{cases}$$

and $\ddot{o}$ represents the aggregated group opinion. Since $|\ddot{o} - \max_i\{\ddot{o}_i\}| \leq \varepsilon$ and $|\ddot{o} - \min_i\{\ddot{o}_i\}| \leq \varepsilon$, then $\{\ddot{o}_1, \ddot{o}_2, \ldots, \ddot{o}_n, \ddot{o}\}$

$\in \Omega_8 / \Omega_{10}$ holds. Moreover,

$$\sum_{i=1}^{n} c_i (o_i - \ddot{o}_i)^2 - \sum_{i=1}^{n} c_i \left(o_i - \overline{o}_i^*\right)^2$$
$$= \sum_{i \in Q} c_i \left(2o_i - \min_i\{o_i\} - \overline{o}_i^*\right)\left(\overline{q}^* - \min_i\{o_i\}\right) < 0,$$

which contradicts the fact that $\left\{\overline{o}_1^*, \overline{o}_2^*, \ldots, \overline{o}_n^*, \overline{o}^*\right\}$ is the optimal solution to model (M-8) or (M-10). Thus, the statement $\min_i\{o_i\} \leq \min_i\{\overline{o}_i^*\}$ holds. Likewise, we have $\max_i\{\overline{o}_i^*\} \leq \max_i\{o_i\}$. This completes the proof of Lemma 4.

Remark 4. Lemma 4 indicates that the adjusted optimal values are among the minimum and maximum values of the original values.

Corollary 1. $\left|\max_i\{\overline{o}_i^*\} - \min_i\{\overline{o}_i^*\}\right| \leq \left|\max_i\{o_i\} - \min_i\{o_i\}\right|$.

Property 1. Let $\left\{\overline{o}_1^*, \overline{o}_2^*, \ldots, \overline{o}_n^*, \overline{o}^*\right\}$ be the optimal solution to (M-8) or (M-10), then

$$\min_i\{o_i\} \leq \overline{o}^* \leq \max_i\{o_i\}.$$

Proof. Since $\overline{o}^* = WA_w\left(\overline{o}_1^*, \overline{o}_2^*, \ldots, \overline{o}_n^*\right)$ in model (M-8) and $\overline{o}^* = OWA_w\left(\overline{o}_1^*, \overline{o}_2^*, \ldots, \overline{o}_n^*\right)$ in model (M-10), $\min_i\{\overline{o}_i^*\} \leq \overline{o}^* \leq \max_i\{\overline{o}_i^*\}$ holds for both models. Based on Lemma 4, we can obtain that $\min_i\{o_i\} \leq \overline{o}^* \leq \max_i\{o_i\}$. This completes the proof of of Property 1.

Property 2. Let $n = 2$ and $\left\{\overline{o}_1^*, \overline{o}_2^*, \overline{o}^*\right\}$ be the optimal solution to (M-8) or (M-10). Then, $\overline{o}_2^* \geq \overline{o}_1^*$ for $o_2 > o_1$.

Proof. We first approve the property of (M-8). Using reduction to absurdity, we assume that $o_2 > o_1$ and $\left\{\overline{o}_1^*, \overline{o}_2^*, \overline{o}^*\right\}$ is the optimal solution to (M-8), where $\overline{o}_1^* > \overline{o}_2^*$. Let $\ddot{o}_1 = \overline{o}_2^*$ and $\ddot{o}_2 = \overline{o}_1^*$, we find that

$$\left|\ddot{o}_1 - \ddot{o}\right| = \left|w_2\left(\overline{o}_1^* - \overline{o}_2^*\right)\right| = \left|\overline{o}_1^* - \overline{o}^*\right| \leq \varepsilon \quad \text{and}$$
$$\left|\ddot{o}_2 - \ddot{o}\right| = \left|w_1\left(\overline{o}_2^* - \overline{o}_1^*\right)\right| = \left|\overline{o}_2^* - \overline{o}^*\right| \leq \varepsilon.$$

Hence, $\{\ddot{o}_1, \ddot{o}_2, \ddot{o}\} \in \Omega_8$. By Lemma 4, we obtain that $2o_1 - \overline{o}_2^* - \overline{o}_1^* < 0$ and $2o_2 - \overline{o}_1^* - \overline{o}_2^* > 0$, therefore

$$\sum_{i=1}^{2} c_i (o_i - \ddot{o}_i)^2 - \sum_{i=1}^{2} c_i \left(o_i - \overline{o}_i^*\right)^2$$
$$= c_1\left(2o_1 - \overline{o}_2^* - \overline{o}_1^*\right)\left(\overline{o}_1^* - \overline{o}_2^*\right) + c_2\left(2o_2 - \overline{o}_1^* - \overline{o}_2^*\right)\left(\overline{o}_2^* - \overline{o}_1^*\right) < 0,$$

which contradicts the assumption that $\left\{\overline{o}_1^*, \overline{o}_2^*, \overline{o}^*\right\}$ is the optimal solution to (M-8). Likewise, we validate Property 2 holds for (M-10). This completes the proof of Property 2.

Remark 5. When $n \geq 3$, the ranking of $\overline{o}_i$ in $\{\overline{o}_1^*, \overline{o}_2^*, ..., \overline{o}_n^*\}$ may be different from that of o_i in $\{o_1, o_2, ..., o_n\}$. Thus, Property 2 is valid only when $n = 2$.

Property 3. Let $\{\overline{o}_1^*, \overline{o}_2^*, ..., \overline{o}_n^*, \overline{o}^*\}$ be the optimal solution to (M-10), $o_h \in \{o_1, o_2, ..., o_n\}$ such that $o_q = \min_i\{o_i\} < o_h < o_p = \max_i\{o_i\}$, $c_h \leq c_p$ and $c_h \leq c_q$. Then, $\overline{o}_q^* \leq \overline{o}_h^* \leq \overline{o}_p^*$.

Proof. We first approve that $\overline{o}_h^* \leq \overline{o}_p^*$. Using reduction to absurdity, we assume that $\overline{o}_h^* > \overline{o}_p^*$. Let a feasible solution be $\{\ddot{o}_1, \ddot{o}_2, ..., \ddot{o}_n, \ddot{o}\}$, where

$$\ddot{o}_i = \begin{cases} \overline{o}_h^*, i = p \\ \overline{o}_p^*, i = h, \\ \overline{o}_i^*, i \neq h, p \end{cases}$$

we find that

$$\sum_{i=1}^n c_i\left(o_i - \ddot{o}_i\right)^2 - \sum_{i=1}^n c_i\left(o_i - \overline{o}_i^*\right)^2$$
$$= c_h\left(o_h - \ddot{o}_h\right)^2 + c_p\left(o_p - \ddot{o}_p\right)^2 - c_h\left(o_h - \overline{o}_h^*\right)^2 - c_p\left(o_p - \overline{o}_p^*\right)^2$$
$$= \left(\overline{o}_p^* - \overline{o}_h^*\right)\left[c_p\left(2o_p - \overline{o}_h^* - \overline{o}_p^*\right) - c_h\left(2o_h - \overline{o}_h^* - \overline{o}_p^*\right)\right]$$

Let $c_p = c_h + \Delta c$ and $\Delta c \geq 0$, then we have.

$$\left(\overline{o}_p^* - \overline{o}_h^*\right)\left[c_p\left(2o_p - \overline{o}_h^* - \overline{o}_p^*\right) - c_h\left(2o_h - \overline{o}_h^* - \overline{o}_p^*\right)\right]$$
$$= \left(\overline{o}_p^* - \overline{o}_h^*\right)\left[\Delta c\left(2o_p - \overline{o}_h^* - \overline{o}_p^*\right) + 2c_h\left(o_p - o_h\right)\right] < 0,$$

which contradicts the fact that $\{\overline{o}_1^*, \overline{o}_2^*, ..., \overline{o}_n^*, \overline{o}^*\}$ is the optimal solution to (M-10).

Therefore, $\overline{o}_h^* \leq \overline{o}_p^*$. Likewise, we prove that $\overline{o}_q^* \leq \overline{o}_h^*$ holds. This completes the proof of Property 3.

Corollary 2. Let $\{o_1, o_2, o_3\}$ be the original opinions such that $o_1 < o_2 < o_3$ and $c_2 = \min_i\{c_i\}$. Then, $\overline{o}_1^* \leq \overline{o}_2^* \leq \overline{o}_3^*$. In this case, (M-10) can be transform into a strictly convex programming model.

Property 4. In (M-10), let o_p, $o_q \in \{o_1, o_2, ..., o_n\}$ and $\{\overline{o}_1^*, \overline{o}_2^*, ..., \overline{o}_n^*, \overline{o}^*\}$ be the optimal solution to (M-10). Then, when $c_p = c_q$,
(1) if $o_p > o_q$, $\overline{o}_p^* \geq \overline{o}_q^*$;
(2) if $o_p = o_q$, $\{\ddot{o}_1, \ddot{o}_2, ..., \ddot{o}_n, \ddot{o}\}$ is also the optimal solution to (M-10), where

$$\ddot{o}_i = \begin{cases} \overline{o}_q^*, i = p \\ \overline{o}_p^*, i = q \\ \overline{o}_i^*, i \neq p, q \end{cases}$$

Proof. Using reduction to absurdity, we first prove part (1). Assuming $\overline{o}_p^* < \overline{o}_q^*$ and $\{o_1', o_2', ..., o_n', o'\}$ is a feasible solution to (M-10), where

$$o_i' = \begin{cases} \overline{o}_q^*, i = p \\ \overline{o}_p^*, i = q \\ \overline{o}_i^*, i \neq p, q \end{cases}.$$

Then we have that $\sum_{i-1}^n c_i\left(o_i - o_i'\right)^2 - \sum_{i-1}^n c_i\left(o_i - \overline{o}_i^*\right)^2 = 2c_p\left(\overline{o}_p^* - \overline{o}_q^*\right)\left(o_p - o_q\right) < 0$, which is contrary to the assumption that $\{\overline{o}_1^*, \overline{o}_2^*, ..., \overline{o}_n^*, \overline{o}^*\}$ is the optimal solution to (M-10). Therefore, part (1) is valid and it is straightforward to prove part (2). This completes the proof of Property 4.

Property 5. *Monotonicity.* For model (M-8) and (M-10), let $\{o_1, o_2\}$ be a set of original opinions and let $\{v_1, v_2\}$ be a second set of original opinions, such that $o_i > v_i$, for $i = 1, 2$. Then, $\overline{o}_i^* \geq \overline{v}_i^*$, for $i = 1, 2$.

Proof. Without loss of generality, we assume that $o_2 > o_1$ and $v_2 > v_1$. Using reduction to absurdity, let $\overline{o}_1^* < \overline{v}_1^*$ or $\overline{o}_2^* < \overline{v}_2^*$. We consider the following three cases:

Case A: $\overline{o}_1^* < \overline{v}_1^*$ and $\overline{o}_2^* < \overline{v}_2^*$. In this case,

$$\sum_{i=1}^2 c_i\left(o_i - \overline{o}_i^*\right)^2 - \sum_{i=1}^2 c_i\left(o_i - \overline{v}_i^*\right)^2$$
$$= c_1\left(2o_1 - \overline{o}_1^* - \overline{v}_1^*\right)\left(\overline{v}_1^* - \overline{o}_1^*\right) + c_2\left(2o_2 - \overline{o}_2^* - \overline{v}_2^*\right)\left(\overline{v}_2^* - \overline{o}_2^*\right)$$
$$> c_1\left(2v_1 - \overline{o}_1^* - \overline{v}_1^*\right)\left(\overline{v}_1^* - \overline{o}_1^*\right) + c_2\left(2v_2 - \overline{o}_2^* - \overline{v}_2^*\right)\left(\overline{v}_2^* - \overline{o}_2^*\right)$$
$$= \sum_{i=1}^2 c_i\left(v_i - \overline{o}_i^*\right)^2 - \sum_{i=1}^2 c_i\left(v_i - \overline{v}_i^*\right)^2 \geq 0,$$

which contradicts the fact that $\{\overline{o}_1^*, \overline{o}_2^*, \overline{o}^*\}$ is the optimal adjusted opinions corresponding to $\{o_1, o_2\}$, when using (M-8) or (M-10).

Case B: $\overline{o}_1^* < \overline{v}_1^*$ and $\overline{o}_2^* \geq \overline{v}_2^*$. Under the circumstance, let $v_1' = \overline{o}_1^*$ and $v_2' = \min\{\overline{o}_2^*, v_2\}$. Then, $\{v_1', v_2', v'\} \in \Omega_8/\Omega_{10}$ and $\sum_{i=1}^2 c_i\left(v_i - v_i'\right)^2 < \sum_{i=1}^2 c_i\left(v_i - \overline{v}_i^*\right)^2$ hold, which is contrary to that $\{\overline{v}_1^*, \overline{v}_2^*, \overline{v}\}$ is the optimal solution corresponding to $\{v_1, v_2\}$, when using (M-8) or (M-10).

Case C: $\overline{o}_1^* \geq \overline{v}_1^*$ and $\overline{o}_2^* < \overline{v}_2^*$. Let $o_1' = \max\{o_1, \overline{v}_1^*\}$ and $o_2' = \overline{v}_2^*$. We prove that $\{o_1', o_2', o'\} \in \Omega_8/\Omega_{10}$ and $\sum_{i=1}^2 c_i\left(o_i - o_i'\right)^2 < \sum_{i=1}^2 c_i\left(o_i - \overline{o}_i^*\right)^2$, which contradicts the fact that $\{\overline{o}_1^*, \overline{o}_2^*, \overline{o}^*\}$ is the optimal solution corresponding to $\{o_1, o_2\}$, when using (M-8) or (M-10).

Based on the above three cases, we get that $\overline{o}_1^* \geq \overline{v}_1^*$ and $\overline{o}_2^* \geq \overline{v}_2^*$. This completes the proof of Property 5.

Remark 6. This paper argues that Property 5 holds when $n > 2$, while it's still an open problem when $n \geq 3$.

Property 6. *Let* $\{d_1, d_2, …, d_n\}$ *and* $\{o_1, o_2, …, o_n\}$ *be two sets of experts' initial opinions,* $\left\{\overline{d}_1^*, \overline{d}_2^*, …, \overline{d}_n^*, \overline{d}^*\right\}$ *and* $\left\{\overline{o}_1^*, \overline{o}_2^*, …, \overline{o}_n^*, \overline{o}^*\right\}$ *be the corresponding optimal solutions in model* (M-10). *If* $c_p = c_q$, *then*

$$\sum_{i=1}^n c_i\left(o_i - \overline{o}_i^*\right)^2 = \sum_{i=1}^n c_i\left(d_i - \overline{d}_i^*\right)^2,$$

where

$$d_i = \begin{cases} o_q, i = p \\ o_p, i = q \\ o_i, i \neq p, q \end{cases} \quad and \quad \overline{d}_i^* = \begin{cases} \overline{o}_q^*, i = p \\ \overline{o}_p^*, i = q \\ \overline{o}_i^*, i \neq p, q \end{cases}.$$

Corollary 3. *Commutativity. Let* $\{d_1, d_2, …, d_n\}$ *be a permutation of* $\{o_1, o_2, …, o_n\}$ *in* (M-11). *Then* $\overline{d}^* = \overline{o}^*$.

Property 7. *Let* $\{o_1, o_2, …, o_n\}$ *be a set of original opinions and* $\{l_1, l_2, …, l_n\}$ *be a second set of original opinions. Denote* $\left\{\overline{o}_1^*, \overline{o}_2^*, …, \overline{o}_n^*, \overline{o}^*\right\}$ *and* $\left\{\overline{l}_1^*, \overline{l}_2^*, …, \overline{l}_n^*, \overline{l}^*\right\}$ *as the corresponding optimal solutions in* (M-8). *If* $o_i + l_i = C$ (*C is a constant*) *for all* $i = 1, 2, …, n$, *then* $\overline{o}^* + \overline{l}^* = C$ *and* $\overline{o}_i^* + \overline{l}_i^* = C$ *for all* $i = 1, 2, …, n$.

Proof. Denote the corresponding feasible sets of $\{o_1, o_2, …, o_n\}$ and $\{l_1, l_2, …, l_n\}$ as Ω_{8o} and Ω_{8l}, respectively. Let $\ddot{l}_i = C - \overline{o}_i^*$ for all $i = 1, 2, …, n$, then we can get that

$$\left|\ddot{l}_i - \ddot{l}\right| = \left|\overline{o}_i^* - \overline{o}^*\right| \leq \varepsilon.$$

Thus $\{\ddot{l}_1, \ddot{l}_2, …, \ddot{l}_n, \ddot{l}\} \in \Omega_{8l}$ and

$$\sum_{i=1}^n c_i\left(l_i - \overline{l}_i^*\right)^2 \leq \sum_{i=1}^n c_i\left(l_i - \ddot{l}_i\right)^2 = \sum_{i=1}^n c_i\left(o_i - \overline{o}_i^*\right)^2.$$

Similarly, let $\ddot{o}_i = C - \overline{l}_i^*$ for all $i = 1, 2, …, n$. Thus, $\{\ddot{o}_1, \ddot{o}_2, …, \ddot{o}_n, \ddot{o}\} \in \Omega_{8o}$ holds and

$$\sum_{i=1}^n c_i\left(o_i - \overline{o}_i^*\right)^2 \leq \sum_{i=1}^n c_i\left(o_i - \ddot{o}_i^*\right)^2 = \sum_{i=1}^n c_i\left(l_i - \overline{l}_i^*\right)^2.$$

Consequently, $\sum_{i=1}^n c_i\left(l_i - \overline{l}_i^*\right)^2 = \sum_{i=1}^n c_i\left(o_i - \overline{o}_i^*\right)^2$. Thus, $\overline{o}_i^* + \overline{l}_i^* = C$ for all $i = 1, 2, …, n$ and $\overline{o}^* + \overline{l}^* = C$. This completes the proof of Property 7.

Property 8. *Let* $\{o_1, o_2, …, o_n\}$ *be a set of original opinions and* $\{d_1, d_2, …, d_n\}$ *be a second set of original opinions. Denote* $\left\{\overline{o}_1^*, \overline{o}_2^*, …, \overline{o}_n^*, \overline{o}^*\right\}$ *and* $\left\{\overline{d}_1^*, \overline{d}_2^*, …, \overline{d}_n^*, \overline{d}^*\right\}$ *as the as the corresponding optimal solutions in* (M-14). *If* $w_i = w_{n+1-i}$ *and*

$o_i + d_i = C$ (*C is a constant.*) *for all* $i = 1, 2, …, n$, *then* $\overline{o}^* + \overline{d}^* = C$ *and* $\overline{o}_i^* + \overline{d}_i^* = C$ *for all* $i = 1, 2, …, n$.

Proof: Denote the corresponding feasible sets of $\{o_1, o_2, …, o_n\}$ and $\{d_1, d_2, …, d_n\}$ as Ω_{14o} and Ω_{14d} in (M-14). Let $\ddot{d}_i = C - \overline{o}_i^*$ for all $i = 1, 2, …, n$, then we can derive that

$$\left|\ddot{d}_i - \ddot{d}\right| = \left|\overline{o}_i^* - \sum_{i=1}^n \left(w_{n+1-i}\overline{o}_{\sigma^*(i)}\right)\right| = \left|\overline{o}_i^* - \overline{o}^*\right| \leq \varepsilon.$$

Thus, $\{\ddot{d}_1, \ddot{d}_2, …, \ddot{d}_n, \ddot{d}\} \in \Omega_{14d}$ and

$$\sum_{i=1}^n c_i\left(d_i - \overline{d}_i^*\right)^2 \leq \sum_{i=1}^n c_i\left(d_i - \ddot{d}_i\right)^2 = \sum_{i=1}^n c_i\left(o_i - \overline{o}_i^*\right)^2.$$

Similarly, let $\ddot{o}_i = C - \overline{d}_i^*$. We prove that $\{\ddot{o}_1, \ddot{o}_2, …, \ddot{o}_n, \ddot{o}\} \in \Omega_{14o}$ and

$$\sum_{i=1}^n c_i\left(o_i - \overline{o}_i^*\right)^2 \leq \sum_{i=1}^n c_i\left(o_i - \ddot{o}_i\right)^2 = \sum_{i=1}^n c_i\left(d_i - \overline{d}_i^*\right)^2.$$

Then we get that $\sum_{i=1}^n c_i\left(d_i - \overline{d}_i^*\right)^2 = \sum_{i=1}^n c_i\left(o_i - \overline{o}_i^*\right)^2$. Therefore $\overline{o}_i^* + \overline{d}_i^* = C$ for all $i = 1, 2, …, n$ and $\overline{o}^* + \overline{d}^* = C$. This completes the proof of Property 8. In the same way, we can prove that $\{\ddot{d}_1, \ddot{d}_2, …, \ddot{d}_n, \ddot{d}\}$ is also the optimal solution to (M-15).

5 Illustrative examples and comparative analyses

In this section, we will offer three illustrative examples to demonstrate how the proposed models work in practice. In order to make a comparison with the methods in Ben-Arieh, Easton and Evans [15], Ben-Arieh and Easton [24], and Zhang, Dong, Xu and Li [31], this paper sets the same parameters and thresholds.

(1) In Example 1, we apply the proposed methods to develop the government's subsidy strategies for encouraging migrants to stay local during Spring Festival. Additionally, the results of CMQC, ε CMQC, and QMEC in Ben-Arieh, Easton and Evans [15] and the proposed methods are also numerically compared.

(2) In Example 2, we mainly make a comparison with the existing linear cost consensus methods introduced by Ben-Arieh and Easton [24] and Zhang, Dong, Xu and Li [31].

(3) The last example involves the GDM problem of house purchase, the data of which is from Zhang, Dong, Xu and Li [31].

5.1 Example 1

In January 2020, COVID-19 broke out in Wuhan, and then swept across the whole country with unprecedented hit to Chinese industries. Since April 29, 2020, the positive momentum in COVID-19 control has been locked in, and nationwide virus control in China is now being conducted on an ongoing basis.

Faced with the approaching Spring Festival in 2021, many places released rules for homecomings and persuaded employees to stay where they work during the holiday to reduce personnel flow and risks of epidemic prevention and control. In response to the call of the state, many people chose to stay put during Spring Festival. Yet they had to sacrifice the annual rights of family gathering. Thus, the subsidy policies introduced by the localities could be vital help in guaranteeing workers' welfare and enterprises' production. Every migrant might expect different subsidy according to their own conditions, such as the affordability of tenancy in large cities, the willingness to work during the holiday and so forth. Thus, a government needs to negotiate with the workers in advance so as to provide reasonable subsidy strategies that satisfy the migrants' expectation.

In the negotiation process, the local government plays the role of a moderator while employees who expect different levels of subsidies are the decision makers with divergent opinions. Suppose that the government could provide the basic allowance of at least 800 (Unit: RMB) to every migrant and there are mainly four different levels of expectations on the subsidy, denoted as e_1, e_2, e_3, and e_4. Compared with the basic allowance, the extra subsidies expected by the four types are: $\{o_1, o_2, o_3, o_4\} = \{0, 3, 6, 10\}$ (Unit: %), and the corresponding weights are $w_1 = 0.3$, $w_2 = 0.1$, $w_3 = 0.4$, and $w_4 = 0.2$, respectively. Suppose the unit adjustment costs of e_1, e_2, e_3, and e_4 are $c_1 = 1$, $c_2 = 2$, $c_3 = 3$, and $c_4 = 1$ (Unit: 10RMB), respectively. To mitigate the difference among subsidy levels $\{o_1, o_2, o_3, o_4\}$, set the consensus threshold $\varepsilon = 0.8$ (Unit: %). Namely, the maximum deviation between any level of subsidy and the collective one is no more than $0.008 \times 800 = 6.4$ (RMB).

(1) Let the final collective opinion and individual opinions after adjustments be o^* and o_i^* ($i = 1, 2, 3, 4$), respectively. If we use the WA operator to assemble the employees' opinions,

according to Theorem 1, (M-8) is equivalent to the following quadratic programming model:

$$\min f = x_1^2 + 2(x_2-3)^2 + 3(x_3-6)^2 + (x_4-10)^2$$
$$\text{s.t.} \begin{cases} 0.7x_1 & -0.1x_2 & -0.4x_3 & -0.2x_4 & \leq 0.8 \\ -0.3x_1 & +0.9x_2 & -0.4x_3 & -0.2x_4 & \leq 0.8 \\ -0.3x_1 & -0.1x_2 & +0.6x_3 & -0.2x_4 & \leq 0.8 \\ -0.3x_1 & -0.1x_2 & -0.4x_3 & +0.8x_4 & \leq 0.8 \\ -0.7x_1 & +0.1x_2 & +0.4x_3 & +0.2x_4 & \leq 0.8 \\ 0.3x_1 & -0.9x_2 & +0.4x_3 & +0.2x_4 & \leq 0.8 \\ 0.3x_1 & +0.1x_2 & -0.6x_3 & +0.2x_4 & \leq 0.8 \\ 0.3x_1 & +0.1x_2 & +0.4x_3 & -0.8x_4 & \leq 0.8 \end{cases}$$

$$(M - 16)$$

The unique optimal solution to (M-16) is $X^* = (4.11, 4.11, 5.31, 5.71)^T$ and the optimal value of the objective function is 39.19. Thus, for the government, the final subsidy policy provided to $\{e_1, e_2, e_3, e_4\}$ is $\{833, 833, 842, 846\}$ (Unit: RMB) and the minimum consensus cost is 391.9 RMB. Meanwhile, since the group consensus opinion is $\bar{o}^* = (0.3, 0.1, 0.4, 0.2)X^* = 4.91\%$, the collective subsidy standard is 839 RMB per person.

Let $\varepsilon = 0.8$. Under the different unit cost vectors (c_1, c_2, c_3, c_4), the final opinions of the employees and the minimum consensus costs in (M-8) are summarized in Table 2.

(2) If the OWA operator with weight vector $w = (0.4, 0.3, 0.2, 0.1)^T$ is used to aggregate the dispersive original opinions, (M-10) can be applied to achieve consensus. In (M-10), the group collective opinion is the ordered weighted average of the individual opinions sorted in descending order and assigned with importance weights of 0.4, 0.3, 0.2, and 0.1, respectively. If $\varepsilon = 0.8$, then grounded on, then grounded on (M-10), we have that

$$\min f = x_1^2 + 2(x_2-3)^2 + 3(x_3-6)^2 + (x_4-10)^2$$
$$\text{s.t.} \begin{cases} 0.6x_{\tau(1)} & -0.3x_{\tau(2)} & -0.2x_{\tau(3)} & -0.1x_{\tau(4)} & \leq 0.8 \\ -0.4x_{\tau(1)} & +0.7x_{\tau(2)} & -0.2x_{\tau(3)} & -0.1x_{\tau(4)} & \leq 0.8 \\ -0.4x_{\tau(1)} & -0.3x_{\tau(2)} & +0.8x_{\tau(3)} & -0.1x_{\tau(4)} & \leq 0.8 \\ -0.4x_{\tau(1)} & -0.3x_{\tau(2)} & -0.2x_{\tau(3)} & +0.9x_{\tau(4)} & \leq 0.8 \\ -0.6x_{\tau(1)} & +0.3x_{\tau(2)} & +0.2x_{\tau(3)} & +0.1x_{\tau(4)} & \leq 0.8 \\ 0.4x_{\tau(1)} & -0.7x_{\tau(2)} & +0.2x_{\tau(3)} & +0.1x_{\tau(4)} & \leq 0.8 \\ 0.4x_{\tau(1)} & +0.3x_{\tau(2)} & -0.8x_{\tau(3)} & +0.1x_{\tau(4)} & \leq 0.8 \\ 0.4x_{\tau(1)} & +0.3x_{\tau(2)} & +0.2x_{\tau(3)} & -0.9x_{\tau(4)} & \leq 0.8 \\ x_{\tau(2)} - x_{\tau(1)} \leq 0 \\ x_{\tau(3)} - x_{\tau(2)} \leq 0 \\ x_{\tau(4)} - x_{\tau(3)} \leq 0 \end{cases}$$

$$(M - 17)$$

Table 2 Optimal solutions to (M-8) under different cost vectors

(c_1, c_2, c_3, c_4)	$\bar{o}_1^*$	$\bar{o}_2^*$	$\bar{o}_3^*$	$\bar{o}_4^*$	$\bar{o}^*$	f_c
(1, 2, 3, 1)	4.11	4.11	5.31	5.71	4.91	39.19
(1, 4, 3, 5)	5.26	5.26	6.46	6.86	6.06	98.03
(6, 3, 4, 1)	2.61	2.61	3.81	4.21	3.41	94.04
(3, 6, 4, 1)	3.26	3.26	4.46	4.86	4.06	68.19

The constraint conditions $x_{\tau(i)} - x_{\tau(i-1)} \leq 0$, $(i = 2, 3, 4)$ guarantee that $x_{\tau(i)}$ is the ith largest value in the adjusted opinions $\{x_1, x_2, x_3, x_4\}$ (i.e., $\{\bar{o}_1, \bar{o}_2, \bar{o}_3, \bar{o}_4\}$). Given that the real rankings of the opinions are unknown after being modified by (M-17), we need to consider all probable cases of order positions. Let P indicate the set of all the permutations on $\{1, 2, 3, 4\}$, then $|P| = 24$. By inputting every possible permutation into (M-17) and comparing the corresponding optimal values, (M-17) is solved. We get that the unique optimal solution is $X^* = (4.38, 4.38, 4.98, 5.92)^T$ and the optimal value is $f_c = 42.68$. Thus, during the whole consensus process, the subsidies paid by the government should increase by $o_1^* = 4.38\%$, $o_2^* = 4.38\%$, $o_3^* = 4.98\%$, and $o_4^* = 5.92\%$, respectively. Consequently, the optimal subsidies paid to e_1, e_2, e_3, and e_4 are 835, 835, 840, and 847 RMB, respectively. In addition, since the final group opinion is $o^* = (0.4, 0.3, 0.2, 0.1)X^* = 5.18\%$, the collective subsidy for every employees equal to 841 RMB.

If we only transform the unit costs and preserve the remaining arguments, the unique optimal opinions provided by (M-10) can also be obtained, as shown in Table 3.

Results in Tables 1 and 2 validate Property 1 of (M-8) and (M-10). That is, the adjusted collective opinion is always within the scope of initial opinions. Based on Table 3, we observe that the order positions of the adjusted opinions are changed when $(c_1, c_2, c_3, c_4) = (6, 3, 4, 1)$. Thus, the property of order preservation is not valid for (M-10), when the unit costs are not uniform among experts. Besides, the result of (M-10) under $(c_1, c_2, c_3, c_4) = (1, 2, 3, 1)$ is consistent with Property 4 (1). To be specific, for any two experts with identical unit costs, if an initial opinion is larger, it will still be the larger one after being modified by (M-10).

(3) In this case, we use model (M-11), the first special case of (M-10), where the unit cost of each expert modifying

his/her opinion is the same as each other. Let the associated weight vector of the OWA operator in (M-11) be $w = (0.3, 0.1, 0.4, 0.2)^T$ and the deviation threshold $\varepsilon = 0.8$. According to the property of order preservation, $\bar{o}_4 \geq \bar{o}_3 \geq \bar{o}_2 \geq \bar{o}_1$ holds, thus $\bar{o} = 0.3\bar{o}_4 + 0.1\bar{o}_3 + 0.4\bar{o}_2 + 0.2\bar{o}_1$. Then, based on Theorem 2, we get the following quadratic programming model:

$$\min f = x_1^2 + 2(x_2 - 3)^2 + (x_3 - 6)^2 + (x_4 - 10)^2$$
$$\text{s.t.} \begin{cases} 0.8x_1 & -0.4x_2 & -0.1x_3 & -0.3x_4 & \leq 0.8 \\ -0.2x_1 & +0.6x_2 & -0.1x_3 & -0.3x_4 & \leq 0.8 \\ -0.2x_1 & -0.4x_2 & +0.9x_3 & -0.3x_4 & \leq 0.8 \\ -0.2x_1 & -0.4x_2 & -0.1x_3 & +0.7x_4 & \leq 0.8 \\ -0.8x_1 & +0.4x_2 & +0.1x_3 & +0.3x_4 & \leq 0.8 \\ 0.2x_1 & -0.6x_2 & +0.1x_3 & +0.3x_4 & \leq 0.8 \\ 0.2x_1 & +0.4x_2 & -0.9x_3 & +0.3x_4 & \leq 0.8 \\ 0.2x_1 & +0.4x_2 & +0.1x_3 & -0.7x_4 & \leq 0.8 \\ x_1 - x_2 \leq 0 \\ x_2 - x_3 \leq 0 \\ x_3 - x_4 \leq 0 \end{cases} \quad (M-18)$$

The unique optimal solution to (M-18) is $X^* = (3.85, 4.25, 5.45, 5.45)^T$ and the optimal value is $f_c = 37.97..$ Therefore, the government could guide the employees to change their opinions into $\bar{o}_1^* = 3.85\%$, $\bar{o}_2^* = 4.25\%$, $\bar{o}_3^* = 5.45\%$, and $\bar{o}_4^* = 5.45\%$ $\{e_1, e_2, e_3, e_4\}$ can receive are $\{831, 834, 844, 844\}$ (Unit: RMB) and the collective level of allowance is 837 RMB based on $\bar{o}^* = 4.65\%$.

Table 4 lists the corresponding results obtained by (M-11) under the different consensus thresholds. Obviously, the lower the consensus degree is, the fewer opinion changes of experts are, which means less consensus cost has to be paid. Thus, the costs also have great impacts on the final individual consensus values.

(4) Here we utilize model (M-15), another special case of the consensus models under OWA operators, where the

Table 3 Optimal solutions to (M-10) under different cost vectors

(c_1, c_2, c_3, c_4)	$\bar{o}_1^*$	$\bar{o}_2^*$	$\bar{o}_3^*$	$\bar{o}_4^*$	$\bar{o}^*$	f_c
(1, 2, 3, 1)	4.38	4.38	4.98	5.92	5.18	42.68
(1, 4, 3, 5)	5.42	5.42	5.95	7.02	6.22	97.21
(6, 3, 4, 1)	2.64	2.64	3.89	3.66	3.44	100.46
(3, 6, 4, 1)	3.31	3.31	4.45	4.45	4.11	73.79

Table 4 Optimal solutions to (M-11) under different consensus thresholds

ε	$\bar{o}_1^*$	$\bar{o}_2^*$	$\bar{o}_3^*$	$\bar{o}_4^*$	$\bar{o}^*$	f_c
0.5	4.19	4.44	5.19	5.19	4.69	43.42
0.6	4.08	4.38	5.28	5.28	4.68	41.35
0.7	3.96	4.31	5.36	5.36	4.66	39.34
0.8	3.85	4.25	5.45	5.45	4.65	37.39

collective opinion is the arithmetic mean of the remaining values after deleting the maximum and minimum of individual opinions. Thus, the aggregation operator used by (M-15) in this case is the OWA operator with weight vector $w = (0, 1/2, 1/2, 0)^T$. Suppose that $x_1 = max_i\{x_i\}$, and $x_2 = min_i\{x_i\}$, then (M-15) can be transformed into the following model:

$$\min f = x_1^2 + 2(x_2-3)^2 + 3(x_3-6)^2 + (x_4-10)^2$$
$$\text{s.t.} \begin{cases} x_1-0.5x_3-0.5x_4 \leq 0.8 \\ x_2-0.5x_3-0.5x_4 \leq 0.8 \\ 0.5x_3-0.5x_4 \leq 0.8 \\ -0.5x_3+0.5x_4 \leq 0.8 \\ -x_1+0.5x_3+0.5x_4 \leq 0.8 \\ -x_2+0.5x_3+0.5x_4 \leq 0.8 \\ -0.5x_3 \quad +0.5x_4 \leq 0.8 \\ 0.5x_3-0.5x_4 \leq 0.8 \\ -x_1+x_i \leq 0, \quad i=2,3,4 \\ x_2-x_i \leq 0, \quad i=1,3,4 \end{cases}$$
$$(M-19)$$

Based on Theorem 3, we need to solve another five models which assume the index tuples of $max_i\{o_i\}$ and $min_i\{o_i\}$ to be (1, 3), (1, 4), (2, 3), (2, 4), and (3, 4), respectively. Through comparing the optimal values of the six models, we get that the optimal solution to (M-19) is $X^* = (3.94, 3.94, 4.54, 4.54)^T$ and the optimal value is $f_c = 37.82$. Thus, the local government would provide the allowances of {832, 832, 836, 836} (RMB) to $\{e_1, e_2, e_3, e_4\}$ and pay the cost of 378.2 RMB to convince the employees. In addition, the collective subsidy obtained by (M-19) is 838 RMB under the adjusted group opinion $o^* = 4.74\%$.

Using (M-15) to aggregate individual opinions under the different unit costs c_i ($i = 1, 2, 3, 4$), we get the corresponding results as shown in Table 5. It's observed that under these four cost vectors, all the order positions of the adjusted opinions $\bar{o}_i^*$ ($i = 1, 2, 3, 4$) in (M-15) coincide with the initial ones. However, it's still an open problem to figure out the particular conditions under which the order preservation property holds for (M-15).

(5) If the relevant importance of the employees is assumed to be the same, then the group collective opinion is the arithmetic mean of the four kinds of opinions. Obviously, the aggregation operator used in this situation is the arithmetical mean operator with weight vector $w = (0.25, 0.25, 0.25, 0.25)^T$. According to (M-8), we have the following quadratic programming model:

$$\min f = x_1^2 + 2(x_2-3)^2 + 3(x_3-6)^2 + (x_4-10)^2$$
$$\text{s.t.} \begin{cases} 0.75x_1 & -0.25x_2 & -0.25x_3 & -0.25x_4 & \leq 0.8 \\ -0.25x_1 & +0.75x_2 & -0.25x_3 & -0.25x_4 & \leq 0.8 \\ -0.25x_1 & -0.25x_2 & +0.75x_3 & -0.25x_4 & \leq 0.8 \\ -0.25x_1 & -0.25x_2 & -0.25x_3 & +0.75x_4 & \leq 0.8 \\ -0.75x_1 & +0.25x_2 & +0.25x_3 & +0.25x_4 & \leq 0.8 \\ 0.25x_1 & -0.75x_2 & +0.25x_3 & +0.25x_4 & \leq 0.8 \\ 0.25x_1 & +0.25x_2 & -0.75x_3 & +0.25x_4 & \leq 0.8 \\ 0.25x_1 & +0.25x_2 & +0.25x_3 & -0.75x_4 & \leq 0.8 \end{cases}$$
$$(M-20)$$

We get that the optimal solution to (M-20) is $X^* = (3.94, 3.94, 5.54, 5.54)^T$ and the optimal value is $f_c = 37.82$. Therefore, when the importance weight of each expert is uniform, e_1, e_2, e_3, and e_4 are suggested to change their opinions to $\bar{o}_1^* = 3.94\%, \bar{o}_2^* = 3.94\%, \bar{o}_3^* = 5.54\%, \bar{o}_4^* = 5.54\%$, and

Table 5 Optimal solutions to (M-15) under different cost vectors

(c_1, c_2, c_3, c_4)	$\bar{o}_1^*$	$\bar{o}_2^*$	$\bar{o}_3^*$	$\bar{o}_4^*$	$\bar{o}^*$	f_c
(1, 2, 3, 1)	3.94	3.94	5.54	5.54	4.74	37.82
(1, 4, 3, 5)	5.17	5.17	6.77	6.77	5.97	99.51
(6, 3, 4, 1)	2.5	2.5	4.1	4.1	3.3	87.5
(3, 6, 4, 1)	3.14	3.14	4.74	4.74	3.94	63.71

Table 6 Optimal solutions to (M-20) under different cost vectors

(c_1, c_2, c_3, c_4)	$\overline{o}_1^*$	$\overline{o}_2^*$	$\overline{o}_3^*$	$\overline{o}_4^*$	$\overline{o}^*$	f_c
(1, 2, 3, 1)	3.94	3.94	5.54	5.54	4.74	37.82
(1, 4, 3, 5)	5.17	5.17	6.77	6.77	5.97	99.51
(6, 3, 4, 1)	2.5	2.5	4.1	4.1	3.30	87.5
(3, 6, 4, 1)	3.14	3.14	4.74	4.74	3.94	63.71

Fig. 2 In (**a**)-(**e**), the vertical axis is the values of deviation threshold ε, and the horizontal axis shows the corresponding optimal opinions in different consensus models

(a) (M-8)

(b) (M-10)

(c) (M-11)

(d) (M-15)

(e) (M-20)
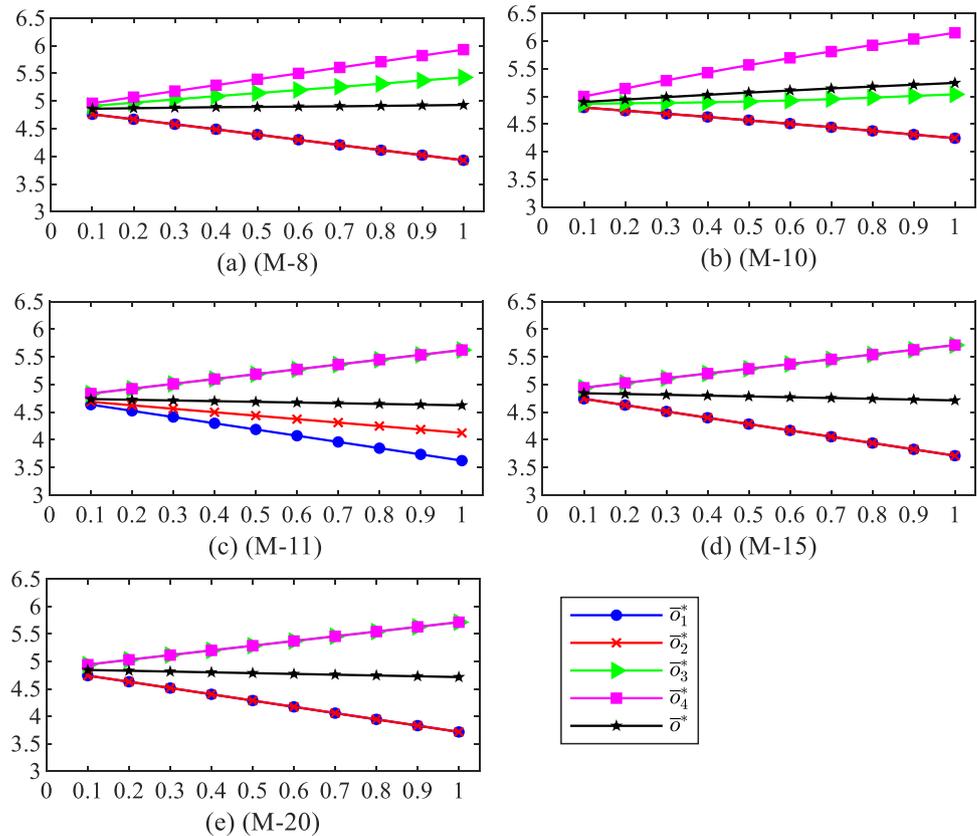
the final collective opinion $\overline{o}^*$ is their arithmetical mean 4.74%. Under the collective allowance level of 838 RMB, the government would finally provide the subsidies of {832, 832, 844, 844} (RMB) to {e_1, e_2, e_3, e_4} and pay the consensus cost of 378.2 RMB.

Under the different unit adjustment costs, the unique optimal solutions and minimum consensus costs in (M-20) are summarized in Table 6.

Fig. 2 show the variation trends of optimal opinions of employees and optimal group opinion in every model as the deviation threshold grows, under the cost vector of $(c_1, c_2, c_3, c_4) = (1, 2, 3, 1)$. Fig. 3 shows the changes of the consensus costs in the five models under the incremental deviation thresholds.

Figure 3 shows that the consensus costs in (M-10) under each value of ε are the highest, and the costs in models (M-15) and (M-20) are completely uniform. It's observed that the distributions and variation trends of optimal opinions are
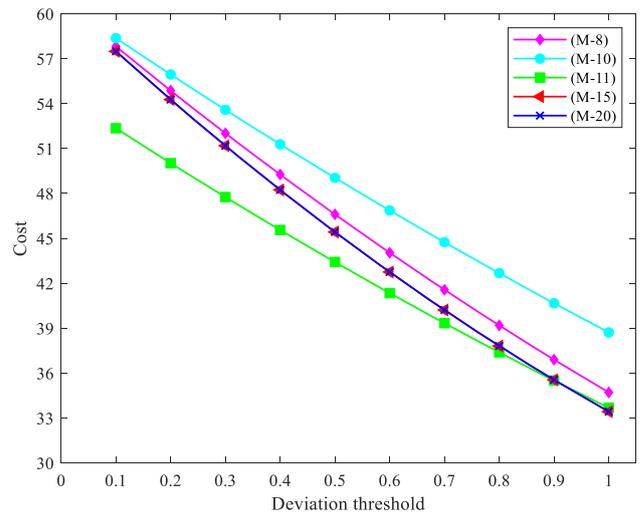

Fig. 3 Changes of the total consensus costs of the five models under different deviation thresholds

influenced by the aggregation operators and deviation thresholds. In the consensus process, the costs paid by a moderator are closely related to the predefined consensus level. It's no doubt that the higher consensus degree requires devoting more resources to be accomplished, yet the lower one can be easily achieved at a lower cost. Thus, the local government could select a proper deviation threshold to maximize the consensus degree under a given budget.

The above five cases match different types of scenes, respectively. In Case (1) (i.e., (M-8)), each expert is assigned with a settled importance weight according to his/her level of expertise (e.g., knowledge, confidence, the ability of making coherent assessments, etc.) with direct or indirect methods. Contrasted with Case (1), implicit in Case (2) (i.e., (M-10)) is the assumption that all experts are of equal importance. The OWA operators lie between the operators requiring all experts and at least one expert to be satisfied. Sometimes, the desired property of generalized commutativity also drives us to use OWA operators instead of the simple WA operators [28]. It should be noted that every importance weight assigned by OWA operators is associated with a particular ordered position rather than some decision maker. However, when the unit cost of each expert is equal (i.e., Case (3)), the weights of all experts are fixed. Case (4) aims at restraining deliberate manipulation in the decision making process by deleting the extreme values among individual opinions. In fact, Case (5) is also a special case of Case (2) that assigns the fixed weight of $1/n$ to each expert.

Ben-Arieh, Easton and Evans [15] devised the methods of CMQC, ε CMQC, and QMEC. In the following, we compare the proposed methods with three existing quadratic cost consensus models using same indices.

We first compute the results of CMQC where adjusted individual opinions have to be uniform (i.e., $\varepsilon = 0$) and aggregation operators are not considered. Under the same cost vectors, the unanimous consensus results derived by CMQC are shown as Table 7.

Then we use the method of ε CMQC to assemble employees' opinions. The optimal adjusted opinions under the different cost vectors are summarized in Table 8.

QMEC aims to maximize the number of experts fitting within consensus given a specified budget. Here we set the unit cost of each expert as $c_1 = 1$, $c_2 = 2$, $c_3 = 3$, $c_4 = 1$ and apply (M-5) to find the optimal adjustment alternatives under descending budgets B (Unit: 10RMB), as shown in Table 9. The consensus group is denoted as set E'. All members belonging to E' share the same group opinion $\bar{o}^*$, while the original opinions of other experts are preserved.

From the results in Tables 2, 3, 4, 5, 6, 7, 8, and 9, we make a comparative analysis of the proposed methods and CMQC, ε CMQC, and QMEC in Ben-Arieh, Easton and Evans [15]:

(1) Compared with the results in other methods, the total costs of CMQC are always the highest. In CMQC, the adjusted opinions of all experts need to be completely uniform. Nevertheless, for the government, the unanimous consensus might be too expensive to be acomplished efficiently. Thus the consensus models acceptable for a proper

Table 7 Optimal solutions of CMQC under different cost vectors

(c_1, c_2, c_3, c_4)	$\bar{o}_1^*$	$\bar{o}_2^*$	$\bar{o}_3^*$	$\bar{o}_4^*$	$\bar{o}^*$	f_c
(1, 2, 3, 1)	4.86	4.86	4.86	4.86	4.86	60.86
(1, 4, 3, 5)	6.15	6.15	6.15	6.15	6.15	151.69
(6, 3, 4, 1)	3.07	3.07	3.07	3.07	3.07	138.93
(3, 6, 4, 1)	3.71	3.71	3.71	3.71	3.71	104.86

Table 8 Optimal solutions of ε CMQC under different cost vectors

(c_1, c_2, c_3, c_4)	$\bar{o}_1^*$	$\bar{o}_2^*$	$\bar{o}_3^*$	$\bar{o}_4^*$	$\bar{o}^*$	f_c
(1, 2, 3, 1)	3.94	3.94	5.54	5.54	4.74	37.82
(1, 4, 3, 5)	5.4	5.4	6	7	6.2	97.20
(6, 3, 4, 1)	2.36	3	3.96	3.96	3.16	86.55
(3, 6, 4, 1)	3.14	3.14	4.74	4.74	3.94	63.71

Table 9 Optimal solutions of QMEC under given budgets

B	$\bar{o}_1^*$	$\bar{o}_2^*$	$\bar{o}_3^*$	$\bar{o}_4^*$	$\bar{o}^*$	E'	f_c
60	4	4	4	10	4	e_2, e_3, e_1	30
30	4	4	4	10	4	e_2, e_3, e_1	30
15	2	2	6	10	2	e_2, e_1	6

deviation between individual opinions and group opinion are more feasible in operation.

(2) In terms of the consensus costs, the method of ε CMQC is superior to the proposed methods considering aggregation operators. It seems that ε CMQC doesn't consider the relationships between the individuals' opinions and the group one. However, we observe that the optimal group collective opinions in ε CMQC are exactly equal to the averages of the maximum and minimum values of adjusted individuals' opinions. In fact, Zhang, Dong, Xu and Li [31] revealed that the inherent aggregation operator of ε CMQC is the OWA operator with weight vector $(1/2, 0, ..., 0, 1/2)^T$. That is, ε CMQC only considers the extreme opinions rather than the majority of decision makers' opinions in pursurance of the minimum consensus cost. Apparently, the group opinion derived by ε CMQC is not representative enough as a result of overlooking the majority of views, especially when there are many decision makers involved. Therefore, compared with of ε CMQC, methods proposed in this paper have wider applications and can aggregate the group opinion from individual ones more resonably.

(3) In Table 9, since the given budgets are lower than the minimum cost of CMQC (i.e., 608.6 RMB), the maximum number of experts that meet consensus is certainly no more than three. We find that in this caes, the method of QMEC can only earn the agreement among e_1, e_2, and e_3 while e_4 can never cross over into the consensus group, when the budget B are within the interval [30,60.86). Thus, every employee would receive the subsidy of 832 RMB even if e_4 is not satisfied with this policy. This is mainly caused by the constraint of unanimous agreemnet. However, in the proposed methods, ε can be tuned to maximize the consensus levels among all employees under the specified budgets, as shown in Fig. 3. Therefore, utilizing the mehtods addressed in this

paper, the opinion of every employee would be considered, and the government could also optimize its resource allocations during the consensus process.

5.2 Example 2

In this example, we use the data from Zhang, Dong, Xu and Li [31]. Assume that there are five experts, and their original opinions are: $o_1 = 0.5$, $o_2 = 1.0$, $o_3 = 2.5$, $o_4 = 3.0$, and $o_5 = 6.0$. Let f_c denote the minimum consensus cost and the deviation threshold ε be 0.8. In the following, we compare the results of the existing linear cost consensus methods (i.e., (M-1), (M-2), and (M-6)) and the proposed methods under WA and OWA operators.

(1) We first use (M-8) to rectify individual expert's opinions, where $\varepsilon = 0.8$ and the aggregation operator is the WA operator with weight vector $w = (0.375, 0.1875, 0.25, 0.0625, 0.125)^T$. Table 10 displays the corresponding optimal individual and collective opinions and the minimum consensus costs obtained from (M-8) under the different values of unit costs c_i ($i = 1, 2, …, 5$).

(2) If the unit costs are uniform among experts, then (M-11) can be used to aggregate the individuals' opinions. Set the associated weight vector of OWA operator be $w = (0.375, 0.1875, 0.25, 0.0625, 0.125)^T$. The corresponding results are listed in Table 11, which shows that under the different consensus thresholds, the magnitude of experts' opinion changes is different.

(3) Utilize (M-15) to facilitate consensus among experts where $\varepsilon = 0.8$ and the OWA operator with weight vector $w = (0, 1/3, 1/3, 1/3, 0)^T$ is used. Table 12 presents the unique optimal solutions to (M-15) under varying cost vectors, which preserve the rankings of original

Table 10 Optimal solutions to (M-8) under different cost vectors

$(c_1, c_2, c_3, c_4, c_5)$	$\bar{o}_1^*$	$\bar{o}_2^*$	$\bar{o}_3^*$	$\bar{o}_4^*$	$\bar{o}_5^*$	$\bar{o}^*$	f_c
(1, 4, 3, 5, 2)	2.16	1.63	2.87	3.06	3.23	2.43	20.13
(6, 3, 4, 1, 2)	1.15	1.65	2.74	2.74	2.74	1.94	25.37
(3, 4, 1, 6, 2)	1.68	1.44	3.02	3.02	3.02	2.22	22.96

Table 11 Optimal solutions to (M-11) under different consensus thresholds

ε	$\bar{o}_1^*$	$\bar{o}_2^*$	$\bar{o}_3^*$	$\bar{o}_4^*$	$\bar{o}_5^*$	$\bar{o}^*$	f_c
0.5	2.27	2.27	2.32	2.87	3.27	2.77	12.25
0.6	2.21	2.21	2.32	2.86	3.41	2.81	11.15
0.7	2.14	2.14	2.31	2.86	3.54	2.84	10.10
0.8	2.08	2.08	2.30	2.85	3.68	2.88	9.11

Table 12 Optimal solutions to (M-15) under different cost vectors

$(c_1, c_2, c_3, c_4, c_5)$	$\bar{o}_1^*$	$\bar{o}_2^*$	$\bar{o}_3^*$	$\bar{o}_4^*$	$\bar{o}_5^*$	$\bar{o}^*$	f_c
(1, 4, 3, 5, 2)	1.72	1.72	2.71	3.13	3.32	2.52	18.14
(6, 3, 4, 1, 2)	1.39	1.39	2.44	2.75	2.99	2.19	23.41
(3, 4, 1, 6, 2)	1.58	1.58	2.55	3.01	3.18	2.38	20.75

Table 13 Optimal solutions to (M-1) under different cost vectors

$(c_1, c_2, c_3, c_4, c_5)$	$\bar{o}_1^*$	$\bar{o}_2^*$	$\bar{o}_3^*$	$\bar{o}_4^*$	$\bar{o}_5^*$	$\bar{o}^*$	f_c
(1, 4, 3, 5, 2)	2.5	2.5	2.5	2.5	2.5	2.5	17.5
(6, 3, 4, 1, 2)	1	1	1	1	1	1	21
(3, 4, 1, 6, 2)	2.5	2.5	2.5	2.5	2.5	2.5	22

Table 14 Optimal solutions to (M-2) under different cost vectors

$(c_1, c_2, c_3, c_4, c_5)$	$\bar{o}_1^*$	$\bar{o}_2^*$	$\bar{o}_3^*$	$\bar{o}_4^*$	$\bar{o}_5^*$	$\bar{o}^*$	f_c
(1, 4, 3, 5, 2)	1.4	1.4	2.5	3	3	2.2	8.5
(6, 3, 4, 1, 2)	0.9	1	2.5	2.5	2.5	1.7	9.9
(3, 4, 1, 6, 2)	1.4	1.4	2.5	3	3	2.2	10.3

Table 15 Optimal solutions to (M-6)

	$\bar{o}_1^*$	$\bar{o}_2^*$	$\bar{o}_3^*$	$\bar{o}_4^*$	$\bar{o}_5^*$	$\bar{o}^*$	f_c
$\dot{o}$	1.27	1	2.6	2.6	2.6	1.8	11.6
$\ddot{o}$	1.67	1.4	3	3	3	2.2	11.6
o	1.47	1.2	2.8	2.8	2.8	2	11.6

opinions. However, the initial order positions were changed in Zhang, Dong, Xu and Li [31], when using the linear consensus cost model (M-6) with the same arguments and OWA operator.

This example was first investigated by Ben-Arieh and Easton [24], where they used the linear cost consensus models (M-1) and (M-2) to reach unanimous consensus and ε consensus, respectively. Let $\varepsilon = 0$. Table 13 shows the optimal solutions and the optimal values obtained by (M-1) under the varying cost vectors.

If we set $\varepsilon = 0.8$, the shifted individual opinions and collective opinion and the minimum consensus costs in (M-2) are shown in Table 14.

In Zhang, Dong, Xu and Li [31], (M-6) was also applied to examine this example, where $\{c_1, c_2, c_3, c_4, c_5\} = \{3, 4, 1, 6, 2\}$, $\varepsilon = 0.8$, and F is a WA operator with weight vector $w = (0.375, 0.1875, 0.25, 0.0625, 0.125)^T$. It should be noted that (M-6) may have multiple optimal solutions. Table 15 shows three of the optimal solutions ($\dot{o}$, $\ddot{o}$ and o) to (M-6).

Both Ben-Arieh and Easton [24] and Zhang, Dong, Xu and Li [31] constructed the consensus methods under the linear cost functions. From Tables 13 and 15, it is obvious that the consensus costs of (M-1) for the unanimous consensus are substantially higher than those of (M-6) that considers consensus degree. When complete agreement is not necessary, based on Tables 14 and 15, it's observed that method (M-2) proposed by Ben-Arieh and Easton [24] has lower costs than (M-6) by Zhang, Dong, Xu and Li [31]. However, Zhang, Dong, Xu and Li [31] prove that when (M-6) uses the OWA operator with weight vector $w = (1/2, 0, ..., 0, 1/2)^T$, (M-6) is equivalent to (M-2). Therefore, as we have pointed out in Sect. 2.2, method in Zhang, Dong, Xu and Li [31] is an extension of Ben-Arieh and Easton [24]'s work.

The main difference between the method in Zhang, Dong, Xu and Li [31] and the proposed method lies in the definition of consensus cost. The first derivative of the quadratic cost function increases as opinion adjustment rises, which reflects the psychological changes of experts. In general, an expert would show more resistance when he/she is required to make more compromises. However, the first derivative of the linear cost function is uniform no matter how much the individual opinion is changed. Thus, the quadratic cost functions could better characterize the consensus costs paid to the experts. Apart from the cost functions, the solving approaches are also distinct. The method in Zhang, Dong, Xu and Li [31] is based on linear programming to obtain the optimal solution, while the proposed methods are grounded on strictly convex quadratic programming. It is widely known that the optimal

Table 16 Values of o_{ij}

	$j=1$	$j=2$	$j=3$	$j=4$
$i=1$	1	3	1	5
$i=2$	3	5	3	2
$i=3$	4	1	2	1
$i=4$	2	3	5	1
$i=5$	2	4	2	3

Table 17 Values of $\bar{o}_{ij}^*$ and $\bar{o}^*$ obtained by (M-8)

	$\bar{o}_{i1}^*$	$\bar{o}_{i2}^*$	$\bar{o}_{i3}^*$	$\bar{o}_{i4}^*$	$\bar{o}_i^*$
$i=1$	1.53	3.2	1.53	3.53	2.53
$i=2$	3.29	4.45	3.37	2.45	3.45
$i=3$	2.86	1.21	2.35	1.35	1.86
$i=4$	2.19	3.14	3.76	1.76	2.76
$i=5$	2.07	3.87	2.09	3.09	2.87

Table 18 Values of $\bar{o}_{ij}^*$ and $\bar{o}^*$ obtained by (M-11)

	$\bar{o}_{i1}^*$	$\bar{o}_{i2}^*$	$\bar{o}_{i3}^*$	$\bar{o}_{i4}^*$	$\bar{o}_i^*$
$i=1$	1.5	3.5	1.5	3.5	2.5
$i=2$	3.25	4.20	3.30	2.25	3.20
$i=3$	2.97	1.32	2.39	1.32	1.97
$i=4$	2.25	3.30	3.73	1.73	2.73
$i=5$	2.09	3.72	2.09	3.11	2.72

Table 19 Values of $\bar{o}_{ij}^*$ and $\bar{o}^*$ obtained by (M-15)

	$\bar{o}_{i1}^*$	$\bar{o}_{i2}^*$	$\bar{o}_{i3}^*$	$\bar{o}_{i4}^*$	$\bar{o}_i^*$
$i=1$	1.57	3.29	1.43	3.43	2.43
$i=2$	3.4	4.4	3.4	2.4	3.4
$i=3$	3	1	2.5	1.5	2
$i=4$	2.29	3.14	3.71	1.71	2.71
$i=5$	2	3.75	2.25	3.25	2.75

Table 20 Values of $\bar{o}_{ij}^*$ and $\bar{o}^*$ obtained by CMQC

	$\bar{o}_{i1}^*$	$\bar{o}_{i2}^*$	$\bar{o}_{i3}^*$	$\bar{o}_{i4}^*$	$\bar{o}_i^*$
$i=1$	2.6	2.6	2.6	2.6	2.6
$i=2$	3.6	3.6	3.6	3.6	3.6
$i=3$	1.8	1.8	1.8	1.8	1.8
$i=4$	2.8	2.8	2.8	2.8	2.8
$i=5$	3	3	3	3	3

Table 21 Values of $\bar{o}_{ij}^*$ and $\bar{o}^*$ obtained by ε CMQC

	$\bar{o}_{i1}^*$	$\bar{o}_{i2}^*$	$\bar{o}_{i3}^*$	$\bar{o}_{i4}^*$	$\bar{o}_i^*$
$i=1$	1.8	3	1.8	3.4	2.6
$i=2$	3	3.93	2.33	2.33	3.13
$i=3$	2.95	1.35	2	1.35	2.15
$i=4$	2.13	3	3.73	2.13	2.93
$i=5$	2.2	3.8	3	2.2	3

solution to the strictly convex quadratic programming exists and is unique. Thus, the proposed models are superior to Zhang, Dong, Xu and Li [31]'s models with respect to the uniqueness of optimal solution.

5.3 Example 3

Suppose there is a family consisting of four members e_j ($j=1$, 2, 3, 4) choosing an apartment from five alternatives A_i ($i=1$, 2, …, 5). Denote o_{ij} the preference of e_j on A_i. The stronger the preference of member e_j for apartment A_i, the higher o_{ij} will be. The preference information of the four members is presented in Table 16.

We observe that e_1, e_2, e_3, and e_4 prefer A_3, A_2, A_4, and A_1, respectively. To this end, minimum cost consensus models are applied to reach consensus among the family members. Denote $\bar{o}_{ij}^*$ as the optimal adjusted opinions of e_j on A_i and $\bar{o}_i^*$ the collective evaluation on A_i.

(1) Model (M-8) is first applied to aggregate the original opinions, where $\{c_1, c_2, c_3, c_4\} = \{1, 2, 1, 1\}$, $\varepsilon = 1$, and the weight vector of WA operator is $w = (0.2, 0.3, 0.25, 0.25)^T$. Table 17 lists the adjusted individual opinions $\bar{o}_{ij}^*$ and collective opinions $\bar{o}_i^*$ derived from (M-8).

(2) Assume that the unit cost of each member is uniform, the weight vector of OWA operator is $w = (0.2, 0.3, 0.25, 0.25)^T$ and $\varepsilon = 1$, then (M-11) is employed to adjust the original individual opinions. Table 18 shows the corresponding results derived from (M-11).

(3) Suppose $\{c_1, c_2, c_3, c_4\} = \{1, 2, 1, 1\}$ and $\varepsilon = 1$, then under the OWA operator with weight vector $w = (0, 1/3, 1/3, 1/3, 0)^T$, model (M-15) is used to facilitate consensus

among the family members. The evaluations adjusted by (M-15) are presented in Table 19.

(4) When $\varepsilon = 0$ and $\{c_1, c_2, c_3, c_4\} = \{1, 2, 1, 1\}$, the proposed method in this paper reduces to CMQC developed by Ben-Arieh, Easton and Evans [15]. CMQC is the quadratic cost consensus method to achieve completely unanimous agreement. The corresponding consensus results are shown in Table 20.

(5) If we use the OWA operator with weight vector $w = (1/2, 0, 0, 0, 1/2)^T$ in (M-10), then the proposed model is equivalent to the model of ε CMQC. Table 21 shows the optimal solutions in ε CMQC.

Table 22 Values of $\overline{o}_{ij}^*$ and $\overline{o}^*$ obtained by (M-1)

	$\overline{o}_{i1}^*$	$\overline{o}_{i2}^*$	$\overline{o}_{i3}^*$	$\overline{o}_{i4}^*$	$\overline{o}_i^*$
$i = 1$	3	3	3	3	3
$i = 2$	3	3	3	3	3
$i = 3$	1	1	1	1	1
$i = 4$	3	3	3	3	3
$i = 5$	3	3	3	3	3

Table 24 Values of $\overline{o}_{ij}^*$ and $\overline{o}^*$ obtained by (M-6)

	$\overline{o}_{i1}^*$	$\overline{o}_{i2}^*$	$\overline{o}_{i3}^*$	$\overline{o}_{i4}^*$	$\overline{o}_i^*$
$i = 1$	1.33	3	1.33	3.33	2.33
$i = 2$	3	4.11	3	2.11	3.11
$i = 3$	2.56	1	2	1	1.56
$i = 4$	2	3	3.6	1.6	2.6
$i = 5$	2	3.79	2	3	2.79

(6) When the quadratic cost function of CMQC is swapped for the linear one, (M-1) addressed by Ben-Arieh and Easton [24] can also be used to achieve complete agreement among family members. We still set $\{c_1, c_2, c_3, c_4\} = \{1, 2, 1, 1\}$, then the optimal solutions obtained by (M-1) are presented in Table 22.

(7) In practice, it's generally expensive and time-wasting to achieve the full and unanimous agreement. Thus, we set $\varepsilon = 1$ and $\{c_1, c_2, c_3, c_4\} = \{1, 2, 1, 1\}$. The linear cost model (M-2) is used to aggregate family members' opinions and the adjusted opinions are shown in Table 23.

(8) Furthermore, incorporating aggregation operators into (M-2), model (M-6) can be applied to attain consensus on the house purchase. Let the aggregation operator F in (M-6) be the WA operator with weight vector $w = (0.2, 0.3, 0.25, 0.25)^T$, $\{c_1, c_2, c_3, c_4\} = \{1, 2, 1, 1\}$, and $\varepsilon = 1$. Table 24 lists the evaluations after being adjusted by (M-6).

According to the optimal collective opinions after modifying, we can obtain the ranking of A_i ($i = 1, 2, \ldots, 5$). In Tables 17, 19, 20, 21 and 24, the ranking is: $A_2 \succ A_5 \succ A_4 \succ A_1 \succ A_3$. In Table 18, the ranking is: $A_2 \succ A_4 \succ A_5 \succ A_1 \succ A_3$. In Tables 22 and 23, the rankings are $A_1 \sim A_2 \sim A_4 \sim A_5 \succ A_3$ and $A_2 \succ A_4 \sim A_5 \succ A_1 \sim A_3$, respectively. Although the priorities of A_i are not exactly the same, each table shows that $\overline{o}_2^* = \max_i\{\overline{o}_i^*\}$. Hence, A_2 is the optimal alternative for the family. Particularly, the prioritization between alternatives A_4 and A_5 is changed under the diverse aggregation operators, which shows the importance of aggregation operators in GDM.

The above numerical comparative analyses reveal the validity and usability of the proposed models. Compared with the consensus methods in Ben-Arieh, Easton and Evans [15], the consensus models proposed in this paper are not only more

Table 23 Values of $\overline{o}_{ij}^*$ and $\overline{o}^*$ obtained by (M-2)

	$\overline{o}_{i1}^*$	$\overline{o}_{i2}^*$	$\overline{o}_{i3}^*$	$\overline{o}_{i4}^*$	$\overline{o}_i^*$
$i = 1$	1	3	1	3	2
$i = 2$	3	5	3	3	4
$i = 3$	3	1	2	1	2
$i = 4$	2	3	4	2	3
$i = 5$	2	4	2	3	3

feasible and efficient in terms of practical applications, but are also more flexble when applied to different scenarios. Moreover, different form refrences [24, 31], this paper also employs quadratic cost functions to capture experts' psychological changes in negotiation process and could provide unique optimal solutions for GDM problems.

6 Conclusions

In GDM, aggregation operators can be applied to reveal the inherent relationships between individuals' opinions and the collective opinion. In the existing minimum cost consensus models, the objective functions are usually assumed to be linear cost functions and the optimal solutions are attained by linear programming. However, there may be infinite optimal solutions to a linear programming model, which may escalate the complexity of GDM problems. Additionally, the unit adjustment costs in the linear cost functions are constant, which ignores the impact on unit costs made by the opinion adjustments. In order to overcome the shortages, this paper develops an extended minimum cost consensus model with quadratic costs under aggregation operators. The proposed model reduces to the consensus model of Ben-Arieh, Easton and Evans [15] when the adjusted individual opinions are equal to the consensus opinion.

In addition, this paper closely examines the proposed models under the WA operator and OWA operators. Corresponding approaches are provided based on strictly convex quadratic programming, the optimal solution of which exists and is unique. Furthermore, these models are in line with some ideal properties.

Fuzzy information is often applied to denote the preference information of experts over a set of alternatives [33, 40–42] when experts' opinions are vague. Both individual consistency and consensus level measures are essential for a rational GDM problem [40]. Thus, how to extend this model to a certain degree of individual consistency under reciprocal fuzzy preference relations is one of our future research topics. In addition, many other operators such as Bonferroni mean [43, 44], Maclaurin symmetric mean [45, 46], etc. are as effective as the WA and OWA operators in decision making problems. Hence, how to apply the proposed model to other

aggregation operators will be also discussed in the future. With great prosperity in information technology, a large spectrum of decision makers could simultaneously participate in decision making processes, such as e-democracy and social networks. Recently, the large-scale group decision making (LSGDM) has become a hotspot among researchers [47–49]. Thus, how to employ the proposed method to practical LSGDM problems, like high-speed rail passenger requirement management [50, 51], systemic risk management of food supply chains [52] and so forth.

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