Applied Intelligence (2019) 49:837–857
https://doi.org/10.1007/s10489-018-1292-1

Programming model-based method for ranking objects from group decision making with interval-valued hesitant fuzzy preference relations

Yuning Zhang[1] · **Jie Tang**[2] · **Fanyong Meng**[2,3]

Published online: 2 October 2018

Abstract

Interval-valued hesitant fuzzy preference relations (IVHFPRs) are useful that allow decision makers to apply several intervals in [0, 1] to denote the uncertain hesitation preference. To derive the reasonable ranking order from group decision making with preference relations, two topics must be considered: consistency and consensus. This paper focuses on group decision making with IVHFPRs. First, a multiplicative consistency concept for IVHFPRs is defined. Then, programming models for judging the consistency of IVHFPRs are constructed. Meanwhile, an approach for deriving the interval fuzzy priority weight vector is introduced that adopts the consistency probability distribution as basis. Subsequently, this paper builds several multiplicative consistency-based programming models for estimating the missing values in incomplete IVHFPRs. A consensus index is introduced to measure the agreement degree between individual IVHFPRs, and a method for increasing the consensus level is presented. Finally, a multiplicative consistency-and-consensus-based group decision-making method with IVHFPRs is offered, and a practical decision-making problem is selected to show the application of the new method.

Keywords Group decision making · IVHFPR · Multiplicative consistency · Consensus · Programming model

1 Introduction

Various types of fuzzy sets are proposed [2, 10, 34] to cope with the increasing complexity of decision-making problems. Each of them has some advantages of denoting decision-making information in some aspects. For example, Atanassov's intuitionistic fuzzy sets [1, 2] can denote the preferred and non-preferred information simultaneously; linguistic variables [34] can express decision makers (DMs)' qualitative preferences, while hesitant fuzzy sets [22] can simply indicate the DMs' hesitant recognitions. Based on these types of fuzzy sets, many decision-making methods are proposed, which are applied in many fields [6, 9, 11, 16, 17, 23, 24, 30].

✉ Fanyong Meng
mengfanyongtjie@163.com

1 School of Management and Economics, Beijing Institute of Technology, Beijing 100081, China

2 School of Business, Central South University, Changsha 410083, China

3 School of Management and Economics, Nanjing University of Information Science and Technology, Nanjing 210044, China

In the current decision-making methods, preference relations or pairwise judgement matrices are good tools. Taking the advantages of preference relations and hesitant fuzzy sets, preference relations with hesitant fuzzy judgements are proposed. According to the structure of elements, they can be classified into three types: hesitant fuzzy preference relations (HFPRs) [20, 31], hesitant multiplicative preference relations (HMPRs) [40], and hesitant fuzzy linguistic preference relations (HFLPRs) [21, 41]. Following the principle of calculating the priority weight vector, there are two main types of methods: the α-normalization methods [25, 26, 32, 35, 42] and the β-normalization methods [36–39]. The former derives the crisp priority weight vector by only considering a crisp preference relation, while the latter uses the normalized hesitant preference relations, and the hesitant fuzzy priority weight vector is obtained from the ordered preference relations. Because the former only considers one value in each hesitant fuzzy element, this type of methods loses information. On the other hand, the latter needs to add values to the shorter hesitant fuzzy judgments, which changes the original hesitant fuzzy preference relations. Furthermore, different ranking orders may be obtained with respect to different added values.

Although preference relations with hesitant fuzzy judgments have some advantages to denote the DMs' hesitancy, they only permit the DMs to use the exact values [27, 31, 40, 41]. This may be still insufficient to express their preferences. To address this issue, interval-valued hesitant fuzzy sets (IVHFSs) introduced by Chen et al. [3] are good choices that permit the DMs to apply several intervals rather than exact ones to denote their inherent hesitancy and uncertainty. Then, the authors developed an approach to decision making with interval-valued hesitant fuzzy information. Considering the interactive characteristics between the weights of criteria and the DMs, Meng and Chen [14] applied the Shapley function to develop a method for interval-valued hesitant fuzzy decision making with incomplete weight information. Furthermore, Chen et al. [4] proposed several correlation coefficients of IVHFSs and applied them to clustering analysis. However, Chen et al.'s correlation coefficients require IVHFSs to have the same length. Otherwise, we need to add values to the shorter ones. In contrast to Chen et al.'s correlation coefficients, Meng et al. [15] defined several other correlation coefficients of IVHFSs that need not to add extra values and consider all information offered by the DMs. Jin et al. [8] introduced a cross entropy of IVHFSs and then developed a TOPSIS method for group decision making with interval-valued hesitant fuzzy information. Menwhile, Yuan et al. [33] used the defined confidence level and the Choquet integral to propose an approach for multi-attribute interval-valued hesitant fuzzy group decision making. More researches about decision making with interval-valued hesitant fuzzy information are available in the literature [5, 6, 18].

Similar to hesitant fuzzy preference relations, Chen et al. [3] introduced the concept of interval-valued hesitant fuzzy preference relations (IVHFPRs) by using interval-valued hesitant fuzzy elements (IVHFEs). Pérez-Fernándeza et al. [18] discussed the application of IVHFPRs in group decision making. However, both of these methods are based on the aggregations operators. Two vertical topics in preference relations, consistency and consensus, are not studied. This makes the ranking orders obtained from these methods be questionable because the illogical ranking orders may be derived from inconsistent IVHFPRs [7, 13, 24, 25, 30]. Furthermore, the final ranking orders cannot reflect the agreement degrees between individual opinions. To address these issues, this paper continues to study IVHFPRs. Considering the consistency of IVHFPRs, a natural and robust consistency concept is introduced. Then, inconsistent and incomplete IVHFPRs are studied. After that, a consensus index is defined, and an interactive method for improving the consensus level is proposed. On the basis of multiplicative consistency and consensus analysis, a group decision-making method with IVHFPRs is developed. To do these, the rest is organized as follows.

Section 2 reviews several related concepts, including hesitant fuzzy sets, interval-valued hesitant fuzzy sets, intervals, basic operations, and IVHFPRs. Section 3 first recalls a multiplicative consistency concept for interval fuzzy preference relations (IFPRs). Then, it defines a multiplicative consistency concept for IVHFPRs. Subsequently, several programming models for judging the multiplicative consistency of IVHFPRs are built. On the basis of the consistency probability distribution and the multiplicatively consistent IFPRs, a method for deriving the interval fuzzy priority weight vector is introduced. Section 4 focuses on incomplete IVHFPRs. Several multiplicative consistency-based programming models for determining missing IVHFEs are constructed. Then, a numerical example is offered to show the concrete application of the developed theoretical results. Section 5 introduces a consensus index and offers an improving consensus method. Then, a multiplicative consistency-and-consensus-based group decision-making method is presented. Section 6 offers a practical group decision-making problem about evaluating the air-conditioning manufacturers to show the concrete application of the new method. Conclusion and future remarks are performed in the last section.

2 Basic concepts

In some situations, decision makers (DMs) may be hesitant on several possible values for a pair of objects. Torra [22] introduced a new type of fuzzy sets called hesitant fuzzy sets (HFSs) to denote the DMs' hesitancy. The facilitate application, Xu and Xia [29] offered the following formal definition of HFSs:

Definition 1 [29] A hesitant fuzzy set (HFS) E on $X = \{x_1, x_2, \ldots x_n\}$ is a finite set, denoted by $E = \{<x_i, h_E(x_i)>|x_i \in X\}$ where $h_E(x_i)$ is a set of several values in [0,1] denoting the possible membership degrees of the element $x_i \in X$ to E.

From the concept of HFSs, we know that HFSs demand the DMs to offer the crisp membership degrees to express their judgments. This restricts the application because it may be still difficult due to the inherent vagueness of the DMs. Considering this issue, Chen et al. [3] further presented interval-valued hesitant fuzzy sets (IVHFSs).

Definition 2 [3] An IVHFS $\tilde{A}$ on $X = \{x_1, x_2, \ldots x_n\}$ is a finite set, denoted by $\tilde{A} = \{< x_i, \tilde{h}_{\tilde{A}}(x_i) > |x_i \in X\}$, where $\tilde{h}_{\tilde{A}}(x_i)$ is a set of all possible interval-valued membership degrees of the element $x_i \in X$ to $\tilde{A}$ in [0,1].

For convenience, $\tilde{h} = \tilde{h}_{\tilde{A}}(x_i)$ is called an interval-valued hesitant fuzzy element (IVHFE).

Using IVHFEs, Chen et al. [3] introduced the following concept of interval-valued hesitant fuzzy preference relations (IVHFPRs):

Definition 3 [3] An IVHFPR $\tilde{H}$ on $X = \{x_1, x_2, \ldots, x_n\}$ is defined as $\tilde{H} = (\tilde{h}_{ij})_{n\times n}$, where $\tilde{h}_{ij}$ is an IVHFE denoting the possible interval-valued preferred degrees of the object x_i over x_j. Elements in $\tilde{H}$ have the following characteristics:

$$\begin{cases} h^-_{l,ij} + h^+_{m_{ji}+1-l,ji} = h^+_{l,ij} + h^-_{m_{ji}+1-l,ji} = 1\ ,\ \ l = 1, 2, ..., m_{ij} \\ h^-_{ii} = h^+_{ii} = 0.5 \end{cases} \quad (1)$$

for all i, j =1, 2, ..., n with $i \neq j$, and m_{ij} is the number of elements in $\tilde{h}_{ij}$.

For example, let $X = \{x_1, x_2, x_3\}$. Then, an IVHFPR $\tilde{H}$ on X may be defined as follows:

$$\tilde{H} = \begin{pmatrix} \{[0.5, 0.5]\} & \{[0.2, 0.3], [0.4, 0.5]\} & \{[0.3, 0.4], [0.5, 0.7], [0.8, 0.9]\} \\ \{[0.5, 0.6], [0.7, 0.8]\} & \{[0.5, 0.5]\} & \{[0.4, 0.6]\} \\ \{[0.1, 0.2], [0.3, 0.5], [0.6, 0.7]\} & \{[0.4, 0.6]\} & \{[0.5, 0.5]\} \end{pmatrix}.$$

From the concept of IVHFPRs, one can check that the IVHFPR $\tilde{H}$ reduces to a hesitant fuzzy preference relation [31] when the endpoints of all intervals in each IVHFE are equal. Furthermore, it reduces to an interval fuzzy preference relation (IFPR) [28] when there is only one interval in each IVHFE. Because IFPRs have a close relationship with our multiplicatively consistent concept for IVHFPRs, it needs to give the concept of IFPRs.

Definition 4 [28] An IFPR $\bar{A}$ on $X = \{x_1, x_2, \ldots, x_n\}$ is defined as follows:

$$\bar{A} = (\bar{a}_{ij})_{n\times n} = \begin{pmatrix} [0.5, 0.5] & [a^-_{12}, a^+_{12}] & \ldots & [a^-_{1n}, a^+_{1n}] \\ [a^-_{21}, a^+_{21}] & [0.5, 0.5] & \ldots & [a^-_{2n}, a^+_{2n}] \\ \vdots & \vdots & \vdots & \vdots \\ [a^-_{n1}, a^+_{n1}] & [a^-_{n2}, a^+_{n2}] & \ldots & [0.5, 0.5] \end{pmatrix}, \quad (2)$$

where $a^-_{ij}, a^+_{ij} > 0$ such that $a^-_{ij} \leq a^+_{ij}$ and $a^-_{ij} + a^+_{ji} = a^+_{ij} + a^-_{ji} = 1$ for all i, j =1, 2, ..., n, and $\bar{a}_{ij}$ is the interval intensity of the object x_i over x_j

Let us review several Minkowski operations on intervals. Let $\bar{a} = [a^-, a^+]$ and $\bar{b} = [b^-, b^+]$ be any two positive intervals, then

(i) $\bar{a} \oplus \bar{b} = [a^- + b^-, a^+ + b^+]$;
(ii) $\bar{a} \otimes \bar{b} = [a^-b^-, a^+b^+]$;
(iii) $\bar{a} / \bar{b} = [a^- / b^+, a^+ / b^-]$;
(iv) $\lambda\bar{a} = [\lambda a^-, \lambda a^+]\lambda \geq 0$;
(v) $\log_\lambda \bar{a} = [\log_\lambda a^-, \log_\lambda a^+]\lambda > 0 \wedge \lambda \neq 1$.

After reviewing the previous consistency concepts for IFPRs, Meng et al. [13] found several limitations and introduced a new one using quasiIFPRs.

Definition 5 [13] Let $\bar{A} = (\bar{a}_{ij})_{n\times n}$ be an IFPR $\bar{B} = (\bar{b}_{ij})_{n\times n}$ is said to be a quasi IFPR (QIFPR) with respect to $\bar{A}$ if $\begin{cases} \bar{b}_{ij} = \bar{a}_{ij} \\ \bar{b}_{ji} = \bar{a}^\circ_{ji} \end{cases}$ or $\begin{cases} \bar{b}_{ij} = \bar{a}^\circ_{ij} \\ \bar{b}_{ji} = \bar{a}_{ji} \end{cases}$ for all $i, j = 1, 2, ..., n$, where $\bar{a}^\circ_{ij}$ is called the quasi interval of $\bar{a}_{ij} = [a^-_{ij}, a^+_{ij}]$ such that $\bar{a}^\circ_{ij} = [a^+_{ij}, a^-_{ij}]$ for all $i, j = 1, 2, ..., n$.

It is worth noting that the operational laws on quasi intervals are the same as those on intervals, where

(i) $\bar{a}^\circ \oplus \bar{b} = [a^+ + b^-, a^- + b^+]$, $\bar{a} \oplus \bar{b}^\circ = [a^- + b^+, a^+ + b^-]$, $\bar{a}^\circ \oplus \bar{b}^\circ = [a^+ + b^+, a^- + b^-]$;
(ii) $\bar{a}^\circ \otimes \bar{b} = [a^+b^-, a^-b^+]$, $\bar{a} \otimes \bar{b}^\circ = [a^-b^+, a^+b^-]$, $\bar{a}^\circ \otimes \bar{b}^\circ = [a^+b^+, a^-b^-]$;
(iii) $\bar{a}^\circ / \bar{b} = [a^+ / b^+, a^- / b^-]$, $\bar{a} / \bar{b}^\circ = [a^- / b^-, a^+ / b^+]$, $\bar{a}^\circ / \bar{b}^\circ = [a^+ / b^-, a^- / b^+]$;
(iv) $\lambda\bar{a}^\circ = [\lambda a^+, \lambda a^-]\lambda \geq 0$;
(v) $\log_\lambda \bar{a}^\circ = [\log_\lambda a^+, \log_\lambda a^-]\lambda > 0 \wedge \lambda \neq 1$

with $\bar{a} = [a^-, a^+]$ and $\bar{b} = [b^-, b^+]$ being any two positive intervals. Because quasi intervals are only applied to judge the consistency of IFPRs, the operational results, intervals or quasi intervals, have no influence on the following discussions.

Definition 6 [13] Let $\bar{A} = (\bar{a}_{ij})_{n\times n}$ be an IFPR. $\bar{A}$ is multiplicatively consistent if there is a multiplicatively consistent QIFPR $\bar{B} = (\bar{b}_{ij})_{n\times n}$, namely,

$$\bar{b}_{ij} \otimes \bar{b}_{jk} \otimes \bar{b}_{ki} = \bar{b}_{ji} \otimes \bar{b}_{ik} \otimes \bar{b}_{kj} \quad (3)$$

for all $i, k, j = 1, 2, ..., n$.

One can check that Definition 6 degenerates to Tanino's multiplicative consistency concept [19] when the IFPR $\bar{A}$ reduces to a reciprocal preference relation.

3 A multiplicative consistency concept for IVHFPRs

Because IFPRs can be seen as a special case of IVHFPRs, it makes us think of extending the multiplicative consistency concept for IFPRs to define the consistency of IVHFPRs. The question is that more than one interval in IVHFEs exists and the number of intervals in different IVHFEs is not equal. As we know, every element in crisp preference relations is used to judge the consistency, and their influence is same. According to this principle, we introduce the following multiplicative consistency concept:

Definition 7 Let $\tilde{H} = (\tilde{h}_{ij})_{n\times n}$ be an IVHFPR on the object set $X = \{x_1, x_2, \ldots, x_n\}$, and let $\bar{h}_{l,i_0 j_0}$ be an interval in $\tilde{h}_{i_0 j_0}$. $\tilde{H}$ is $\bar{h}_{l,i_0 j_0}$-multiplicatively consistent if there is a multiplicatively consistent IFPR $\bar{A} = (\bar{a}_{ij})_{n\times n}$, where $\bar{a}_{ij} \in \tilde{h}_{ij}$ for all $i, j = 1, 2, \ldots, n$ with $\bar{a}_{i_0 j_0} = \bar{h}_{l,i_0 j_0}$ and $l \in \{1, 2, ..., m_{i_0 j_0}\}$.

One can check that Definition 7 degenerates to Definition 6 when there is only one interval in every IVHFE. Similarly, we can further extend Definition 7 to give the following multiplicative consistency concept:

Definition 8 Let $\tilde{H} = (\tilde{h}_{ij})_{n\times n}$ be an IVHFPR on the object set $X = \{x_1, x_2, \ldots, x_n\}$. $\tilde{H}$ is $\tilde{h}_{i_0 j_0}$-multiplicatively consistent if there is a multiplicatively consistent IFPR for each interval in $\tilde{h}_{i_0 j_0}$, namely, for any interva l $\bar{h}_{l,i_0 j_0} \in \tilde{h}_{i_0 j_0} l = 1, 2, ..., m_{i_0 j_0}$, there is a multiplicatively consistent IFPR $\bar{A} = (\bar{a}_{ij})_{n\times n}$ such that $\bar{a}_{ij} \in \tilde{h}_{ij}$ for all $i, j = 1, 2, \ldots, n$ with $\bar{a}_{i_0 j_0} = \bar{h}_{l,i_0 j_0}$.

Definition 6 shows that the IFPR $\bar{A} = (\bar{a}_{ij})_{n\times n}$ is multiplicatively consistent if and only if it is $\bar{a}_{ij}$-multiplicatively consistent. However, this conclusion will not necessarily hold for IVHFPRs. For example, let $\tilde{H}$ be an IVHFPR on $X = \{x_1, x_2, x_3\}$, where

$$\tilde{H} = \begin{pmatrix} \left\{\left[\frac{1}{2}, \frac{1}{2}\right]\right\} & \left\{\left[\frac{1}{5}, \frac{3}{10}\right]\right\} & \left\{\left[\frac{1}{7}, \frac{3}{10}\right], \left[\frac{2}{5}, \frac{1}{2}\right]\right\} \\ \left\{\left[\frac{7}{10}, \frac{4}{5}\right]\right\} & \left\{\left[\frac{1}{2}, \frac{1}{2}\right]\right\} & \left\{\left[\frac{2}{5}, \frac{1}{2}\right], \left[\frac{2}{3}, \frac{3}{4}\right]\right\} \\ \left\{\left[\frac{1}{2}, \frac{3}{5}\right], \left[\frac{7}{10}, \frac{6}{7}\right]\right\} & \left\{\left[\frac{1}{4}, \frac{1}{3}\right], \left[\frac{1}{2}, \frac{3}{5}\right]\right\} & \left\{\left[\frac{1}{2}, \frac{1}{2}\right]\right\} \end{pmatrix}.$$

One can check that $\tilde{H}$ is $\tilde{h}_{12}$-multiplicatively consistent, where the multiplicatively consistent IFPR is $\bar{A} = \begin{pmatrix} \left[\frac{1}{2}, \frac{1}{2}\right] & \left[\frac{1}{5}, \frac{3}{10}\right] & \left[\frac{1}{7}, \frac{3}{10}\right] \\ \left[\frac{7}{10}, \frac{4}{5}\right] & \left[\frac{1}{2}, \frac{1}{2}\right] & \left[\frac{2}{5}, \frac{1}{2}\right] \\ \left[\frac{7}{10}, \frac{6}{7}\right] & \left[\frac{1}{2}, \frac{3}{5}\right] & \left[\frac{1}{2}, \frac{1}{2}\right] \end{pmatrix}$, but it is inconsistent for $\tilde{h}_{13}$ and $\tilde{h}_{23}$ Thus, we need some extra conditions to define the multiplicative consistency of IVHFPRs.

Definition 9 Let $\tilde{H} = (\tilde{h}_{ij})_{n\times n}$ be an IVHFPR on the object set $X = \{x_1, x_2, \ldots, x_n\}$. $\tilde{H}$is multiplicatively consistent if for each interval in $\tilde{h}_{ij}$, there is a multiplicatively consistent IFPR following Definition 6 for all $i, j = 1, 2, \ldots, n$, namely, $\tilde{H}$is $\tilde{h}_{ij}$-multiplicatively consistent for all $i, j = 1, 2, \ldots, n$.

Example 3.1 Let $X = \{x_1, x_2, x_3, x_4\}$ be the object set, and let the IVHFPR $\tilde{H}$ on X be defined as follows:

$$\tilde{H} = \begin{pmatrix} \left\{\left[\frac{1}{2}, \frac{1}{2}\right]\right\} & \left\{\left[\frac{2}{3}, \frac{3}{4}\right]\right\} & \left\{\left[\frac{2}{5}, \frac{3}{5}\right]\right\} & \left\{\left[\frac{4}{13}, \frac{3}{5}\right], \left[\frac{3}{4}, \frac{4}{5}\right]\right\} \\ \left\{\left[\frac{1}{4}, \frac{1}{3}\right]\right\} & \left\{\left[\frac{1}{2}, \frac{1}{2}\right]\right\} & \left\{\left[\frac{1}{4}, \frac{1}{3}\right]\right\} & \left\{\left[\frac{2}{11}, \frac{1}{3}\right], \left[\frac{1}{2}, \frac{2}{3}\right]\right\} \\ \left\{\left[\frac{2}{5}, \frac{3}{5}\right]\right\} & \left\{\left[\frac{2}{3}, \frac{3}{4}\right]\right\} & \left\{\left[\frac{1}{2}, \frac{1}{2}\right]\right\} & \left\{\left[\frac{2}{5}, \frac{1}{2}\right], \left[\frac{2}{3}, \frac{6}{7}\right]\right\} \\ \left\{\left[\frac{1}{5}, \frac{1}{4}\right], \left[\frac{2}{5}, \frac{9}{13}\right]\right\} & \left\{\left[\frac{1}{3}, \frac{1}{2}\right], \left[\frac{2}{3}, \frac{9}{11}\right]\right\} & \left\{\left[\frac{1}{7}, \frac{1}{3}\right], \left[\frac{1}{2}, \frac{3}{5}\right]\right\} & \left\{\left[\frac{1}{2}, \frac{1}{2}\right]\right\} \end{pmatrix}.$$

One can check that the IVHFPR $\tilde{H}$ is multiplicatively consistent.

As an extension of Definition 6, one can check that Definition 9 satisfies robustness, namely, the consistency conclusion is independent of the compared orders of objects. However, it is not an easy thing to judge the consistency of IVHFPRs because there may be several intervals in each IVHFE. To address this issue, we introduce a programming model based method for judging the multiplicative consistency of IVIFPRs.

Let $\alpha_{l,ij} = \begin{cases} 1 & \text{if } \bar{h}_{l,ij} \text{ is chosen} \\ 0 & \text{otherwise} \end{cases}$ and $\alpha_{l,ij} = \alpha_{m_{ij}+1-l,ji}$ for all i, j =1, 2, ..., n and all $l = 1, 2, \ldots, m_{ij}$. Then, each interval in the IVHFE $\tilde{h}_{ij}$ can be denoted as: $\otimes_{l=1}^{m_{ij}} \left(\bar{h}_{l,ij}\right)^{\alpha_{l,ij}}$ with $\sum_{l=1}^{m_{ij}} \alpha_{l,ij} = 1$. Thus, any IFPR $\bar{A} = (\bar{a}_{ij})_{n\times n}$ obtained from the IVHFPR $\tilde{H} = (\tilde{h}_{ij})_{n\times n}$ can be expressed as follows:

$$\bar{A} = \begin{pmatrix} [0.5, 0.5] & \otimes_{l=1}^{m_{12}} \left(\bar{h}_{l,12}\right)^{\alpha_{l,12}} & \cdots & \otimes_{l=1}^{m_{1n}} \left(\bar{h}_{l,1n}\right)^{\alpha_{l,1n}} \\ \otimes_{l=1}^{m_{12}} \left(\bar{h}_{l,21}\right)^{\alpha_{l,21}} & [0.5, 0.5] & \cdots & \otimes_{l=1}^{m_{2n}} \left(\bar{h}_{l,2n}\right)^{\alpha_{l,2n}} \\ \vdots & \vdots & \vdots & \vdots \\ \otimes_{l=1}^{m_{1n}} \left(\bar{h}_{l,n1}\right)^{\alpha_{l,n1}} & \otimes_{l=1}^{m_{2n}} \left(\bar{h}_{l,n2}\right)^{\alpha_{l,n2}} & & [0.5, 0.5] \end{pmatrix} \quad (4)$$

with $\sum_{l=1}^{m_{ij}} \alpha_{l,ij} = 1$ and $\alpha_{l,ij} = \alpha_{m_{ij}+1-l,ji}$ for all $i, j = 1, 2, \ldots, n$.

To judge the multiplicative consistency of the IVHFPR $\tilde{H}$, one can check whether the IFPR $\bar{A}$ is multiplicatively consistent, where $\alpha_{l,ij} = 1$ for all i, j =1, 2, ..., n and all $l = 1, 2, \ldots, m_{ij}$.

When the IFPR $\bar{A}$ defined by formula (4) is multiplicatively consistent, there are the 0-1 indicator variables $\delta_{ij} = \begin{cases} 1 & \text{if } \bar{a}_{ij} = \otimes_{l=1}^{m_{ij}} \left(\bar{h}_{l,ij}\right)^{\alpha_{l,ij}} \\ 0 & \text{if } \bar{a}_{ij} = \left(\otimes_{l=1}^{m_{ij}} \left(\bar{h}_{l,ij}\right)^{\alpha_{l,ij}}\right)^{\circ} \end{cases}$ with $\delta_{ij}+\delta_{ji} = 1$ for all i, j =1, 2, ..., n satisfying formula (3), namely,

$$\left(\left(\otimes_{l=1}^{m_{ij}} \left(\bar{h}_{l,ij}\right)^{\alpha_{l,ij}}\right)^{\delta_{ij}} \otimes \left(\left(\otimes_{l=1}^{m_{ij}} \left(\bar{h}_{l,ij}\right)^{\alpha_{l,ij}}\right)^{\circ}\right)^{(1-\delta_{ij})}\right) \otimes \left(\left(\otimes_{l=1}^{m_{jk}} \left(\bar{h}_{l,jk}\right)^{\alpha_{l,jk}}\right)^{\delta_{jk}} \otimes \left(\left(\otimes_{l=1}^{m_{jk}} \left(\bar{h}_{l,jk}\right)^{\alpha_{l,jk}}\right)^{\circ}\right)^{(1-\delta_{jk})}\right)$$
$$\otimes \left(\left(\otimes_{l=1}^{m_{ki}} \left(\bar{h}_{l,ki}\right)^{\alpha_{l,ki}}\right)^{\delta_{ki}} \otimes \left(\left(\otimes_{l=1}^{m_{ki}} \left(\bar{h}_{l,ki}\right)^{\alpha_{l,ki}}\right)^{\circ}\right)^{(1-\delta_{ki})}\right) = \left(\left(\otimes_{l=1}^{m_{ji}} \left(\bar{h}_{l,ji}\right)^{\alpha_{l,ji}}\right)^{\delta_{ji}} \otimes \left(\left(\otimes_{l=1}^{m_{ji}} \left(\bar{h}_{l,ji}\right)^{\alpha_{l,ji}}\right)^{\circ}\right)^{(1-\delta_{ji})}\right)$$
$$\otimes \left(\left(\otimes_{l=1}^{m_{ik}} \left(\bar{h}_{l,ik}\right)^{\alpha_{l,ik}}\right)^{\delta_{ik}} \otimes \left(\left(\otimes_{l=1}^{m_{ik}} \left(\bar{h}_{l,ik}\right)^{\alpha_{l,ik}}\right)^{\circ}\right)^{(1-\delta_{ik})}\right) \otimes \left(\left(\otimes_{l=1}^{m_{kj}} \left(\bar{h}_{l,kj}\right)^{\alpha_{l,kj}}\right)^{\delta_{kj}} \otimes \left(\left(\otimes_{l=1}^{m_{kj}} \left(\bar{h}_{l,kj}\right)^{\alpha_{l,kj}}\right)^{\circ}\right)^{(1-\delta_{kj})}\right), \quad (5)$$

where i, k, j =1, 2, ..., n.

We take the logarithm on both sides of formula (5) and get

$$\delta_{ij} \left(\sum_{l=1}^{m_{ij}} \alpha_{l,ij} \log\left(\bar{h}_{l,ij}\right)\right) \oplus (1-\delta_{ij}) \left(\sum_{l=1}^{m_{ij}} \alpha_{l,ij} \log\left(\bar{h}_{l,ij}\right)\right)^{\circ} \oplus \delta_{jk} \left(\sum_{l=1}^{m_{jk}} \alpha_{l,jk} \log\left(\bar{h}_{l,jk}\right)\right) \oplus (1-\delta_{jk})$$
$$\left(\sum_{l=1}^{m_{jk}} \alpha_{l,jk} \log\left(\bar{h}_{l,jk}\right)\right)^{\circ} \oplus \delta_{ki} \left(\sum_{l=1}^{m_{ki}} \alpha_{l,ki} \log\left(\bar{h}_{l,ki}\right)\right) \oplus (1-\delta_{ki}) \left(\sum_{l=1}^{m_{ki}} \alpha_{l,ki} \log\left(\bar{h}_{l,ki}\right)\right)^{\circ}$$
$$= \delta_{ji} \left(\sum_{l=1}^{m_{ji}} \alpha_{l,ji} \log\left(\bar{h}_{l,ji}\right)\right) \oplus (1-\delta_{ji}) \left(\sum_{l=1}^{m_{ji}} \alpha_{l,ji} \log\left(\bar{h}_{l,ji}\right)\right)^{\circ} \oplus \delta_{ik} \left(\sum_{l=1}^{m_{ik}} \alpha_{l,ik} \log\left(\bar{h}_{l,ik}\right)\right)$$
$$\oplus (1-\delta_{ik}) \left(\sum_{l=1}^{m_{ik}} \alpha_{l,ik} \log\left(\bar{h}_{l,ik}\right)\right)^{\circ} \oplus \delta_{kj} \left(\sum_{l=1}^{m_{kj}} \alpha_{l,kj} \log\left(\bar{h}_{l,kj}\right)\right) \oplus (1-\delta_{kj}) \left(\sum_{l=1}^{m_{kj}} \alpha_{l,kj} \log\left(\bar{h}_{l,kj}\right)\right)^{\circ}, \quad (6)$$

where i, k, j =1, 2, ..., n.

For per (6), we further derive

$$\begin{cases} \delta_{ij} \left(\sum_{l=1}^{m_{ij}} \alpha_{l,ij} \log\left(h_{l,ij}^{-}\right)\right) + (1-\delta_{ij}) \left(\sum_{l=1}^{m_{ij}} \alpha_{l,ij} \log\left(h_{l,ij}^{+}\right)\right) + \delta_{jk} \left(\sum_{l=1}^{m_{jk}} \alpha_{l,jk} \log\left(h_{l,jk}^{-}\right)\right) + (1-\delta_{jk}) \left(\sum_{l=1}^{m_{jk}} \alpha_{l,jk} \log\left(h_{l,jk}^{+}\right)\right) + \\ \delta_{ki} \left(\sum_{l=1}^{m_{ki}} \alpha_{l,ki} \log\left(h_{l,ki}^{-}\right)\right) + (1-\delta_{ki})\delta_{ki} \left(\sum_{l=1}^{m_{ki}} \alpha_{l,ki} \log\left(h_{l,ki}^{+}\right)\right) = \delta_{ji} \left(\sum_{l=1}^{m_{ji}} \alpha_{l,ji} \log\left(h_{l,ji}^{-}\right)\right) + (1-\delta_{ji}) \left(\sum_{l=1}^{m_{ji}} \alpha_{l,ji} \log\left(h_{l,ji}^{+}\right)\right) + \\ \delta_{ik} \left(\sum_{l=1}^{m_{ik}} \alpha_{l,ik} \log\left(h_{l,ik}^{-}\right)\right) + (1-\delta_{ik}) \left(\sum_{l=1}^{m_{ik}} \alpha_{l,ik} \log\left(h_{l,ik}^{+}\right)\right) + \delta_{kj} \left(\sum_{l=1}^{m_{kj}} \alpha_{l,kj} \log\left(h_{l,kj}^{-}\right)\right) + (1-\delta_{kj}) \left(\sum_{l=1}^{m_{kj}} \alpha_{l,kj} \log\left(h_{l,kj}^{+}\right)\right) \\ \delta_{ij} \left(\sum_{l=1}^{m_{ij}} \alpha_{l,ij} \log\left(h_{l,ij}^{+}\right)\right) + (1-\delta_{ij}) \left(\sum_{l=1}^{m_{ij}} \alpha_{l,ij} \log\left(h_{l,ij}^{-}\right)\right) + \delta_{jk} \left(\sum_{l=1}^{m_{jk}} \alpha_{l,jk} \log\left(h_{l,jk}^{+}\right)\right) + (1-\delta_{jk}) \left(\sum_{l=1}^{m_{jk}} \alpha_{l,jk} \log\left(h_{l,jk}^{-}\right)\right) + \\ \delta_{ki} \left(\sum_{l=1}^{m_{ki}} \alpha_{l,ki} \log\left(h_{l,ki}^{+}\right)\right) + (1-\delta_{ki})\delta_{ki} \left(\sum_{l=1}^{m_{ki}} \alpha_{l,ki} \log\left(h_{l,ki}^{-}\right)\right) = \delta_{ji} \left(\sum_{l=1}^{m_{ji}} \alpha_{l,ji} \log\left(h_{l,ji}^{+}\right)\right) + (1-\delta_{ji}) \left(\sum_{l=1}^{m_{ji}} \alpha_{l,ji} \log\left(h_{l,ji}^{-}\right)\right) + \\ \delta_{ik} \left(\sum_{l=1}^{m_{ik}} \alpha_{l,ik} \log\left(h_{l,ik}^{+}\right)\right) + (1-\delta_{ik}) \left(\sum_{l=1}^{m_{ik}} \alpha_{l,ik} \log\left(h_{l,ik}^{-}\right)\right) + \delta_{kj} \left(\sum_{l=1}^{m_{kj}} \alpha_{l,kj} \log\left(h_{l,kj}^{+}\right)\right) + (1-\delta_{kj}) \left(\sum_{l=1}^{m_{kj}} \alpha_{l,kj} \log\left(h_{l,kj}^{-}\right)\right) \end{cases}. \quad (7)$$

Because formula (7) will not necessarily hold for any given IVHFPR $\tilde{H} = (\tilde{h}_{ij})_{n\times n}$, we relax it by introducing the positive and negative derivation values $\varepsilon_{ij}^{-}, \varepsilon_{ij}^{+}, \tau_{ij}^{-}, \tau_{ij}^{+}$ such that $\varepsilon_{ij}^{-}, \varepsilon_{ij}^{+}, \tau_{ij}^{-}, \tau_{ij}^{+} \geq 0$ for all i, j =1, 2, ..., n, where

$$\begin{cases} \delta_{ij} \left(\sum_{l=1}^{m_{ij}} \alpha_{l,ij} \log\left(h_{l,ij}^{-}\right)\right) + (1-\delta_{ij}) \left(\sum_{l=1}^{m_{ij}} \alpha_{l,ij} \log\left(h_{l,ij}^{+}\right)\right) + \delta_{jk} \left(\sum_{l=1}^{m_{jk}} \alpha_{l,jk} \log\left(h_{l,jk}^{-}\right)\right) + (1-\delta_{jk}) \left(\sum_{l=1}^{m_{jk}} \alpha_{l,jk} \log\left(h_{l,jk}^{+}\right)\right) + \\ \delta_{ki} \left(\sum_{l=1}^{m_{ki}} \alpha_{l,ki} \log\left(h_{l,ki}^{-}\right)\right) + (1-\delta_{ki})\delta_{ki} \left(\sum_{l=1}^{m_{ki}} \alpha_{l,ki} \log\left(h_{l,ki}^{+}\right)\right) - \left(\delta_{ji} \left(\sum_{l=1}^{m_{ji}} \alpha_{l,ji} \log\left(h_{l,ji}^{-}\right)\right) + (1-\delta_{ji}) \left(\sum_{l=1}^{m_{ji}} \alpha_{l,ji} \log\left(h_{l,ji}^{+}\right)\right) + \right. \\ \left. \delta_{ik} \left(\sum_{l=1}^{m_{ik}} \alpha_{l,ik} \log\left(h_{l,ik}^{-}\right)\right) + (1-\delta_{ik}) \left(\sum_{l=1}^{m_{ik}} \alpha_{l,ik} \log\left(h_{l,ik}^{+}\right)\right) + \delta_{kj} \left(\sum_{l=1}^{m_{kj}} \alpha_{l,kj} \log\left(h_{l,kj}^{-}\right)\right) + (1-\delta_{kj}) \left(\sum_{l=1}^{m_{kj}} \alpha_{l,kj} \log\left(h_{l,kj}^{+}\right)\right)\right) + \varepsilon_{ij}^{-} - \varepsilon_{ij}^{+} = 0 \\ \delta_{ij} \left(\sum_{l=1}^{m_{ij}} \alpha_{l,ij} \log\left(h_{l,ij}^{+}\right)\right) + (1-\delta_{ij}) \left(\sum_{l=1}^{m_{ij}} \alpha_{l,ij} \log\left(h_{l,ij}^{-}\right)\right) + \delta_{jk} \left(\sum_{l=1}^{m_{jk}} \alpha_{l,jk} \log\left(h_{l,jk}^{+}\right)\right) + (1-\delta_{jk}) \left(\sum_{l=1}^{m_{jk}} \alpha_{l,jk} \log\left(h_{l,jk}^{-}\right)\right) + \\ \delta_{ki} \left(\sum_{l=1}^{m_{ki}} \alpha_{l,ki} \log\left(h_{l,ki}^{+}\right)\right) + (1-\delta_{ki})\delta_{ki} \left(\sum_{l=1}^{m_{ki}} \alpha_{l,ki} \log\left(h_{l,ki}^{-}\right)\right) - \left(\delta_{ji} \left(\sum_{l=1}^{m_{ji}} \alpha_{l,ji} \log\left(h_{l,ji}^{+}\right)\right) + (1-\delta_{ji}) \left(\sum_{l=1}^{m_{ji}} \alpha_{l,ji} \log\left(h_{l,ji}^{-}\right)\right) + \right. \\ \left. \delta_{ik} \left(\sum_{l=1}^{m_{ik}} \alpha_{l,ik} \log\left(h_{l,ik}^{+}\right)\right) + (1-\delta_{ik}) \left(\sum_{l=1}^{m_{ik}} \alpha_{l,ik} \log\left(h_{l,ik}^{-}\right)\right) + \delta_{kj} \left(\sum_{l=1}^{m_{kj}} \alpha_{l,kj} \log\left(h_{l,kj}^{+}\right)\right) + (1-\delta_{kj}) \left(\sum_{l=1}^{m_{kj}} \alpha_{l,kj} \log\left(h_{l,kj}^{-}\right)\right)\right) + \tau_{ij}^{-} - \tau_{ij}^{+} = 0 \end{cases}. \quad (8)$$

Let $\alpha_{l_0,i_0 j_0} = 1$ for all $i_0,\ j_0 = 1, 2, \ldots, n$ and all $l_0 =1, 2, \ldots, m_{i_0 j_0}$. We construct the following programming model to judge the $\bar{h}_{l_0,i_0 j_0}$-multiplicative consistency of the IVHFPR $\tilde{H} = (\tilde{h}_{ij})_{n\times n}$:

$$J^*_{l_0,i_0 j_0} = \min \sum\nolimits_{i,j=1}^{n} \left(\varepsilon^-_{ij} + \varepsilon^+_{ij} + \tau^+_{ij} + \tau^+_{ij}\right)$$

$$s.t. \begin{cases} \delta_{ij}\left(\sum_{l=1}^{m_{ij}} \alpha_{l,ij} \log\left(h^-_{l,ij}\right)\right) + (1-\delta_{ij})\left(\sum_{l=1}^{m_{ij}} \alpha_{l,ij} \log\left(h^+_{l,ij}\right)\right) + \delta_{jk}\left(\sum_{l=1}^{m_{jk}} \alpha_{l,jk} \log\left(h^-_{l,jk}\right)\right) + (1-\delta_{jk})\left(\sum_{l=1}^{m_{jk}} \alpha_{l,jk} \log\left(h^+_{l,jk}\right)\right) + \\ \delta_{ki}\left(\sum_{l=1}^{m_{ki}} \alpha_{l,ki} \log\left(h^-_{l,ki}\right)\right) + (1-\delta_{ki})\delta_{ki}\left(\sum_{l=1}^{m_{ki}} \alpha_{l,ki} \log\left(h^+_{l,ki}\right)\right) - \left(\delta_{ji}\left(\sum_{l=1}^{m_{ji}} \alpha_{l,ji} \log\left(h^-_{l,ji}\right)\right) + (1-\delta_{ji})\left(\sum_{l=1}^{m_{ji}} \alpha_{l,ji} \log\left(h^+_{l,ji}\right)\right) + \right. \\ \left. \delta_{ik}\left(\sum_{l=1}^{m_{ik}} \alpha_{l,ik} \log\left(h^-_{l,ik}\right)\right) + (1-\delta_{ik})\left(\sum_{l=1}^{m_{ik}} \alpha_{l,ik} \log\left(h^+_{l,ik}\right)\right) + \delta_{kj}\left(\sum_{l=1}^{m_{kj}} \alpha_{l,kj} \log\left(h^-_{l,kj}\right)\right) + (1-\delta_{kj})\left(\sum_{l=1}^{m_{kj}} \alpha_{l,kj} \log\left(h^+_{l,kj}\right)\right)\right) \\ +\varepsilon^-_{ij} - \varepsilon^+_{ij} = 0, i, k, j = 1, 2, ..., n \\ \delta_{ij}\left(\sum_{l=1}^{m_{ij}} \alpha_{l,ij} \log\left(h^+_{l,ij}\right)\right) + (1-\delta_{ij})\left(\sum_{l=1}^{m_{ij}} \alpha_{l,ij} \log\left(h^-_{l,ij}\right)\right) + \delta_{jk}\left(\sum_{l=1}^{m_{jk}} \alpha_{l,jk} \log\left(h^+_{l,jk}\right)\right) + (1-\delta_{jk})\left(\sum_{l=1}^{m_{jk}} \alpha_{l,jk} \log\left(h^-_{l,jk}\right)\right) + \\ \delta_{ki}\left(\sum_{l=1}^{m_{ki}} \alpha_{l,ki} \log\left(h^+_{l,ki}\right)\right) + (1-\delta_{ki})\delta_{ki}\left(\sum_{l=1}^{m_{ki}} \alpha_{l,ki} \log\left(h^-_{l,ki}\right)\right) - \left(\delta_{ji}\left(\sum_{l=1}^{m_{ji}} \alpha_{l,ji} \log\left(h^+_{l,ji}\right)\right) + (1-\delta_{ji})\left(\sum_{l=1}^{m_{ji}} \alpha_{l,ji} \log\left(h^-_{l,ji}\right)\right) + \right. \\ \left. \delta_{ik}\left(\sum_{l=1}^{m_{ik}} \alpha_{l,ik} \log\left(h^+_{l,ik}\right)\right) + (1-\delta_{ik})\left(\sum_{l=1}^{m_{ik}} \alpha_{l,ik} \log\left(h^-_{l,ik}\right)\right) + \delta_{kj}\left(\sum_{l=1}^{m_{kj}} \alpha_{l,kj} \log\left(h^+_{l,kj}\right)\right) + (1-\delta_{kj})\left(\sum_{l=1}^{m_{kj}} \alpha_{l,kj} \log\left(h^-_{l,kj}\right)\right)\right) \\ +\tau^-_{ij} - \tau^+_{ij} = 0, i, k, j = 1, 2, ..., n \\ \alpha_{l,ij} = 0 \vee 1, i, j = 1, 2, ..., n, l = 1, 2, ..., m_{ij} \\ \alpha_{l,ij} = \alpha_{m_{ij}+1-l,ji}, i, j = 1, 2, ..., n, l = 1, 2, ..., m_{ij} \\ \sum_{l=1}^{m_{ij}} \alpha_{l,ij} = 1, i, j = 1, 2, ..., n \\ \delta_{ij} = 0 \vee 1, i, j = 1, 2, ..., n \\ \delta_{ij} + \delta_{ji} = 1, i, j = 1, 2, ..., n \\ \alpha_{l_0,i_0 j_0} = 1 \end{cases} . \quad (9)$$

From the robustness of Definition 9, we can only apply the upper triangular part to judge the multiplicative consistency. Thus, we further construct the following programming model to judge the $\bar{h}_{l_0,i_0 j_0}$-multiplicative consistency:

$$K^*_{l_0,i_0 j_0} = \min \sum\nolimits_{i,j=1,i<j}^{n} \left(\varepsilon^-_{ij} + \varepsilon^+_{ij} + \tau^+_{ij} + \tau^+_{ij}\right)$$

$$s.t. \begin{cases} \delta_{ij}\left(\sum_{l=1}^{m_{ij}} \alpha_{l,ij} \log\left(h^-_{l,ij}\right)\right) + (1-\delta_{ij})\left(\sum_{l=1}^{m_{ij}} \alpha_{l,ij} \log\left(h^+_{l,ij}\right)\right) + \delta_{jk}\left(\sum_{l=1}^{m_{jk}} \alpha_{l,jk} \log\left(h^-_{l,jk}\right)\right) + (1-\delta_{jk})\left(\sum_{l=1}^{m_{jk}} \alpha_{l,jk} \log\left(h^+_{l,jk}\right)\right) + \\ \delta_{ki}\left(\sum_{l=1}^{m_{ki}} \alpha_{l,ki} \log\left(h^-_{l,ki}\right)\right) + (1-\delta_{ki})\delta_{ki}\left(\sum_{l=1}^{m_{ki}} \alpha_{l,ki} \log\left(h^+_{l,ki}\right)\right) - \left(\delta_{ji}\left(\sum_{l=1}^{m_{ji}} \alpha_{l,ji} \log\left(h^-_{l,ji}\right)\right) + (1-\delta_{ji})\left(\sum_{l=1}^{m_{ji}} \alpha_{l,ji} \log\left(h^+_{l,ji}\right)\right) + \right. \\ \left. \delta_{ik}\left(\sum_{l=1}^{m_{ik}} \alpha_{l,ik} \log\left(h^-_{l,ik}\right)\right) + (1-\delta_{ik})\left(\sum_{l=1}^{m_{ik}} \alpha_{l,ik} \log\left(h^+_{l,ik}\right)\right) + \delta_{kj}\left(\sum_{l=1}^{m_{kj}} \alpha_{l,kj} \log\left(h^-_{l,kj}\right)\right) + (1-\delta_{kj})\left(\sum_{l=1}^{m_{kj}} \alpha_{l,kj} \log\left(h^+_{l,kj}\right)\right)\right) \\ +\varepsilon^-_{ij} - \varepsilon^+_{ij} = 0, i, k, j = 1, 2, ..., n, i < k < j \\ \delta_{ij}\left(\sum_{l=1}^{m_{ij}} \alpha_{l,ij} \log\left(h^+_{l,ij}\right)\right) + (1-\delta_{ij})\left(\sum_{l=1}^{m_{ij}} \alpha_{l,ij} \log\left(h^-_{l,ij}\right)\right) + \delta_{jk}\left(\sum_{l=1}^{m_{jk}} \alpha_{l,jk} \log\left(h^+_{l,jk}\right)\right) + (1-\delta_{jk})\left(\sum_{l=1}^{m_{jk}} \alpha_{l,jk} \log\left(h^-_{l,jk}\right)\right) + \\ \delta_{ki}\left(\sum_{l=1}^{m_{ki}} \alpha_{l,ki} \log\left(h^+_{l,ki}\right)\right) + (1-\delta_{ki})\delta_{ki}\left(\sum_{l=1}^{m_{ki}} \alpha_{l,ki} \log\left(h^-_{l,ki}\right)\right) - \left(\delta_{ji}\left(\sum_{l=1}^{m_{ji}} \alpha_{l,ji} \log\left(h^+_{l,ji}\right)\right) + (1-\delta_{ji})\left(\sum_{l=1}^{m_{ji}} \alpha_{l,ji} \log\left(h^-_{l,ji}\right)\right) + \right. \\ \left. \delta_{ik}\left(\sum_{l=1}^{m_{ik}} \alpha_{l,ik} \log\left(h^+_{l,ik}\right)\right) + (1-\delta_{ik})\left(\sum_{l=1}^{m_{ik}} \alpha_{l,ik} \log\left(h^-_{l,ik}\right)\right) + \delta_{kj}\left(\sum_{l=1}^{m_{kj}} \alpha_{l,kj} \log\left(h^+_{l,kj}\right)\right) + (1-\delta_{kj})\left(\sum_{l=1}^{m_{kj}} \alpha_{l,kj} \log\left(h^-_{l,kj}\right)\right)\right) \\ +\tau^-_{ij} - \tau^+_{ij} = 0, i, k, j = 1, 2, ..., n, i < k < j \\ \alpha_{l,ij} = 0 \vee 1, i, j = 1, 2, ..., n, l = 1, 2, ..., m_{ij}, i < j \\ \alpha_{l,ij} = \alpha_{m_{ij}+1-l,ji}, i, j = 1, 2, ..., n, l = 1, 2, ..., m_{ij}, i < j \\ \sum_{l=1}^{m_{ij}} \alpha_{l,ij} = 1, i, j = 1, 2, ..., n, i < j \\ \delta_{ij} = 0 \vee 1, i, j = 1, 2, ..., n, i < j \\ \delta_{ij} + \delta_{ji} = 1, i, j = 1, 2, ..., n, i < j \\ \alpha_{l_0,i_0 j_0} = 1 \end{cases} . \quad (10)$$

Solving model (10), if $K^*_{l_0,i_0 j_0} = 0$, the IVHFPR $\tilde{H} = (\tilde{h}_{ij})_{n\times n}$ is $\bar{h}_{l_0,i_0 j_0}$-multiplicatively consistent. Furthermore, if $K^*_{l_0,i_0 j_0} = 0$ for all $l_0 =1, 2, \ldots, m_{i_0 j_0}$, it is $\bar{h}_{i_0 j_0}$-multiplicatively consistent. Moreover, if $K^*_{l_0,i_0 j_0} = 0$ for all $i_0,\ j_0 = 1, 2, \ldots, n$ with $i_0 < j_0$ and all $l_0 =1, 2, \ldots, m_{i_0 j_0}$, it is multiplicatively consistent.

On the other hand, if $K^*_{l_0,i_0 j_0} \neq 0$, then the IVHFPR $\tilde{H}$ is inconsistent. According to the obtained 0-1 indicator variables $\alpha^*_{l,ij}$ and δ^*_{ij} for all $i,\ j = 1, 2, \ldots, n$ with $i < j$ and all $l =1, 2, \ldots, m_{ij}$, we obtain the QIFPR $\bar{B}_{l_0} = (\bar{b}_{l_0,ij})_{n\times n}$, where $\bar{b}_{l_0,ij} = (\otimes_{l=1}^{m_{ij}} (\bar{h}_{l,ij})^{\alpha^*_{l,ij}})^{\delta^*_{ij}} \otimes ((\otimes_{l=1}^{m_{ij}} (\bar{h}_{l,ij})^{\alpha^*_{l,ij}})^{\circ})^{(1-\delta^*_{ij})}$ for all $i,\ j = 1, 2, \ldots, n$, which has the highest consistency level with respect to $\bar{h}_{l_0,i_0 j_0}$ Then, we can adopt the method in Property 6 [13] to derive the multiplicatively consistent QIFPR $\bar{C}_{l_0} = (\bar{c}_{l_0,ij})_{n\times n}$, where

$$\bar{c}_{l_0,ij} = \left[\frac{\sqrt[n]{\Pi_{k=1}^{n} b_{l_0,ik}^{-} b_{l_0,kj}^{-}}}{\sqrt[n]{\Pi_{k=1}^{n} b_{l_0,ik}^{-} b_{l_0,kj}^{-}} + \sqrt[n]{\Pi_{k=1}^{n} b_{l_0,jk}^{-} b_{l_0,ki}^{-}}}, \frac{\sqrt[n]{\Pi_{k=1}^{n} b_{l_0,ik}^{+} b_{l_0,kj}^{+}}}{\sqrt[n]{\Pi_{k=1}^{n} b_{l_0,ik}^{+} b_{l_0,kj}^{+}} + \sqrt[n]{\Pi_{k=1}^{n} b_{l_0,jk}^{+} b_{l_0,ki}^{+}}} \right] \tag{11}$$

for all $ij = 1, 2, ..., n$

Next, we introduce a method for deriving the collectively multiplicatively consistent QIFPR $\bar{C} = (\bar{c}_{ij})_{n\times n}$, by which we can obtain the interval fuzzy priority weight vector. The DM applies several intervals rather than one to denote his/her uncertain judgment for a pair of objects because all of these intervals are regarded as the uncertain preferred degrees. Thus, when the DM does not offer the importance of intervals in an IVHFE, it is reasonable to consider that there is a uniform probability distribution on these intervals, namely, intervals in each IVHFE have the same importance, and their chosen probabilities are equal. From this point of view, we determine the consistency probability distribution on the derived QIFPRs.

Let $\tilde{H} = (\tilde{h}_{ij})_{n\times n}$ be an IVHFPR, and let $\bar{C}_q = (\bar{c}_{q,ij})_{n\times n}$, $q = 1, 2, \ldots, \Delta$, be the multiplicatively consistent QIFPR obtained from formula (11), where Δ is the number of QIFPRs. Let p_q be the consistency probability of the multiplicatively consistent QIFPR $\bar{C}_q$ such that $\sum_{q=1}^{\Delta} p_q = 1$ and $p_q \geq 0$.

To derive the collectively multiplicatively consistent QIFPR $\bar{C} = (\bar{c}_{ij})_{n\times n}$, we apply the interval weighted geometric mean aggregation (IWGMA) operator to calculate the collective interval preference relation $\bar{S} = (\bar{s}_{ij})_{n\times n}$, where $\bar{s}_{ij} = \text{IWGMA}(\bar{c}_{1,ij}, \bar{c}_{2,ij}, ..., \bar{c}_{q,ij}) = \Pi_{q=1}^{\Delta} \left(\bar{c}_{q,ij}\right)^{p_q}$ for all $i, j = 1, 2, \ldots, n$. Because we usually have $\begin{cases} s_{ij}^{-} + s_{ji}^{-} \neq 1 \\ s_{ij}^{+} + s_{ji}^{+} \neq 1 \end{cases}$ for $ij = 1, 2, ..., n$, this means that $\bar{S}$ is not a QIFPR. To address this issue, we apply formula (11) to obtain the collectively multiplicatively consistent QIFPR $\bar{C} = (\bar{c}_{ij})_{n\times n}$.

Following $\bar{C} = (\bar{c}_{ij})_{n\times n}$ we obtain the multiplicatively consistent IFPR $\bar{D} = (\bar{d}_{ij})_{n\times n}$, where $\bar{d}_{ij} = \begin{cases} \bar{c}_{ij} & \bar{c}_{ij} \text{ is an interval} \\ \bar{c}_{ij}^{\circ} & \bar{c}_{ij} \text{ is a quasi-interval} \end{cases}$ for all $ij = 1, 2, ..., n$

To show the above principle clearly, we consider Example 1 again. Using model (10), the following two multiplicatively consistent QIFPRs are derived:

$$\bar{B}_1 = \begin{pmatrix} \left[\frac{1}{2}, \frac{1}{2}\right] & \left[\frac{2}{3}, \frac{3}{4}\right] & \left[\frac{2}{5}, \frac{3}{5}\right] & \left[\frac{4}{13}, \frac{3}{5}\right] \\ \left[\frac{1}{3}, \frac{1}{4}\right] & \left[\frac{1}{2}, \frac{1}{2}\right] & \left[\frac{1}{4}, \frac{1}{3}\right] & \left[\frac{2}{11}, \frac{1}{3}\right] \\ \left[\frac{3}{5}, \frac{2}{5}\right] & \left[\frac{3}{4}, \frac{2}{3}\right] & \left[\frac{1}{2}, \frac{1}{2}\right] & \left[\frac{2}{5}, \frac{1}{2}\right] \\ \left[\frac{9}{13}, \frac{2}{5}\right] & \left[\frac{9}{11}, \frac{2}{3}\right] & \left[\frac{3}{5}, \frac{1}{2}\right] & \left[\frac{1}{2}, \frac{1}{2}\right] \end{pmatrix}$$

for intervals $\bar{h}_{1,14}$, $\bar{h}_{1,24}$ and $\bar{h}_{1,34}$, and

$$\bar{B}_2 = \begin{pmatrix} \left[\frac{1}{2}, \frac{1}{2}\right] & \left[\frac{3}{4}, \frac{2}{3}\right] & \left[\frac{3}{5}, \frac{2}{5}\right] & \left[\frac{3}{4}, \frac{4}{5}\right] \\ \left[\frac{1}{4}, \frac{1}{3}\right] & \left[\frac{1}{2}, \frac{1}{2}\right] & \left[\frac{1}{3}, \frac{1}{4}\right] & \left[\frac{1}{2}, \frac{2}{3}\right] \\ \left[\frac{2}{5}, \frac{3}{5}\right] & \left[\frac{2}{3}, \frac{3}{4}\right] & \left[\frac{1}{2}, \frac{1}{2}\right] & \left[\frac{2}{3}, \frac{6}{7}\right] \\ \left[\frac{1}{4}, \frac{1}{5}\right] & \left[\frac{1}{2}, \frac{1}{3}\right] & \left[\frac{1}{3}, \frac{1}{7}\right] & \left[\frac{1}{2}, \frac{1}{2}\right] \end{pmatrix}$$

for intervals $\bar{h}_{2,14}$, $\bar{h}_{2,24}$ and $\bar{h}_{2,34}$.

Because the QIFPRs $\bar{B}_1$ and $\bar{B}_2$ have the same chosen probability, the consistency probability distribution on them is {1/2, 1/2} . Using the IWGMA operator and formula (11), the collectively multiplicatively consistent QIFPR is

$$\bar{C} = \begin{pmatrix} [0.5000, 0.5000] & [0.6778, 0.7404] & [0.4246, 0.5754] & [0.4979, 0.7404] \\ [0.3222, 0.2596] & [0.5000, 0.5000] & [0.2596, 0.3222] & [0.3204, 0.5000] \\ [0.5754, 0.4246] & [0.7404, 0.6778] & [0.5000, 0.5000] & [0.5734, 0.6778] \\ [0.5021, 0.2596] & [0.6796, 0.5000] & [0.4266, 0.3222] & [0.5000, 0.5000] \end{pmatrix}$$

by which the multiplicatively consistent IFPR is

$$\bar{D} = \begin{pmatrix} [0.5000, 0.5000] & [0.6778, 0.7404] & [0.4246, 0.5754] & [0.4979, 0.7404] \\ [0.2596, 0.3222] & [0.5000, 0.5000] & [0.2596, 0.3222] & [0.3204, 0.5000] \\ [0.4246, 0.5754] & [0.6778, 0.7404] & [0.5000, 0.5000] & [0.5734, 0.6778] \\ [0.2596, 0.5021] & [0.5000, 0.6796] & [0.3222, 0.4266] & [0.5000, 0.5000] \end{pmatrix}$$

4 Programming models for determining missing values

In some cases, we can only obtain incomplete preference relations or judgment matrices for various reasons [11, 12, 39]. Considering this situation, this section focuses on incomplete IVHFPRs, namely, some values in an IVHFPR are missing. Several multiplicative consistency-based programming models for estimating missing values are constructed. First, we consider the multiplicative consistency concept for incomplete IVHFPRs.

Definition 10 Let $\tilde{H}=(\tilde{h}_{ij})_{n\times n}$ be an incomplete IVHFPR on the object set X, namely, there are missing IVHFEs. It is multiplicatively consistent if there are IVHFEs for missing judgements that make $\tilde{H}$ be multiplicatively consistent.

When the incomplete IVHFPR $\tilde{H}$ is multiplicatively consistent, by formula (5) we obtain:

$$\begin{aligned}&\prod_{k=1}^{n}\left(\left(\left(\otimes_{l=1}^{m_{ij}}\left(\bar{h}_{l,ij}\right)^{\alpha_{l,ij}}\right)^{\delta_{ij}}\otimes\left(\left(\otimes_{l=1}^{m_{ij}}\left(\bar{h}_{l,ij}\right)^{\alpha_{l,ij}}\right)^{\circ}\right)^{(1-\delta_{ij})}\right)\otimes\left(\left(\otimes_{l=1}^{m_{jk}}\left(\bar{h}_{l,jk}\right)^{\alpha_{l,jk}}\right)^{\delta_{jk}}\otimes\left(\left(\otimes_{l=1}^{m_{jk}}\left(\bar{h}_{l,jk}\right)^{\alpha_{l,jk}}\right)^{\circ}\right)^{(1-\delta_{jk})}\right)\right.\\&\left.\otimes\left(\left(\otimes_{l=1}^{m_{ki}}\left(\bar{h}_{l,ki}\right)^{\alpha_{l,ki}}\right)^{\delta_{ki}}\otimes\left(\left(\otimes_{l=1}^{m_{ki}}\left(\bar{h}_{l,ki}\right)^{\alpha_{l,ki}}\right)^{\circ}\right)^{(1-\delta_{ki})}\right)\right)=\prod_{k=1}^{n}\left(\left(\left(\otimes_{l=1}^{m_{ji}}\left(\bar{h}_{l,ji}\right)^{\alpha_{l,ji}}\right)^{\delta_{ji}}\otimes\left(\left(\otimes_{l=1}^{m_{ji}}\left(\bar{h}_{l,ji}\right)^{\alpha_{l,ji}}\right)^{\circ}\right)^{(1-\delta_{ji})}\right)\right.\\&\left.\otimes\left(\left(\otimes_{l=1}^{m_{ik}}\left(\bar{h}_{l,ik}\right)^{\alpha_{l,ik}}\right)^{\delta_{ik}}\otimes\left(\left(\otimes_{l=1}^{m_{ik}}\left(\bar{h}_{l,ik}\right)^{\alpha_{l,ik}}\right)^{\circ}\right)^{(1-\delta_{ik})}\right)\otimes\left(\left(\otimes_{l=1}^{m_{kj}}\left(\bar{h}_{l,kj}\right)^{\alpha_{l,kj}}\right)^{\delta_{kj}}\otimes\left(\left(\otimes_{l=1}^{m_{kj}}\left(\bar{h}_{l,kj}\right)^{\alpha_{l,kj}}\right)^{\circ}\right)^{(1-\delta_{kj})}\right)\right)\end{aligned}\tag{12}$$

for all $i,\ j=1,2,\ldots,n$ and all $l=1,2,\ldots,m_{ij}$, where the notations as shown in formulae (4) and (5).

From formula (12), we derive:

$$\begin{aligned}&\left(\left(\otimes_{l=1}^{m_{ij}}\left(\bar{h}_{l,ij}\right)^{\alpha_{l,ij}}\right)^{\delta_{ij}}\otimes\left(\left(\otimes_{l=1}^{m_{ij}}\left(\bar{h}_{l,ij}\right)^{\alpha_{l,ij}}\right)^{\circ}\right)^{(1-\delta_{ij})}\right)^{n-2}\otimes\prod_{k=1,k\neq i,j}^{n}\left(\left(\left(\otimes_{l=1}^{m_{jk}}\left(\bar{h}_{l,jk}\right)^{\alpha_{l,jk}}\right)^{\delta_{jk}}\right.\right.\\&\left.\left.\otimes\left(\left(\otimes_{l=1}^{m_{jk}}\left(\bar{h}_{l,jk}\right)^{\alpha_{l,jk}}\right)^{\circ}\right)^{(1-\delta_{jk})}\right)\otimes\left(\left(\otimes_{l=1}^{m_{ki}}\left(\bar{h}_{l,ki}\right)^{\alpha_{l,ki}}\right)^{\delta_{ki}}\otimes\left(\left(\otimes_{l=1}^{m_{ki}}\left(\bar{h}_{l,ki}\right)^{\alpha_{l,ki}}\right)^{\circ}\right)^{(1-\delta_{ki})}\right)\right)\\&=\left(\left(\otimes_{l=1}^{m_{ji}}\left(\bar{h}_{l,ji}\right)^{\alpha_{l,ji}}\right)^{\delta_{ji}}\otimes\left(\left(\otimes_{l=1}^{m_{ji}}\left(\bar{h}_{l,ji}\right)^{\alpha_{l,ji}}\right)^{\circ}\right)^{(1-\delta_{ji})}\right)^{n-2}\otimes\prod_{k=1,k\neq i,j}^{n}\left(\left(\left(\otimes_{l=1}^{m_{ik}}\left(\bar{h}_{l,ik}\right)^{\alpha_{l,ik}}\right)^{\delta_{ik}}\right.\right.\\&\left.\left.\otimes\left(\left(\otimes_{l=1}^{m_{ik}}\left(\bar{h}_{l,ik}\right)^{\alpha_{l,ik}}\right)^{\circ}\right)^{(1-\delta_{ik})}\right)\otimes\left(\left(\otimes_{l=1}^{m_{kj}}\left(\bar{h}_{l,kj}\right)^{\alpha_{l,kj}}\right)^{\delta_{kj}}\otimes\left(\left(\otimes_{l=1}^{m_{kj}}\left(\bar{h}_{l,kj}\right)^{\alpha_{l,kj}}\right)^{\circ}\right)^{(1-\delta_{kj})}\right)\right)\end{aligned}\tag{13}$$

for all $i,\ j=1,2,\ldots,n$ and all $l=1,2,\ldots,m_{ij}$.

We take the logarithm on both sides of formula (13) and get

$$\begin{aligned}&(n-2)\left(\delta_{ij}\left(\sum_{l=1}^{m_{ij}}\alpha_{l,ij}\log\left(\bar{h}_{l,ij}\right)\right)\oplus(1-\delta_{ij})\left(\sum_{l=1}^{m_{ij}}\alpha_{l,ij}\log\left(\bar{h}_{l,ij}\right)\right)^{\circ}\right)\oplus\\&\sum_{k=1,k\neq i,j}^{n}\left(\delta_{jk}\left(\sum_{l=1}^{m_{jk}}\alpha_{l,jk}\log\left(\bar{h}_{l,jk}\right)\right)\oplus(1-\delta_{jk})\left(\sum_{l=1}^{m_{jk}}\alpha_{l,jk}\log\left(\bar{h}_{l,jk}\right)\right)^{\circ}\right)\oplus\\&\sum_{k=1,k\neq i,j}^{n}\left(\delta_{ki}\left(\sum_{l=1}^{m_{ki}}\alpha_{l,ki}\log\left(\bar{h}_{l,ki}\right)\right)\oplus(1-\delta_{ki})\left(\sum_{l=1}^{m_{ki}}\alpha_{l,ki}\log\left(\bar{h}_{l,ki}\right)\right)^{\circ}\right)\\&=(n-2)\left(\delta_{ji}\left(\sum_{l=1}^{m_{ji}}\alpha_{l,ji}\log\left(\bar{h}_{l,ji}\right)\right)\oplus(1-\delta_{ji})\left(\sum_{l=1}^{m_{ji}}\alpha_{l,ji}\log\left(\bar{h}_{l,ji}\right)\right)^{\circ}\right)\oplus\\&\sum_{k=1,k\neq i,j}^{n}\left(\delta_{ik}\left(\sum_{l=1}^{m_{ik}}\alpha_{l,ik}\log\left(\bar{h}_{l,ik}\right)\right)\oplus(1-\delta_{ik})\left(\sum_{l=1}^{m_{ik}}\alpha_{l,ik}\log\left(\bar{h}_{l,ik}\right)\right)^{\circ}\right)\oplus\\&\sum_{k=1,k\neq i,j}^{n}\left(\delta_{kj}\left(\sum_{l=1}^{m_{kj}}\alpha_{l,kj}\log\left(\bar{h}_{l,kj}\right)\right)\oplus(1-\delta_{kj})\left(\sum_{l=1}^{m_{kj}}\alpha_{l,kj}\log\left(\bar{h}_{l,kj}\right)\right)^{\circ}\right)\end{aligned}\tag{14}$$

for all $i,\ j\ =1,\ 2,\ \ldots,\ n$, which can be equivalently expressed as follows:

$$\begin{cases} (n-2)\Big(\delta_{ij}\Big(\sum_{l=1}^{m_{ij}}\alpha_{l,ij}\log\big(h_{l,ij}^{-}\big)\Big)+(1-\delta_{ij})\Big(\sum_{l=1}^{m_{ij}}\alpha_{l,ij}\log\big(h_{l,ij}^{+}\big)\Big)\Big)+\sum_{k=1,k\neq i,j}^{n}\Big(\delta_{jk}\Big(\sum_{l=1}^{m_{jk}}\alpha_{l,jk}\log\big(h_{l,jk}^{-}\big)\Big)+(1-\delta_{jk})\Big(\sum_{l=1}^{m_{jk}}\alpha_{l,jk}\log\big(h_{l,jk}^{+}\big)\Big)+ \\ \delta_{ki}\Big(\sum_{l=1}^{m_{ki}}\alpha_{l,ki}\log\big(h_{l,ki}^{-}\big)\Big)+(1-\delta_{ki})\delta_{ki}\Big(\sum_{l=1}^{m_{ki}}\alpha_{l,ki}\log\big(h_{l,ki}^{+}\big)\Big)\Big)=(n-2)\Big(\delta_{ji}\Big(\sum_{l=1}^{m_{ji}}\alpha_{l,ji}\log\big(h_{l,ji}^{-}\big)\Big)+(1-\delta_{ji})\Big(\sum_{l=1}^{m_{ji}}\alpha_{l,ji}\log\big(h_{l,ji}^{+}\big)\Big)\Big)+ \\ \sum_{k=1,k\neq i,j}^{n}\Big(\delta_{ik}\Big(\sum_{l=1}^{m_{ik}}\alpha_{l,ik}\log\big(h_{l,ik}^{-}\big)\Big)+(1-\delta_{ik})\Big(\sum_{l=1}^{m_{ik}}\alpha_{l,ik}\log\big(h_{l,ik}^{+}\big)\Big)+\delta_{kj}\Big(\sum_{l=1}^{m_{kj}}\alpha_{l,kj}\log\big(h_{l,kj}^{-}\big)\Big)+(1-\delta_{kj})\Big(\sum_{l=1}^{m_{kj}}\alpha_{l,kj}\log\big(h_{l,kj}^{+}\big)\Big)\Big) \\ (n-2)\Big(\delta_{ij}\Big(\sum_{l=1}^{m_{ij}}\alpha_{l,ij}\log\big(h_{l,ij}^{+}\big)\Big)+(1-\delta_{ij})\Big(\sum_{l=1}^{m_{ij}}\alpha_{l,ij}\log\big(h_{l,ij}^{-}\big)\Big)\Big)+\sum_{k=1,k\neq i,j}^{n}\Big(\delta_{jk}\Big(\sum_{l=1}^{m_{jk}}\alpha_{l,jk}\log\big(h_{l,jk}^{+}\big)\Big)+(1-\delta_{jk})\Big(\sum_{l=1}^{m_{jk}}\alpha_{l,jk}\log\big(h_{l,jk}^{-}\big)\Big)+ \\ \delta_{ki}\Big(\sum_{l=1}^{m_{ki}}\alpha_{l,ki}\log\big(h_{l,ki}^{+}\big)\Big)+(1-\delta_{ki})\delta_{ki}\Big(\sum_{l=1}^{m_{ki}}\alpha_{l,ki}\log\big(h_{l,ki}^{-}\big)\Big)\Big)=(n-2)\Big(\delta_{ji}\Big(\sum_{l=1}^{m_{ji}}\alpha_{l,ji}\log\big(h_{l,ji}^{+}\big)\Big)+(1-\delta_{ji})\Big(\sum_{l=1}^{m_{ji}}\alpha_{l,ji}\log\big(h_{l,ji}^{-}\big)\Big)\Big)+ \\ \sum_{k=1,k\neq i,j}^{n}\Big(\delta_{ik}\Big(\sum_{l=1}^{m_{ik}}\alpha_{l,ik}\log\big(h_{l,ik}^{+}\big)\Big)+(1-\delta_{ik})\Big(\sum_{l=1}^{m_{ik}}\alpha_{l,ik}\log\big(h_{l,ik}^{-}\big)\Big)+\delta_{kj}\Big(\sum_{l=1}^{m_{kj}}\alpha_{l,kj}\log\big(h_{l,kj}^{+}\big)\Big)+(1-\delta_{kj})\Big(\sum_{l=1}^{m_{kj}}\alpha_{l,kj}\log\big(h_{l,kj}^{-}\big)\Big)\Big) \end{cases} \quad (15)$$

for each pair of $(i,\ j)$.

When the multiplicative consistency of the incomplete IVHFPR $\tilde{H}$ cannot be guaranteed, we add the positive and negative derivation values $\varepsilon_{ij}^{-},\ \varepsilon_{ij}^{+},\ \tau_{ij}^{-},\ \tau_{ij}^{+}$ into formula (15), where $\varepsilon_{ij}^{-},\ \varepsilon_{ij}^{+},\ \tau_{ij}^{-},\ \tau_{ij}^{+}\geq 0$ for all $i,\ j\ =1,2,\ldots,n$. Then,

$$\begin{cases} (n-2)\Big(\delta_{ij}\Big(\sum_{l=1}^{m_{ij}}\alpha_{l,ij}\log\big(h_{l,ij}^{-}\big)\Big)+(1-\delta_{ij})\Big(\sum_{l=1}^{m_{ij}}\alpha_{l,ij}\log\big(h_{l,ij}^{+}\big)\Big)-\Big(\delta_{ji}\Big(\sum_{l=1}^{m_{ji}}\alpha_{l,ji}\log\big(h_{l,ji}^{-}\big)\Big)+(1-\delta_{ji})\Big(\sum_{l=1}^{m_{ji}}\alpha_{l,ji}\log\big(h_{l,ji}^{+}\big)\Big)\Big)\Big)+ \\ \sum_{k=1,k\neq i,j}^{n}\Big(\delta_{jk}\Big(\sum_{l=1}^{m_{jk}}\alpha_{l,jk}\log\big(h_{l,jk}^{-}\big)\Big)+(1-\delta_{jk})\Big(\sum_{l=1}^{m_{jk}}\alpha_{l,jk}\log\big(h_{l,jk}^{+}\big)\Big)+\delta_{ki}\Big(\sum_{l=1}^{m_{ki}}\alpha_{l,ki}\log\big(h_{l,ki}^{-}\big)\Big)+(1-\delta_{ki})\delta_{ki}\Big(\sum_{l=1}^{m_{ki}}\alpha_{l,ki}\log\big(h_{l,ki}^{+}\big)\Big)\Big)- \\ \Big(\sum_{k=1,k\neq i,j}^{n}\Big(\delta_{ik}\Big(\sum_{l=1}^{m_{ik}}\alpha_{l,ik}\log\big(h_{l,ik}^{-}\big)\Big)+(1-\delta_{ik})\Big(\sum_{l=1}^{m_{ik}}\alpha_{l,ik}\log\big(h_{l,ik}^{+}\big)\Big)+\delta_{kj}\Big(\sum_{l=1}^{m_{kj}}\alpha_{l,kj}\log\big(h_{l,kj}^{-}\big)\Big)+(1-\delta_{kj})\Big(\sum_{l=1}^{m_{kj}}\alpha_{l,kj}\log\big(h_{l,kj}^{+}\big)\Big)\Big)\Big)+ \\ \varepsilon_{ij}^{-}-\varepsilon_{ij}^{+}=0 \\ (n-2)\Big(\Big(\delta_{ij}\Big(\sum_{l=1}^{m_{ij}}\alpha_{l,ij}\log\big(h_{l,ij}^{+}\big)\Big)+(1-\delta_{ij})\Big(\sum_{l=1}^{m_{ij}}\alpha_{l,ij}\log\big(h_{l,ij}^{-}\big)\Big)\Big)-\Big(\delta_{ji}\Big(\sum_{l=1}^{m_{ji}}\alpha_{l,ji}\log\big(h_{l,ji}^{+}\big)\Big)+(1-\delta_{ji})\Big(\sum_{l=1}^{m_{ji}}\alpha_{l,ji}\log\big(h_{l,ji}^{-}\big)\Big)\Big)\Big)+ \\ \sum_{k=1,k\neq i,j}^{n}\Big(\delta_{jk}\Big(\sum_{l=1}^{m_{jk}}\alpha_{l,jk}\log\big(h_{l,jk}^{+}\big)\Big)+(1-\delta_{jk})\Big(\sum_{l=1}^{m_{jk}}\alpha_{l,jk}\log\big(h_{l,jk}^{-}\big)\Big)+\delta_{ki}\Big(\sum_{l=1}^{m_{ki}}\alpha_{l,ki}\log\big(h_{l,ki}^{+}\big)\Big)+(1-\delta_{ki})\delta_{ki}\Big(\sum_{l=1}^{m_{ki}}\alpha_{l,ki}\log\big(h_{l,ki}^{-}\big)\Big)\Big)- \\ \sum_{k=1,k\neq i,j}^{n}\Big(\delta_{ik}\Big(\sum_{l=1}^{m_{ik}}\alpha_{l,ik}\log\big(h_{l,ik}^{+}\big)\Big)+(1-\delta_{ik})\Big(\sum_{l=1}^{m_{ik}}\alpha_{l,ik}\log\big(h_{l,ik}^{-}\big)\Big)+\delta_{kj}\Big(\sum_{l=1}^{m_{kj}}\alpha_{l,kj}\log\big(h_{l,kj}^{+}\big)\Big)+(1-\delta_{kj})\Big(\sum_{l=1}^{m_{kj}}\alpha_{l,kj}\log\big(h_{l,kj}^{-}\big)\Big)\Big)+ \\ \tau_{ij}^{-}-\tau_{ij}^{+}=0 \end{cases} \quad (16)$$

for each pair of $(i,\ j)$.

Because the higher the consistency is, the better will be. We build the following programming model to determine the missing values:

$$\varphi_{l_0,i_0j_0}^{*}=\min\sum_{i,j=1}^{n}\Big(\varepsilon_{ij}^{-}+\varepsilon_{ij}^{+}+\tau_{ij}^{+}+\tau_{ij}^{+}\Big)$$

$$s.t.\begin{cases} (n-2)\Big(\delta_{ij}\Big(\sum_{l=1}^{m_{ij}}\alpha_{l,ij}\log\big(h_{l,ij}^{-}\big)\Big)+(1-\delta_{ij})\Big(\sum_{l=1}^{m_{ij}}\alpha_{l,ij}\log\big(h_{l,ij}^{+}\big)\Big)-\Big(\delta_{ji}\Big(\sum_{l=1}^{m_{ji}}\alpha_{l,ji}\log\big(h_{l,ji}^{-}\big)\Big)+(1-\delta_{ji})\Big(\sum_{l=1}^{m_{ji}}\alpha_{l,ji}\log\big(h_{l,ji}^{+}\big)\Big)\Big)\Big)+ \\ \sum_{k=1,k\neq i,j}^{n}\Big(\delta_{jk}\Big(\sum_{l=1}^{m_{jk}}\alpha_{l,jk}\log\big(h_{l,jk}^{-}\big)\Big)+(1-\delta_{jk})\Big(\sum_{l=1}^{m_{jk}}\alpha_{l,jk}\log\big(h_{l,jk}^{+}\big)\Big)+\delta_{ki}\Big(\sum_{l=1}^{m_{ki}}\alpha_{l,ki}\log\big(h_{l,ki}^{-}\big)\Big)+(1-\delta_{ki})\delta_{ki}\Big(\sum_{l=1}^{m_{ki}}\alpha_{l,ki}\log\big(h_{l,ki}^{+}\big)\Big)\Big)- \\ \Big(\sum_{k=1,k\neq i,j}^{n}\Big(\delta_{ik}\Big(\sum_{l=1}^{m_{ik}}\alpha_{l,ik}\log\big(h_{l,ik}^{-}\big)\Big)+(1-\delta_{ik})\Big(\sum_{l=1}^{m_{ik}}\alpha_{l,ik}\log\big(h_{l,ik}^{+}\big)\Big)+\delta_{kj}\Big(\sum_{l=1}^{m_{kj}}\alpha_{l,kj}\log\big(h_{l,kj}^{-}\big)\Big)+(1-\delta_{kj})\Big(\sum_{l=1}^{m_{kj}}\alpha_{l,kj}\log\big(h_{l,kj}^{+}\big)\Big)\Big)\Big)+ \\ \varepsilon_{ij}^{-}-\varepsilon_{ij}^{+}=0,\ i,\ j=1,2,...,n \\ (n-2)\Big(\Big(\delta_{ij}\Big(\sum_{l=1}^{m_{ij}}\alpha_{l,ij}\log\big(h_{l,ij}^{+}\big)\Big)+(1-\delta_{ij})\Big(\sum_{l=1}^{m_{ij}}\alpha_{l,ij}\log\big(h_{l,ij}^{-}\big)\Big)\Big)-\Big(\delta_{ji}\Big(\sum_{l=1}^{m_{ji}}\alpha_{l,ji}\log\big(h_{l,ji}^{+}\big)\Big)+(1-\delta_{ji})\Big(\sum_{l=1}^{m_{ji}}\alpha_{l,ji}\log\big(h_{l,ji}^{-}\big)\Big)\Big)\Big)+ \\ \sum_{k=1,k\neq i,j}^{n}\Big(\delta_{jk}\Big(\sum_{l=1}^{m_{jk}}\alpha_{l,jk}\log\big(h_{l,jk}^{+}\big)\Big)+(1-\delta_{jk})\Big(\sum_{l=1}^{m_{jk}}\alpha_{l,jk}\log\big(h_{l,jk}^{-}\big)\Big)+\delta_{ki}\Big(\sum_{l=1}^{m_{ki}}\alpha_{l,ki}\log\big(h_{l,ki}^{+}\big)\Big)+(1-\delta_{ki})\delta_{ki}\Big(\sum_{l=1}^{m_{ki}}\alpha_{l,ki}\log\big(h_{l,ki}^{-}\big)\Big)\Big)- \\ \sum_{k=1,k\neq i,j}^{n}\Big(\delta_{ik}\Big(\sum_{l=1}^{m_{ik}}\alpha_{l,ik}\log\big(h_{l,ik}^{+}\big)\Big)+(1-\delta_{ik})\Big(\sum_{l=1}^{m_{ik}}\alpha_{l,ik}\log\big(h_{l,ik}^{-}\big)\Big)+\delta_{kj}\Big(\sum_{l=1}^{m_{kj}}\alpha_{l,kj}\log\big(h_{l,kj}^{+}\big)\Big)+(1-\delta_{kj})\Big(\sum_{l=1}^{m_{kj}}\alpha_{l,kj}\log\big(h_{l,kj}^{-}\big)\Big)\Big)+ \\ \tau_{ij}^{-}-\tau_{ij}^{+}=0,\ i,\ j=1,2,...,n \\ h_{l,ij}^{-}+h_{l,ji}^{+}=h_{l,ij}^{+}+h_{l,ji}^{-}=1,\ [h_{l,ij}^{-},h_{l,ij}^{+}]\subseteq[0,1],\ \ \tilde{h}_{ij}\in U=\{\tilde{h}_{ij}|\tilde{h}_{ij},\text{ is missing } i,\ j=1,2,...,n\} \\ \alpha_{l,ij}=0\vee 1,\ \alpha_{l,ij}=\alpha_{m_{ij}+1-l,ji},\ i,\ j=1,2,...,n,\ l=1,2,...,m_{ij} \\ \sum_{l=1}^{m_{ij}}\alpha_{l,ij}=1,\ i,\ j=1,2,...,n \\ \delta_{ij}=0\vee 1,\ i,\ j=1,2,...,n \\ \delta_{ij}+\delta_{ji}=1,\ i,\ j=1,2,...,n \\ \alpha_{l_0,i_0j_0}=1,\ i_0,\ j_0=1,2,...,n,\ \ l_0=1,2,...,m_{i_0j_0}\ \text{ with } \tilde{h}_{i_0j_0}\notin U \end{cases}. \quad (17)$$

The first two constraints are constants if $\tilde{h}_{ij}\ \notin\ U$ for all $i\ =1,\ 2,\ \ldots,\ n$ or all $j\ =1,\ 2,\ \ldots,\ n$, which have no influence on missing values. Thus, we can disregard such constraints. To do this, we further introduce the 0-1 indictor variables $\beta_{ij}=\begin{cases}0 & \tilde{h}_{ij}\notin U, i=1,2,..,n\vee j=1,2,..,n\\ 1 & \text{otherwise}\end{cases}$

for each pair of (i, j), and construct the following programming model:

$$\psi^*_{l_0,i_0j_0} = \min \sum_{i,j=1}^{n} \left(\varepsilon^-_{ij} + \varepsilon^+_{ij} + \tau^+_{ij} + \tau^+_{ij}\right)$$

$$s.t. \begin{cases} x\beta_{ij} ((n-2) (\delta_{ij} (\sum_{l=1}^{m_{ij}} \alpha_{l,ij} \log (h^-_{l,ij})) + (1-\delta_{ij}) (\sum_{l=1}^{m_{ij}} \alpha_{l,ij} \log (h^+_{l,ij})) - (\delta_{ji} (\sum_{l=1}^{m_{ji}} \alpha_{l,ji} \log (h^-_{l,ji})) + (1-\delta_{ji}) (\sum_{l=1}^{m_{ji}} \alpha_{l,ji} \log (h^+_{l,ji}))))+ \\ \sum_{k=1,k\neq i,j}^{n} (\delta_{jk} (\sum_{l=1}^{m_{jk}} \alpha_{l,jk} \log (h^-_{l,jk})) + (1-\delta_{jk}) (\sum_{l=1}^{m_{jk}} \alpha_{l,jk} \log (h^+_{l,jk})) + \delta_{ki} (\sum_{l=1}^{m_{ki}} \alpha_{l,ki} \log (h^-_{l,ki})) + (1-\delta_{ki})\delta_{ki} (\sum_{l=1}^{m_{ki}} \alpha_{l,ki} \log (h^+_{l,ki}))) - \\ (\sum_{k=1,k\neq i,j}^{n} (\delta_{ik} (\sum_{l=1}^{m_{ik}} \alpha_{l,ik} \log (h^-_{l,ik})) + (1-\delta_{ik}) (\sum_{l=1}^{m_{ik}} \alpha_{l,ik} \log (h^+_{l,ik})) + \delta_{kj} (\sum_{l=1}^{m_{kj}} \alpha_{l,kj} \log (h^-_{l,kj})) + (1-\delta_{kj}) (\sum_{l=1}^{m_{kj}} \alpha_{l,kj} \log (h^+_{l,kj})))) + \\ \varepsilon^-_{ij} - \varepsilon^+_{ij}) = 0, i, j = 1, 2, ..., n \\ \beta_{ij} ((n-2) ((\delta_{ij} (\sum_{l=1}^{m_{ij}} \alpha_{l,ij} \log (h^+_{l,ij})) + (1-\delta_{ij}) (\sum_{l=1}^{m_{ij}} \alpha_{l,ij} \log (h^-_{l,ij}))) - (\delta_{ji} (\sum_{l=1}^{m_{ji}} \alpha_{l,ji} \log (h^+_{l,ji})) + (1-\delta_{ji}) (\sum_{l=1}^{m_{ji}} \alpha_{l,ji} \log (h^-_{l,ji}))))+ \\ \sum_{k=1,k\neq i,j}^{n} (\delta_{jk} (\sum_{l=1}^{m_{jk}} \alpha_{l,jk} \log (h^+_{l,jk})) + (1-\delta_{jk}) (\sum_{l=1}^{m_{jk}} \alpha_{l,jk} \log (h^-_{l,jk})) + \delta_{ki} (\sum_{l=1}^{m_{ki}} \alpha_{l,ki} \log (h^+_{l,ki})) + (1-\delta_{ki})\delta_{ki} (\sum_{l=1}^{m_{ki}} \alpha_{l,ki} \log (h^-_{l,ki}))) - \\ \sum_{k=1,k\neq i,j}^{n} (\delta_{ik} (\sum_{l=1}^{m_{ik}} \alpha_{l,ik} \log (h^+_{l,ik})) + (1-\delta_{ik}) (\sum_{l=1}^{m_{ik}} \alpha_{l,ik} \log (h^-_{l,ik})) + \delta_{kj} (\sum_{l=1}^{m_{kj}} \alpha_{l,kj} \log (h^+_{l,kj})) + (1-\delta_{kj}) (\sum_{l=1}^{m_{kj}} \alpha_{l,kj} \log (h^-_{l,kj}))))+ \\ \tau^-_{ij} - \tau^+_{ij}) = 0, i, j = 1, 2, ..., n \\ h^-_{l,ij} + h^+_{l,ji} = h^+_{l,ij} + h^-_{l,ji} = 1, [h^-_{l,ij}, h^+_{l,ij}] \subseteq [0, 1],\ \tilde{h}_{ij} \in U = \{\tilde{h}_{ij} | \tilde{h}_{ij} \text{ is missing},\ i, j = 1, 2, ..., n\} \\ \alpha_{l,ij} = 0 \vee 1, \alpha_{l,ij} = \alpha_{m_{ij}+1-l,ji}, i, j = 1, 2, ..., n, l = 1, 2, ..., m_{ij} \\ \sum_{l=1}^{m_{ij}} \alpha_{l,ij} = 1, i, j = 1, 2, ..., n \\ \delta_{ij} = 0 \vee 1, i, j = 1, 2, ..., n \\ \delta_{ij} + \delta_{ji} = 1, i, j = 1, 2, ..., n \\ \alpha_{l_0,i_0j_0} = 1, i_0, j_0 = 1, 2, ..., n,\ l_0 = 1, 2, ..., m_{i_0j_0} \text{ with } \tilde{h}_{i_0j_0} \notin U \end{cases} . \quad (18)$$

According to the construction of the elements in IVHFPRs, we can only determine missing IVHFEs in the upper triangular part. Thus, model (18) can be further transformed into the following model:

$$\phi^*_{l_0,i_0j_0} = \min \sum_{i,j=1}^{n} \left(\varepsilon^-_{ij} + \varepsilon^+_{ij} + \tau^+_{ij} + \tau^+_{ij}\right)$$

$$s.t. \begin{cases} \beta_{ij} ((n-2) (\delta_{ij} (\sum_{l=1}^{m_{ij}} \alpha_{l,ij} \log (h^-_{l,ij})) + (1-\delta_{ij}) (\sum_{l=1}^{m_{ij}} \alpha_{l,ij} \log (h^+_{l,ij})) - (\delta_{ji} (\sum_{l=1}^{m_{ji}} \alpha_{l,ji} \log (h^-_{l,ji})) + (1-\delta_{ji}) (\sum_{l=1}^{m_{ji}} \alpha_{l,ji} \log (h^+_{l,ji}))))+ \\ \sum_{k=1,k\neq i,j}^{n} (\delta_{jk} (\sum_{l=1}^{m_{jk}} \alpha_{l,jk} \log (h^-_{l,jk})) + (1-\delta_{jk}) (\sum_{l=1}^{m_{jk}} \alpha_{l,jk} \log (h^+_{l,jk})) + \delta_{ki} (\sum_{l=1}^{m_{ki}} \alpha_{l,ki} \log (h^-_{l,ki})) + (1-\delta_{ki})\delta_{ki} (\sum_{l=1}^{m_{ki}} \alpha_{l,ki} \log (h^+_{l,ki}))) - \\ (\sum_{k=1,k\neq i,j}^{n} (\delta_{ik} (\sum_{l=1}^{m_{ik}} \alpha_{l,ik} \log (h^-_{l,ik})) + (1-\delta_{ik}) (\sum_{l=1}^{m_{ik}} \alpha_{l,ik} \log (h^+_{l,ik})) + \delta_{kj} (\sum_{l=1}^{m_{kj}} \alpha_{l,kj} \log (h^-_{l,kj})) + (1-\delta_{kj}) (\sum_{l=1}^{m_{kj}} \alpha_{l,kj} \log (h^+_{l,kj})))) + \\ \varepsilon^-_{ij} - \varepsilon^+_{ij}) = 0, i, j = 1, 2, ..., n, i < j \\ \beta_{ij} ((n-2) ((\delta_{ij} (\sum_{l=1}^{m_{ij}} \alpha_{l,ij} \log (h^+_{l,ij})) + (1-\delta_{ij}) (\sum_{l=1}^{m_{ij}} \alpha_{l,ij} \log (h^-_{l,ij}))) - (\delta_{ji} (\sum_{l=1}^{m_{ji}} \alpha_{l,ji} \log (h^+_{l,ji})) + (1-\delta_{ji}) (\sum_{l=1}^{m_{ji}} \alpha_{l,ji} \log (h^-_{l,ji}))))+ \\ \sum_{k=1,k\neq i,j}^{n} (\delta_{jk} (\sum_{l=1}^{m_{jk}} \alpha_{l,jk} \log (h^+_{l,jk})) + (1-\delta_{jk}) (\sum_{l=1}^{m_{jk}} \alpha_{l,jk} \log (h^-_{l,jk})) + \delta_{ki} (\sum_{l=1}^{m_{ki}} \alpha_{l,ki} \log (h^+_{l,ki})) + (1-\delta_{ki})\delta_{ki} (\sum_{l=1}^{m_{ki}} \alpha_{l,ki} \log (h^-_{l,ki}))) - \\ \sum_{k=1,k\neq i,j}^{n} (\delta_{ik} (\sum_{l=1}^{m_{ik}} \alpha_{l,ik} \log (h^+_{l,ik})) + (1-\delta_{ik}) (\sum_{l=1}^{m_{ik}} \alpha_{l,ik} \log (h^-_{l,ik})) + \delta_{kj} (\sum_{l=1}^{m_{kj}} \alpha_{l,kj} \log (h^+_{l,kj})) + (1-\delta_{kj}) (\sum_{l=1}^{m_{kj}} \alpha_{l,kj} \log (h^-_{l,kj}))))+ \\ \tau^-_{ij} - \tau^+_{ij}) = 0, i, j = 1, 2, ..., n, i < j \\ h^-_{l,ij} + h^+_{l,ji} = h^+_{l,ij} + h^-_{l,ji} = 1, [h^-_{l,ij}, h^+_{l,ij}] \subseteq [0, 1],\ \tilde{h}_{ij} \in U' = \{\tilde{h}_{ij} | \tilde{h}_{ij} \text{ is missing},\ i, j = 1, 2, ..., n, i < j\} \\ \alpha_{l,ij} = 0 \vee 1, \alpha_{l,ij} = \alpha_{m_{ij}+1-l,ji}, i, j = 1, 2, ..., n, l = 1, 2, ..., m_{ij}, i < j \\ \sum_{l=1}^{m_{ij}} \alpha_{l,ij} = 1, i, j = 1, 2, ..., n, i < j \\ \delta_{ij} = 0 \vee 1, i, j = 1, 2, ..., n, i < j \\ \delta_{ij} + \delta_{ji} = 1, i, j = 1, 2, ..., n, i < j \\ \alpha_{l_0,i_0j_0} = 1, i_0, j_0 = 1, 2, ..., n,\ l_0 = 1, 2, ..., m_{i_0j_0} \text{ with } \tilde{h}_{i_0j_0} \notin U' \end{cases} . \quad (19)$$

Solving model (19) with respect to each interval in every IVHFE $\tilde{h}_{i_0j_0} \notin U'$, we derive the missing IVHFEs in the upper triangular part, and the missing IVHFEs in the lower triangular part can be obtained following the additive reciprocity. On the basis of the above analysis, we introduce the following decision-making method with IVHFPRs:

Algorithm 1 Ranking objects from decision making with IVHFPRs

Step 1: Let $\tilde{H} = (\tilde{h}_{ij})_{n\times n}$ be an IVHFPR on the object set $X = \{x_1, x_2, \ldots, x_n\}$. If it is incomplete, we apply model (19) to determine missing IVHFEs. Otherwise, go to the next step;

Step 2: With respect to the complete IVHFPR $\tilde{H}$, we apply model (10) to judge its multiplicative consistency and to derive the 0-1 indictor variables, the associated QIFPRs $\bar{B}_q = (\bar{b}_{q,ij})_{n\times n}$, and the forming times $n_q, q = 1, 2, \ldots, \Delta$. When all of the obtained QIFPRs are multiplicatively consistent, go to Step 4. Otherwise, turn to the next step;

Step 3: For any inconsistent QIFPR $\bar{B}_q = (\bar{b}_{q,ij})_{n\times n}$, $q = 1, 2, \ldots, \Delta$, we adopt formula (11) to derive the multiplicatively consistent QIFPR $\bar{C}_q = (\bar{c}_{q,ij})_{n\times n}$;

Step 4: According to the forming times of each QIFPR, we obtain the consistency probability distribution $P = (p_1, p_2, \ldots., p)$, where $p_q = \frac{n_q}{\sum_{z=1}^{\Delta} n_z}$, $q = 1, 2, \ldots, \Delta$. Then, we use the IWGMA operator and formula (11) to calculate the collectively multiplicatively consistent QIFPR $\bar{C} = (\bar{c}_{ij})_{n\times n}$ by which we can obtain the multiplicatively consistent IFPR $\bar{D} = (\bar{d}_{ij})_{n\times n}$;

Step 5: Following $\bar{D}$, we calculate the interval fuzzy priority weight vector $\bar{\omega} = (\bar{\omega}_1, \bar{\omega}_2, ..., \bar{\omega}_n)$, where

$$\bar{\omega}_i = [\omega_i^-, \omega_i^+] = \left[\frac{\sqrt[n]{\Pi_{j=1}^n d_{ij}^-}}{\sqrt[n]{\Pi_{j=1}^n d_{ij}^-} + \sum_{k=1,k\neq i}^n \sqrt[n]{\Pi_{j=1}^n d_{kj}^+}}, \frac{\sqrt[n]{\Pi_{j=1}^n d_{ij}^+}}{\sqrt[n]{\Pi_{j=1}^n d_{ij}^+} + \sum_{k=1,k\neq i}^n \sqrt[n]{\Pi_{j=1}^n d_{kj}^-}}\right], \tag{20}$$

for all $i = 1, 2, ..., n$

Step 6: We apply the interval formula in [15] to compare $\bar{\omega}_i$, $i = 1, 2, ..., n$ and to rank objects x_1, $x_2, \ldots, x_n$ following the order relationship of $\bar{\omega}_i$, $i = 1, 2, \ldots., n$

Remark 1 Following previous theoretical research results, we offer algorithm I for ranking objects from IVHFPRs. Its main principle includes: (i) ascertaining missing judgments, (ii) judging the multiplicative consistency of IVHFPRs using QIFPRs, (iii) deriving multiplicatively consistent QIFPRs, (iv) calculating the collectively multiplicatively consistent QIFPR based on the consistency probability distribution and the IWGMA operator, (v) obtaining the multiplicatively consistent IFPR, (vi) calculating the interval fuzzy priority weight vector using formula (20), and (vii) ranking objects. There are several features of Algorithm I, such as it is based on the multiplicative consistency analysis, it neither adds extra value nor disregards any information, and it can address incomplete and inconsistent IVHFPRs.

Example 4.1 Let $X = \{x_1, x_2, x_3, x_4\}$ be the set of the compared objects, and let the incomplete IVHFPR $\tilde{H}$ be defined as follows:

$$\tilde{H} = \begin{pmatrix} \left\{\left[\frac{1}{2}, \frac{1}{2}\right]\right\} & x & \left\{\left[\frac{3}{5}, \frac{3}{4}\right]\right\} & \left\{\left[\frac{1}{7}, \frac{2}{7}\right], \left[\frac{1}{3}, \frac{2}{5}\right], \left[\frac{9}{20}, \frac{1}{2}\right]\right\} \\ x & \left\{\left[\frac{1}{2}, \frac{1}{2}\right]\right\} & x & x \\ \left\{\left[\frac{1}{4}, \frac{2}{5}\right]\right\} & x & \left\{\left[\frac{1}{2}, \frac{1}{2}\right]\right\} & \left\{\left[\frac{1}{4}, \frac{1}{3}\right], \left[\frac{2}{5}, \frac{1}{2}\right]\right\} \\ \left\{\left[\frac{1}{2}, \frac{11}{20}\right], \left[\frac{3}{5}, \frac{2}{3}\right], \left[\frac{5}{7}, \frac{6}{7}\right]\right\} & x & \left\{\left[\frac{1}{2}, \frac{3}{5}\right], \left[\frac{2}{3}, \frac{3}{4}\right]\right\} & \left\{\left[\frac{1}{2}, \frac{1}{2}\right]\right\} \end{pmatrix}.$$

To rank the objects x_1, x_2, x_3, and x_4, the following steps are needed:

Step 1: With respect to the incomplete IVHFPR $\tilde{H}$, missing IVHFEs based on model (19) are determined as shown in Table 1.

According to the additive reciprocity of intervals in IVHFEs, the missing IVHFEs $\tilde{h}_{21}$, $\tilde{h}_{32}$, and $\tilde{h}_{42}$ are obtained as shown in Table 2.

Step 2: With respect to the complete IVHFPR $\tilde{H}$, we apply model (10) to judge its multiplicative consistency, where the objective function values, 0-1 indicator variables, QIFPRs, and their forming times are derived as shown in Table 3.

Step 3: Table 3 shows that all of QIFPRs are inconsistent. Thus, formula (11) is used to obtain the multiplicatively consistent QIFPRs, which are shown in Table 4.

Step 4: According to the forming times of QIFPRs, the consistency probability distribution is derived as follows:

$$P = \left\{\frac{1}{20}, \frac{1}{20}, \frac{3}{20}, \frac{5}{20}, \frac{1}{20}, \frac{1}{20}, \frac{1}{20}, \frac{1}{20}, \frac{1}{20}, \frac{1}{20}, \frac{1}{20}, \frac{1}{20}, \frac{1}{20}, \frac{1}{20}\right\},$$

by which the collectively multiplicatively consistent QIFPR is

Table 1 Determined missing IVHFEs

Missing IVHFEs	Determined values
$\tilde{h}_{12}$	f [0.3075, 0.4519] , [0.4545, 0.4799] , [0.5030, 0.5527] , [0.5478, 0.5478] , [0.4825, 0.5285] g
$\tilde{h}_{23}$	f [0.7008.0.7008] , [0.6429, 0.7127] , [0.6040, 0.7082] , [0.5603, 0.7123] , [0.6008, 0.6498] g
$\tilde{h}_{24}$	f [0.3512, 0.4298] , [0.3750, 0.4863] , [0.4398, 0.4472] , [0.3961, 0.4522] , [0.4841, 0.5627] g

$$\bar{C} = \begin{pmatrix} [0.5000, 0.5000] & [0.5152, 0.5122] & [0.6917, 0.6533] & [0.4565, 0.4594] \\ [0.4848, 0.4878] & [0.5000, 0.5000] & [0.6786, 0.6421] & [0.4415, 0.4472] \\ [0.3083, 0.3467] & [0.3214, 0.3579] & [0.5000, 0.5000] & [0.2724, 0.3108] \\ [0.5435, 0.5406] & [0.5585, 0.5528] & [0.7276, 0.6892] & [0.5000, 0.5000] \end{pmatrix},$$

and the collectively multiplicatively consistent IFPR is

$$\bar{D} = \begin{pmatrix} [0.5000, 0.5000] & [0.5122, 0.5152] & [0.6533, 0.6917] & [0.4565, 0.4594] \\ [0.4848, 0.4878] & [0.5000, 0.5000] & [0.6421, 0.6786] & [0.4415, 0.4472] \\ [0.3083, 0.3467] & [0.3214, 0.3579] & [0.5000, 0.5000] & [0.2724, 0.3108] \\ [0.5406, 0.5435] & [0.5528, 0.5585] & [0.6892, 0.7276] & [0.5000, 0.5000] \end{pmatrix}.$$

Step 5: Using formula (20), the interval fuzzy priority weight vector is

$$\bar{\omega} = ([0.2633, 0.2737], [0.2565, 0.2668], [0.1727, 0.1885], [0.2839, 0.2949]).$$

Step 6: Adopting the formula for comparing intervals in [13], we obtain $\rho(\bar{\omega}_4 > \bar{\omega}_1) = 1$, $\rho(\bar{\omega}_1 > \bar{\omega}_2) = 0.9424$, $\rho(\bar{\omega}_2 > \bar{\omega}_3) = 1$. Thus, the ranking order is $x_4 \succ x_1 \succ x_2 \succ x_3$.

5 A group decision-making method with IVHFPRs

In Section 4, we offer a method for ranking objects from incomplete and inconsistent IVHFPRs. However, it is insufficient to address group decision making. Therefore, this section further researches group decision making with IVHFPRs.

In general, for a group decision-making problem, there are n objects $X = \{x_1, x_2, \ldots, x_n\}$, which are evaluated by m DMs $E = \{e_1, e_2, \ldots, e_m\}$. Let $\tilde{H}^k = (\tilde{h}^k_{ij})_{n\times n}$ be the IVHFPR offered by the DM e_k, $k = 1, 2, \ldots, m$.

To measure the agreement degree between individual IVHFPRs, the consensus analysis is necessary [11, 13, 29]. Next, we apply the collectively multiplicatively consistent QIFPRs to define a distance measure between IVHFPRs, by which a consensus index is derived.

Definition 11 Let $\tilde{H}^k = (\tilde{h}^k_{ij})_{n\times n}$ and $\tilde{H}^g = (\tilde{h}^g_{ij})_{n\times n}$ be any two IVHFPRs, and let $\bar{C}^k = (\bar{c}^k_{ij})_{n\times n}$ and $\bar{C}^g = (\bar{c}^g_{ij})_{n\times n}$ be their collectively multiplicatively consistent QIFPRs obtained from Algorithm I. Then, the distance measure between the QIFPRs $\bar{C}^k$ and $\bar{C}^g$ is defined as follows:

$$V\left(\bar{C}^k, \bar{C}^g\right) = \frac{1}{n(n-1)} \sum\nolimits_{i,j=1,i<j}^{n} \times\left(\left|c^{k-}_{ij} - c^{g-}_{ij}\right| + \left|c^{k+}_{ij} - c^{g+}_{ij}\right|\right). \tag{21}$$

Property 1 Let $\tilde{H}^k = (\tilde{h}^k_{ij})_{n\times n}$ and $\tilde{H}^g = (\tilde{h}^g_{ij})_{n\times n}$ be any two IVHFPRs, and let $\bar{C}^k = (\bar{c}^k_{ij})_{n\times n}$ and $\bar{C}^g = (\bar{c}^g_{ij})_{n\times n}$ be

Table 2 Determined missing IVHFEs

Missing IVHFEs	Determined values
$\tilde{h}_{21}$	{[0.5481, 0.6925] , [0.5201, 0.5455] , [0.4473, 0.4970] , [0.4522, 0.4522] , [0.4715, 0.5175]}
$\tilde{h}_{32}$	{[0.2992.0.2992] , [0.2873, 0.3571] , [0.2918, 0.3960] , [0.2877, 0.4397] , [0.3502, 0.3992]}
$\tilde{h}_{42}$	{[0.5702, 0.6488] , [0.5137, 0.6250] , [0.5528, 0.5602] , [0.5478, 0.6039] , [0.4373, 0.5159]}

Table 3 Objective function values, 0-1 indicator variables, QIFPRs, and forming times

Objective function values	0-1 indicator variables	QIFPRs	Forming times
$K^*_{1,12} = 1.7004$	$\delta_{12} = \delta_{13} = \delta_{14} = \delta_{23} = 1, \delta_{24} = \delta_{34} = 0$	$\bar{B}_1 = \begin{pmatrix} [0.5000, 0.5000] & [0.3075, 0.4519] & [0.6000, 0.7500] & [0.4500, 0.5000] \\ [0.6925, 0.5481] & [0.5000, 0.5000] & [0.7008, 0.7008] & [0.5627, 0.4841] \\ [0.4000, 0.2500] & [0.2992, 0.2992] & [0.5000, 0.5000] & [0.3333, 0.2500] \\ [0.5500, 0.5000] & [0.4373, 0.5159] & [0.6667, 0.7500] & [0.5000, 0.5000] \end{pmatrix}$	1
$K^*_{2,12} = 0.9455$	$\delta_{12} = \delta_{13} = \delta_{14} = \delta_{23} = 1, \delta_{24} = \delta_{34} = 0$	$\bar{B}_2 = \begin{pmatrix} [0.5000, 0.5000] & [0.4545, 0.4799] & [0.6000, 0.7500] & [0.4500, 0.5000] \\ [0.5455, 0.5201] & [0.5000, 0.5000] & [0.6429, 0.7127] & [0.4472, 0.4398] \\ [0.4000, 0.2500] & [0.3571, 0.2873] & [0.5000, 0.5000] & [0.3333, 0.2500] \\ [0.5500, 0.5000] & [0.5528, 0.5602] & [0.6667, 0.7500] & [0.5000, 0.5000] \end{pmatrix}$	1
$K^*_{3,12} = K^*_{3,23} = K^*_{3,24} = 0.0878$	$\delta_{12} = \delta_{13} = \delta_{14} = \delta_{23} = \delta_{24} = 1, \delta_{34} = 0$	$\bar{B}_3 = \begin{pmatrix} [0.5000, 0.5000] & [0.5030, 0.5527] & [0.6000, 0.7500] & [0.4500, 0.5000] \\ [0.4970, 0.4473] & [0.5000, 0.5000] & [0.6040, 0.7082] & [0.4398, 0.4472] \\ [0.4000, 0.2500] & [0.3960, 0.2918] & [0.5000, 0.5000] & [0.3333, 0.2500] \\ [0.5500, 0.5000] & [0.5602, 0.5528] & [0.6667, 0.7500] & [0.5000, 0.5000] \end{pmatrix}$	3
$K^*_{4,12} = K^*_{4,23} = K^*_{1,34} = K^*_{3,14} = K^*_{4,24} = 0.0875$	$\delta_{12} = \delta_{13} = \delta_{14} = \delta_{23} = \delta_{24} = 1, \delta_{34} = 0$	$\bar{B}_4 = \begin{pmatrix} [0.5000, 0.5000] & [0.5478, 0.5478] & [0.6000, 0.7500] & [0.4500, 0.5000] \\ [0.4522, 0.4522] & [0.5000, 0.5000] & [0.5603, 0.7123] & [0.3961, 0.4522] \\ [0.4000, 0.2500] & [0.4397, 0.2877] & [0.5000, 0.5000] & [0.3333, 0.2500] \\ [0.5500, 0.5000] & [0.6039, 0.5478] & [0.6667, 0.7500] & [0.5000, 0.5000] \end{pmatrix}$	5
$K^*_{5,12} = 0.3894$	$\delta_{12} = \delta_{13} = \delta_{14} = \delta_{23} = \delta_{24} = 1, \delta_{34} = 0$	$\bar{B}_5 = \begin{pmatrix} [0.5000, 0.5000] & [0.4825, 0.5285] & [0.6000, 0.7500] & [0.4500, 0.5000] \\ [0.5175, 0.4715] & [0.5000, 0.5000] & [0.6040, 0.7082] & [0.4398, 0.4472] \\ [0.4000, 0.2500] & [0.3960, 0.2918] & [0.5000, 0.5000] & [0.3333, 0.2500] \\ [0.5500, 0.5000] & [0.5602, 0.5528] & [0.6667, 0.7500] & [0.5000, 0.5000] \end{pmatrix}$	1
$K^*_{1,23} = 0.9001$	$\delta_{12} = \delta_{13} = \delta_{14} = 1, \delta_{23} = \delta_{24} = \delta_{34} = 0$	$\bar{B}_6 = \begin{pmatrix} [0.5000, 0.5000] & [0.5030, 0.5527] & [0.6000, 0.7500] & [0.4500, 0.5000] \\ [0.4970, 0.4473] & [0.5000, 0.5000] & [0.7008, 0.7008] & [0.4472, 0.4398] \\ [0.4000, 0.2500] & [0.2992, 0.2992] & [0.5000, 0.5000] & [0.3333, 0.2500] \\ [0.5500, 0.5000] & [0.5528, 0.5602] & [0.6667, 0.7500] & [0.5000, 0.5000] \end{pmatrix}$	1
$K^*_{2,23} = 0.4044$	$\delta_{12} = \delta_{13} = \delta_{14} = \delta_{23} = \delta_{24} = 1, \delta_{34} = 0$	$\bar{B}_7 = \begin{pmatrix} [0.5000, 0.5000] & [0.5030, 0.5527] & [0.6000, 0.7500] & [0.4500, 0.5000] \\ [0.4970, 0.4473] & [0.5000, 0.5000] & [0.6429, 0.7127] & [0.4398, 0.4472] \\ [0.4000, 0.2500] & [0.3571, 0.2873] & [0.5000, 0.5000] & [0.3333, 0.2500] \\ [0.5500, 0.5000] & [0.5602, 0.5528] & [0.6667, 0.7500] & [0.5000, 0.5000] \end{pmatrix}$	1
$K^*_{5,23} = 0.6247$	$\delta_{12} = \delta_{13} = \delta_{14} = \delta_{23} = \delta_{24} = 1, \delta_{34} = 0$	$\bar{B}_8 = \begin{pmatrix} [0.5000, 0.5000] & [0.5030, 0.5527] & [0.6000, 0.7500] & [0.4500, 0.5000] \\ [0.4970, 0.4473] & [0.5000, 0.5000] & [0.6008, 0.6498] & [0.4398, 0.4472] \\ [0.4000, 0.2500] & [0.3992, 0.3502] & [0.5000, 0.5000] & [0.3333, 0.2500] \\ [0.5500, 0.5000] & [0.5602, 0.5528] & [0.6667, 0.7500] & [0.5000, 0.5000] \end{pmatrix}$	1
$K^*_{1,24} = 0.5948$	$\delta_{12} = \delta_{13} = \delta_{14} = \delta_{23} = \delta_{24} = 1, \delta_{34} = 0$	$\bar{B}_9 = \begin{pmatrix} [0.5000, 0.5000] & [0.5478, 0.5478] & [0.6000, 0.7500] & [0.4500, 0.5000] \\ [0.4522, 0.4522] & [0.5000, 0.5000] & [0.5603, 0.7123] & [0.3512, 0.4298] \\ [0.4000, 0.2500] & [0.4397, 0.2877] & [0.5000, 0.5000] & [0.3333, 0.2500] \\ [0.5500, 0.5000] & [0.6488, 0.5702] & [0.6667, 0.7500] & [0.5000, 0.5000] \end{pmatrix}$	1
$K^*_{2,24} = 0.4816$	$\delta_{12} = \delta_{13} = \delta_{14} = \delta_{23} = \delta_{24} = 1, \delta_{34} = 0$	$\bar{B}_{10} = \begin{pmatrix} [0.5000, 0.5000] & [0.5478, 0.5478] & [0.6000, 0.7500] & [0.4500, 0.5000] \\ [0.4522, 0.4522] & [0.5000, 0.5000] & [0.5603, 0.7123] & [0.3750, 0.4863] \\ [0.4000, 0.2500] & [0.4397, 0.2877] & [0.5000, 0.5000] & [0.3333, 0.2500] \\ [0.5500, 0.5000] & [0.6250, 0.5137] & [0.6667, 0.7500] & [0.5000, 0.5000] \end{pmatrix}$	1
$K^*_{5,24} = 0.9714$	$\delta_{12} = \delta_{13} = \delta_{14} = \delta_{23} = \delta_{24} = 1, \delta_{34} = 0$	$\bar{B}_{11} = \begin{pmatrix} [0.5000, 0.5000] & [0.4545, 0.4799] & [0.6000, 0.7500] & [0.4500, 0.5000] \\ [0.5455, 0.5201] & [0.5000, 0.5000] & [0.6429, 0.7127] & [0.4841, 0.5627] \\ [0.4000, 0.2500] & [0.3571, 0.2873] & [0.5000, 0.5000] & [0.3333, 0.2500] \\ [0.5500, 0.5000] & [0.5159, 0.4373] & [0.6667, 0.7500] & [0.5000, 0.5000] \end{pmatrix}$	1
$K^*_{1,14} = 4.8407$	$\delta_{12} = \delta_{13} = \delta_{14} = \delta_{23} = \delta_{24} = \delta_{34} = 1$	$\bar{B}_{12} = \begin{pmatrix} [0.5000, 0.5000] & [0.3705, 0.4519] & [0.6000, 0.7500] & [0.1428, 0.2857] \\ [0.6295, 0.5481] & [0.5000, 0.5000] & [0.7008, 0.7008] & [0.3512, 0.4298] \\ [0.4000, 0.2500] & [0.2992, 0.2992] & [0.5000, 0.5000] & [0.2500, 0.3333] \\ [0.8572, 0.7143] & [0.6488, 0.5702] & [0.7500, 0.6667] & [0.5000, 0.5000] \end{pmatrix}$	1

Table 3 (continued)

Objective function values	0-1 indicator variables	QIFPRs	Forming times
$K^*_{2,14} = 1.6219$	$\delta_{12} = \delta_{13} = \delta_{23} = 0$, $\delta_{14} = \delta_{24} = \delta_{34} = 1$	$\bar{B}_{13} = \begin{pmatrix} [0.5000, 0.5000] & [0.5285, 0.4825] & [0.7500, 0.6000] & [0.3333, 0.4000] \\ [0.4715, 0.5175] & [0.5000, 0.5000] & [0.7082, 0.6040] & [0.3512, 0.4298] \\ [0.2500, 0.4000] & [0.2918, 0.3960] & [0.5000, 0.5000] & [0.2500, 0.3333] \\ [0.6667, 0.6000] & [0.6488, 0.5702] & [0.7500, 0.6667] & [0.5000, 0.5000] \end{pmatrix}$	1
$K^*_{2,34} = 2.5986$	$\delta_{12} = \delta_{13} = \delta_{14} =$ $\delta_{23} = \delta_{24} = \delta_{34} = 1$	$\bar{B}_{14} = \begin{pmatrix} [0.5000, 0.5000] & [0.4825, 0.5285] & [0.6000, 0.7500] & [0.4500, 0.5000] \\ [0.5175, 0.4715] & [0.5000, 0.5000] & [0.6040, 0.7082] & [0.4841, 0.5627] \\ [0.4000, 0.2500] & [0.3960, 0.2918] & [0.5000, 0.5000] & [0.4000, 0.5000] \\ [0.5500, 0.5000] & [0.5159, 0.4373] & [0.6000, 0.5000] & [0.5000, 0.5000] \end{pmatrix}$	1

their collectively multiplicatively consistent QIFPRs. Then, their distance measure defined by formula (21) has the following characteristics:

(i) $V\left(\bar{C}^k, \bar{C}^g\right) = V\left(\bar{C}^g, \bar{C}^k\right)$;
(ii) $0 \leq V\left(\bar{C}^k, \bar{C}^g\right) \leq 1$;
(iii) $V\left(\bar{C}^k, \bar{C}^g\right) = 0$ if and only if $\bar{C}^k = \bar{C}^g$;
(iv) Let $\tilde{H}^t = (\tilde{h}^t_{ij})_{n\times n}$ be any another IVHFPR, and let $\bar{C}^t = (\bar{c}^t_{ij})_{n\times n}$ be its collectively multiplicatively consistent QIFPR. Then, $V\left(\bar{C}^k, \bar{C}^g\right) \leq V\left(\bar{C}^g, \bar{C}^t\right) + V\left(\bar{C}^t, \bar{C}^k\right)$.

Proof From formula (21), one can easily derive the conclusions. □

Definition 12 Let $\tilde{H}^k = (\tilde{h}^k_{ij})_{n\times n}$, $k = 1, 2, \ldots, m$, be any m IVHFPRs, and let $\bar{C}^k = (\bar{c}^k_{ij})_{n\times n}$ be their collectively multiplicatively consistent QIFPR. Furthermore, let $\bar{C} = \left(\bar{c}_{ij}\right)_{n\times n}$ be the comprehensively multiplicatively consistent QIFPR. Then, the consensus measure of $\bar{C}^k$ is defined as follows:

$$COI\left(\bar{C}^k\right) = 1 - \frac{1}{n(n-1)} \sum\nolimits_{i,j=1, i<j}^{n} \times\left(\left|c^{k-}_{ij} - c^{-}_{ij}\right| + \left|c^{k+}_{ij} - c^{+}_{ij}\right|\right), \tag{22}$$

where $k = 1, 2, \ldots, m$.

When we calculate the comprehensively multiplicatively consistent QIFPR, the weights of the DMs are used. To determine the weighting vector on the DM set, we establish the following maximum consensus-based programming model:

$$\Lambda^* = \min \sum\nolimits_{k=1}^{m} \varepsilon^-_k + \varepsilon^+_k$$
$$s.t.\begin{cases} \sum\nolimits_{i,j=1,i<j}^{n} \left(\left(c^{k-}_{ij} - \frac{\sqrt[n]{\Pi_{k=1}^{n}\left(\left(\Pi_{g=1}^{m}\left(c^{g-}_{ik}\right)^{w_{eg}}\right)\left(\Pi_{g=1}^{m}\left(c^{g-}_{kj}\right)^{w_{eg}}\right)\right)}}{\sqrt[n]{\Pi_{k=1}^{n}\left(\left(\Pi_{g=1}^{m}\left(c^{g-}_{ik}\right)^{w_{eg}}\right)\left(\Pi_{g=1}^{m}\left(c^{g-}_{kj}\right)^{w_{eg}}\right)\right)} + \sqrt[n]{\Pi_{k=1}^{n}\left(\left(\Pi_{g=1}^{m}\left(c^{g-}_{jk}\right)^{w_{eg}}\right)\left(\Pi_{g=1}^{m}\left(c^{g-}_{ki}\right)^{w_{eg}}\right)\right)}}\right)^2 + \right. \\ \left. \left(c^{k+}_{ij} - \frac{\sqrt[n]{\Pi_{k=1}^{n}\left(\left(\Pi_{g=1}^{m}\left(c^{g+}_{ik}\right)^{w_{eg}}\right)\left(\Pi_{g=1}^{m}\left(c^{g+}_{kj}\right)^{w_{eg}}\right)\right)}}{\sqrt[n]{\Pi_{k=1}^{n}\left(\left(\Pi_{g=1}^{m}\left(c^{g+}_{ik}\right)^{w_{eg}}\right)\left(\Pi_{g=1}^{m}\left(c^{g+}_{kj}\right)^{w_{eg}}\right)\right)} + \sqrt[n]{\Pi_{k=1}^{n}\left(\left(\Pi_{g=1}^{m}\left(c^{g+}_{jk}\right)^{w_{eg}}\right)\left(\Pi_{g=1}^{m}\left(c^{g+}_{ki}\right)^{w_{eg}}\right)\right)}}\right)^2\right) - \varepsilon^-_k + \varepsilon^+_k = 0, k = 1, 2, ..., m \\ \sum\nolimits_{k=1}^{m} w_{e_k} = 1, \\ w_{e_k} \in [0, 1], k = 1, 2, ..., m \end{cases} . \tag{23}$$

One can check that $\Lambda^* = 0$ if and only if $\bar{C}^k = \bar{C}^g$ for all $k, g = 1, 2, \ldots, m$ with $k \neq g$.

Following multiplicative consistency and consensus analysis, we further present the following algorithm for group decision making with IVHFPRs that can address incomplete and inconsistent cases.

Table 4 Multiplicatively consistent QIFPRs

Multiplicatively consistent QIFPRs	Multiplicatively consistent QIFPRs
$\bar{C}_1 = \begin{pmatrix} [0.5000, 0.5000] & [0.3474, 0.4953] & [0.5831, 0.7288] & [0.4226, 0.4840] \\ [0.6526, 0.5047] & [0.5000, 0.5000] & [0.7244, 0.7325] & [0.5789, 0.4886] \\ [0.4169, 0.2712] & [0.2756, 0.2675] & [0.5000, 0.5000] & [0.3435, 0.2587] \\ [0.5774, 0.5160] & [0.4211, 0.5114] & [0.6565, 0.7413] & [0.5000, 0.5000] \end{pmatrix}$	$\bar{C}_2 = \begin{pmatrix} [0.5000, 0.5000] & [0.4665, 0.5169] & [0.6052, 0.7371] & [0.4327, 0.4799] \\ [0.5335, 0.4831] & [0.5000, 0.5000] & [0.6368, 0.7238] & [0.4658, 0.4630] \\ [0.3948, 0.2629] & [0.3632, 0.2762] & [0.5000, 0.5000] & [0.3322, 0.2476] \\ [0.5673, 0.5201] & [0.5342, 0.5370] & [0.6678, 0.7524] & [0.5000, 0.5000] \end{pmatrix}$
$\bar{C}_3 = \begin{pmatrix} [0.5000, 0.5000] & [0.5030, 0.5527] & [0.6069, 0.7500] & [0.4428, 0.5000] \\ [0.4970, 0.4473] & [0.5000, 0.5000] & [0.6040, 0.7082] & [0.4398, 0.4472] \\ [0.3931, 0.2500] & [0.3960, 0.2918] & [0.5000, 0.5000] & [0.3398, 0.2500] \\ [0.5572, 0.5000] & [0.5602, 0.5528] & [0.6602, 0.7500] & [0.5000, 0.5000] \end{pmatrix}$	$\bar{C}_4 = \begin{pmatrix} [0.5000, 0.5000] & [0.5478, 0.5478] & [0.6069, 0.7500] & [0.4428, 0.5000] \\ [0.4522, 0.4522] & [0.5000, 0.5000] & [0.5603, 0.7123] & [0.3961, 0.4522] \\ [0.3931, 0.2500] & [0.4397, 0.2877] & [0.5000, 0.5000] & [0.3398, 0.2500] \\ [0.5572, 0.5000] & [0.6039, 0.5478] & [0.6602, 0.7500] & [0.5000, 0.5000] \end{pmatrix}$
$\bar{C}_5 = \begin{pmatrix} [0.5000, 0.5000] & [0.4928, 0.5407] & [0.6020, 0.7454] & [0.4378, 0.4939] \\ [0.5072, 0.4593] & [0.5000, 0.5000] & [0.6089, 0.7132] & [0.4449, 0.4533] \\ [0.3980, 0.2546] & [0.3911, 0.2868] & [0.5000, 0.5000] & [0.3398, 0.2500] \\ [0.5622, 0.5061] & [0.5551, 0.5467] & [0.6602, 0.7500] & [0.5000, 0.5000] \end{pmatrix}$	$\bar{C}_6 = \begin{pmatrix} [0.5000, 0.5000] & [0.4744, 0.5568] & [0.6322, 0.7483] & [0.4447, 0.4981] \\ [0.5256, 0.4432] & [0.5000, 0.5000] & [0.6557, 0.7030] & [0.4701, 0.4413] \\ [0.3678, 0.2517] & [0.3443, 0.2970] & [0.5000, 0.5000] & [0.3178, 0.2503] \\ [0.5553, 0.5019] & [0.5299, 0.5587] & [0.6822, 0.7497] & [0.5000, 0.5000] \end{pmatrix}$
$\bar{C}_7 = \begin{pmatrix} [0.5000, 0.5000] & [0.4927, 0.5514] & [0.6168, 0.7510] & [0.4428, 0.5000] \\ [0.5073, 0.4486] & [0.5000, 0.5000] & [0.6237, 0.7105] & [0.4500, 0.4486] \\ [0.3832, 0.2490] & [0.3763, 0.2895] & [0.5000, 0.5000] & [0.3306, 0.2490] \\ [0.5572, 0.5000] & [0.5500, 0.5514] & [0.6694, 0.7510] & [0.5000, 0.5000] \end{pmatrix}$	$\bar{C}_8 = \begin{pmatrix} [0.5000, 0.5000] & [0.5039, 0.5693] & [0.6061, 0.7372] & [0.4428, 0.5000] \\ [0.4961, 0.4307] & [0.5000, 0.5000] & [0.6024, 0.6797] & [0.4390, 0.4307] \\ [0.3939, 0.2628] & [0.3976, 0.3203] & [0.5000, 0.5000] & [0.3405, 0.2628] \\ [0.5572, 0.5000] & [0.5610, 0.5693] & [0.6595, 0.7372] & [0.5000, 0.5000] \end{pmatrix}$
$\bar{C}_9 = \begin{pmatrix} [0.5000, 0.5000] & [0.5597, 0.5534] & [0.6069, 0.7500] & [0.4310, 0.4943] \\ [0.4403, 0.4466] & [0.5000, 0.5000] & [0.5485, 0.7076] & [0.3734, 0.4410] \\ [0.3931, 0.2500] & [0.4515, 0.2924] & [0.5000, 0.5000] & [0.3291, 0.2458] \\ [0.5690, 0.5057] & [0.6266, 0.5590] & [0.6709, 0.7542] & [0.5000, 0.5000] \end{pmatrix}$	$\bar{C}_{10} = \begin{pmatrix} [0.5000, 0.5000] & [0.5533, 0.5393] & [0.6069, 0.7500] & [0.4373, 0.5086] \\ [0.4467, 0.4607] & [0.5000, 0.5000] & [0.5548, 0.7193] & [0.3855, 0.4692] \\ [0.3931, 0.2500] & [0.4452, 0.2807] & [0.5000, 0.5000] & [0.3348, 0.2565] \\ [0.5627, 0.4914] & [0.6145, 0.5308] & [0.6652, 0.7435] & [0.5000, 0.5000] \end{pmatrix}$
$\bar{C}_{11} = \begin{pmatrix} [0.5000, 0.5000] & [0.4573, 0.4861] & [0.6052, 0.7371] & [0.4418, 0.5107] \\ [0.5427, 0.5139] & [0.5000, 0.5000] & [0.6453, 0.7478] & [0.4843, 0.5246] \\ [0.3948, 0.2629] & [0.3547, 0.2522] & [0.5000, 0.5000] & [0.3405, 0.2713] \\ [0.5582, 0.4893] & [0.5157, 0.4754] & [0.6595, 0.7287] & [0.5000, 0.5000] \end{pmatrix}$	$\bar{C}_{12} = \begin{pmatrix} [0.5000, 0.5000] & [0.3075, 0.4519] & [0.5098, 0.6588] & [0.1938, 0.3833] \\ [0.6925, 0.5481] & [0.5000, 0.5000] & [0.7008, 0.7008] & [0.3512, 0.4298] \\ [0.4902, 0.3412] & [0.2992, 0.2992] & [0.5000, 0.5000] & [0.1877, 0.2435] \\ [0.8062, 0.6167] & [0.6488, 0.5702] & [0.8123, 0.7565] & [0.5000, 0.5000] \end{pmatrix}$
$\bar{C}_{13} = \begin{pmatrix} [0.5000, 0.5000] & [0.5225, 0.4825] & [0.7111, 0.5897] & [0.3842, 0.4103] \\ [0.4775, 0.5175] & [0.5000, 0.5000] & [0.6923, 0.6065] & [0.3631, 0.4273] \\ [0.2889, 0.4103] & [0.3077, 0.3935] & [0.5000, 0.5000] & [0.2022, 0.3262] \\ [0.6158, 0.5897] & [0.6369, 0.5727] & [0.7978, 0.6738] & [0.5000, 0.5000] \end{pmatrix}$	$\bar{C}_{14} = \begin{pmatrix} [0.5000, 0.5000] & [0.4816, 0.5118] & [0.5847, 0.6899] & [0.4666, 0.5905] \\ [0.5184, 0.4882] & [0.5000, 0.5000] & [0.6024, 0.6797] & [0.4849, 0.5791] \\ [0.4153, 0.3101] & [0.3976, 0.3203] & [0.5000, 0.5000] & [0.3833, 0.3933] \\ [0.5334, 0.4095] & [0.5151, 0.4209] & [0.6167, 0.6067] & [0.5000, 0.5000] \end{pmatrix}$

Algorithm 2 Ranking objects from group decision making with inconsistent and incomplete IVHFPRs

Step 1: With respect to each individual IVHFPR $\tilde{H}^k = \left(\tilde{h}_{ij}^k\right)_{n\times n}, k = 1, 2, \ldots, m$, Steps 2-4 in Algorithm I are adopted to derive the multiplicatively consistent QIFPRs $\bar{C}^k = (\bar{c}_{ij}^k)_{n\times n}, k = 1, 2, \ldots, m$;

Step 2: Model (23) is used to determine the weights of the DMs, and the comprehensively multiplicatively consistent QIFPR $\bar{C} = (\bar{c}_{ij})_{n\times n}$ is calculated following the IWGMA operator and formula (11);

Step 3: Letπ be the given consensus threshold. Formula (22) is adopted to judge the consensus of the multiplicatively consistent QIFPR $\bar{C}^k$. If $COI\left(\bar{C}^k\right) \geq \pi$ for all $k = 1, 2, \ldots, m$, then turn to Step 5. Otherwise, skip to the next step;

Step 4: Let$COI\left(\tilde{H}^k\right) = \min_{1\leq g\leq m} COI\left(\tilde{H}^g\right)$, and let $\bar{C}^{k,(p+1)} = (\bar{c}_{ij}^{k,(p+1)})_{n\times n}$, where

$$\bar{c}_{ij}^{k,(p+1)} = \left[\frac{\sqrt[n]{\left(c_{ij}^{k,(p)-}\right)^{\alpha}\left(c_{ij}^{(p)-}\right)^{1-\alpha}}}{\sqrt[n]{\left(c_{ij}^{k,(p)-}\right)^{\alpha}\left(c_{ij}^{(p)-}\right)^{1-\alpha}}+\sqrt[n]{\left(c_{ji}^{k,(p)-}\right)^{\alpha}\left(c_{ji}^{(p)-}\right)^{1-\alpha}}},\right.$$
$$\left.\frac{\sqrt[n]{\left(c_{ij}^{k,(p)+}\right)^{\alpha}\left(c_{ij}^{(p)+}\right)^{1-\alpha}}}{\sqrt[n]{\left(c_{ij}^{k,(p)+}\right)^{\alpha}\left(c_{ij}^{(p)+}\right)^{1-\alpha}}+\sqrt[n]{\left(c_{ji}^{k,(p)+}\right)^{\alpha}\left(c_{ji}^{(p)+}\right)^{1-\alpha}}}\right] \tag{24}$$

for all $i, j = 1, 2, \ldots, n$. Then, return to Step 2;

Step 5: Following the comprehensively multiplicatively consistent QIFPR $\bar{C} = (\bar{c}_{ij})_{n\times n}$, the comprehensively multiplicatively consistent IFPR $\bar{D} = (\bar{d}_{ij})_{n\times n}$ is obtained. Then, Steps 5-6 in Algorithm I are utilized to rank objects.

To see the procedure of Algorithm II intuitively, please see Fig. 1.

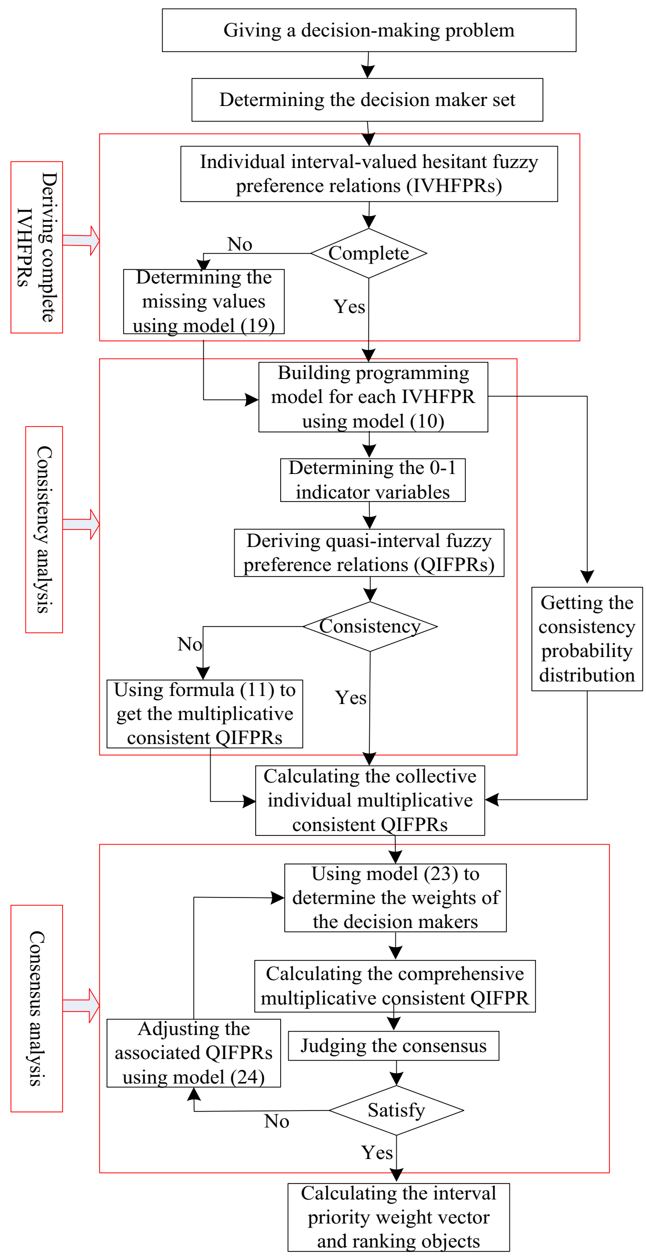

Fig. 1 The procedure of Algorithm II

Remark 2 Following algorithm II and Fig. 1, one can find that algorithms I and II have the similar procedure for ranking objects, and there are two more steps of algorithm II than Algorithm I: determining the weights of the DMs, and measuring and improving the consensus level.

Example 5.1 The technology of air conditioning has made extraordinary progress since Dr. Carrier invented the world's first air-conditioning in 1902. According to the development of air conditioning, it can be summarized as four development phases: centrifugal air conditioning, solar air conditioning, inverter air conditioning, and gas air conditioning. At present, air conditioning has become one of the most important office equipment and appliances.

The global air conditioning market is huge. According to the report of NIKKEI in 2015, the shipments of China's domestic air conditioning to the world are 7398.8 million units. To gain more market share, the competition among air conditioning manufacturers is fierce.

In the Chinese market, there are about 38 brands of air conditioning. According to the data of internet consumer research center in 2014, there are four main brands of air conditioning: Gree, Haier, Hisense, and Midea. There is an air conditioning industry assessment panel that is composed by three experts. They were invited to compare these four brands of air conditioning in the next five years of developments. Because there are many factors, such as enterprise management, enterprise R D capability, corporate reputation, and brand utility, it is not an easy thing to give their comparisons using one exact or fuzzy variable. In this case, the experts can apply IVHFEs to express their uncertain hesitancy preferences. Furthermore, they are allowed to only give partial judgments, namely, the missing judgements are permitted. According to their expertise and known information, the individual IVHFPRs are listed in Tables 5, 6 and 7.

To rank these four air conditioning brands, the following procedure is needed:

Step 1: Because all of these three individual IVHFPRs are incomplete, model (19) is applied to determine the missing IVHFEs, which are obtained as shown in Table 8.

With respect to each complete IVHFPR, the multiplicatively consistent QIFPRs, and their consistency probability distributions are derived as shown in Tables 9, 10 and 11.

Using the IWGMA operator and formula (11), the multiplicatively consistent QIFPRs are:

$$\bar{C}^1 = \begin{pmatrix} [0.5000, 0.5000] & [0.7271, 0.7973] & [0.7844,0.7003] & [0.8484,0.9060] \\ [0.2729, 0.2027] & [0.5000, 0.5000] & [0.5772, 0.3726] & [0.6775,0.7102] \\ [0.2156, 0.2997] & [0.4228, 0.6274] & [0.5000, 0.5000] & [0.6060,0.8049] \\ [0.1516, 0.0940] & [0.3225, 0.2898] & [0.3940, 0.1951] & [0.5000,0.5000] \end{pmatrix},$$

$$\bar{C}^2 = \begin{pmatrix} [0.5000,0.5000] & [0.8287,0.7206] & [0.8433,0.7336] & [0.8892,0.7101] \\ [0.1713,0.2794] & [0.5000,0.5000] & [0.5267,0.5163] & [0.6239,0.4871] \\ [0.1567,0.2664] & [0.4733,0.4837] & [0.5000,0.5000] & [0.5985,0.4708] \\ [0.1108,0.2899] & [0.3761,0.5129] & [0.4015,0.5292] & [0.5000,0.5000] \end{pmatrix},$$

$$\bar{C}^3 = \begin{pmatrix} [0.5000,0.5000] & [0.7984,0.7990] & [0.8099,0.7746] & [0.8041,0.8488] \\ [0.2016,0.2010] & [0.5000,0.5000] & [0.5182,0.4637] & [0.5089,0.5854] \\ [0.1901,0.2254] & [0.4818,0.5363] & [0.5000,0.5000] & [0.4907,0.6202] \\ [0.1959,0.1512] & [0.4911,0.4146] & [0.5093,0.3798] & [0.5000,0.5000] \end{pmatrix}.$$

Step 2: Following model (23), the weights of the DMs are $w_{e_1} = 0.3063$, $w_{e_2} = 0.3632$, $w_{e_3} = 0.3305$. Furthermore, the comprehensively multiplicatively consistent QIFPR based on the IWGMA operator and formula (11) is

$$\bar{C} = \begin{pmatrix} [0.5000, 0.5000] & [0.7905, 0.7720] & [0.8155, 0.7380] & [0.8520, 0.8306] \\ [0.2095, 0.2280] & [0.5000, 0.5000] & [0.5395, 0.4542] & [0.6042, 0.5914] \\ [0.1845, 0.2620] & [0.4605, 0.5458] & [0.5000, 0.5000] & [0.5658, 0.6350] \\ [0.1480, 0.1694] & [0.3958, 0.4086] & [0.4342, 0.3650] & [0.5000, 0.5000] \end{pmatrix}.$$

Step 3: Let$\pi = 0.9$. From formula (22), we have
$$\begin{cases} COI(\tilde{H}^1) = 0.9468 \\ COI(\tilde{H}^1) = 0.9500 \\ COI(\tilde{H}^1) = 0.9722 \end{cases}.$$

Step 4: With respect to the multiplicatively consistent QIFPR $\bar{C}$, the comprehensively multiplicatively consistent IFPR is

Table 5 IVHFPR $\tilde{H}^1$ offered by the DM e_1

	Gree	Haier	Hisense	Midea
Gree	{[0.5, 0.5]}	{[0.6, 0.7] , [0.75, 0.9]}	{[0.6, 0.85]}	x
Haier	{[0.1, 0.25] , [0.3, 0.4]}	{[0.5, 0.5]}	x	{[0.5, 0.6] , [0.7.0.8]}
Hisense	{[0.15, 0.4]}	x	{[0.5, 0.5]}	x
Midea	x	{[0.2, 0.3] , [0.4, 0.5]}	x	{[0.5, 0.5]}

Table 6 IVHFPR $\tilde{H}^2$offered by the DM e_2

	Gree	Haier	Hisense	Midea
Gree	{[0.5, 0.5]}	x	{[0.5, 0.6] , [0.8, 0.9]}	{[0.7, 0.9]}
Haier	x	{[0.5, 0.5]}	{[0.4, 0.65]}	x
Hisense	{[0.1, 0.2] , [0.4, 0.5]}	{[0.35, 0.6]}	{[0.5, 0.5]}	{[0.4, 0.5] , [0.6.0.7]}
Midea	{[0.1, 0.3]}	x	{[0.3, 0.4] , [0.5.0.6]}	{[0.5, 0.5]}

$$\bar{D} = \begin{pmatrix} [0.5000, 0.5000] & [0.7720, 0.7905] & [0.7380, 0.8155] & [0.8306, 0.8520] \\ [0.2095, 0.2280] & [0.5000, 0.5000] & [0.4542, 0.5395] & [0.5914, 0.6042] \\ [0.1845, 0.2620] & [0.4605, 0.5458] & [0.5000, 0.5000] & [0.5658, 0.6350] \\ [0.1480, 0.1694] & [0.3958, 0.4086] & [0.3650, 0.4342] & [0.5000, 0.5000] \end{pmatrix}.$$

Furthermore, the interval fuzzy priority weight vector is

$$\bar{\omega} = ([0.3580, 0.3916], [0.2106,0.2371], [0.2065,0.2442], [0.1652,0.1892]).$$

Using the formula for comparing intervals in [13], the ranking is Gree ≻ Hisense ≻ Haier ≻ Midea.

Note that the previous group decision-making methods with IVHFPRs [3, 18] only considered the complete case. Thus, none of them can be applied in this example directly.

Following the complete IVHFPRs obtained from our method, when Pérez-Fernándeza et al.'s method [18] is used in Example 5.1, the ranking is Gree ≻ Hisense ≻ Haier ≻ Midea. It is the same as the above ranking. Notably, the aggregation operator uses the arithmetic mean and α equals 0.25, which are adopted by Pérez-Fernandez et al. [18].

Furthermore, when Chen et al.'s method [3] is adopted in Example 5.1, the weights of the DMs are $w_{e_1} = 0.2787$, $w_{e_2} = 0.3895$, and $w_{e_3} = 0.3318$. Moreover, the score values of objects are

$$s(\tilde{h}_1) = [0.73,0.89], s(\tilde{h}_2) = [0.47,0.59], s(\tilde{h}_3) = [0.42,0.59], s(\tilde{h}_4) = [0.32,0.46],$$

by which the ranking is Gree ≻ Haier ≻ Hisense ≻ Midea, which is different from the above ranking. Notably, the used aggregation operators are the same as those adopted by Chen et al. [3].

Remark 3 There are several limitations of methods in [3, 18]: (i) they cannot address decision making with incomplete IVHFPRs, (ii) they did not study the consistency of IVHFPRs, which may lead to illogical ranking, (iii) they did not consider the consensus of individual IVHFPRs. Therefore, the final ranking cannot reflect the agreement degree of individual opinions, (iv) method in [18] is based on the assumption that all of individual IVHFPRs have the same important, while method in [3] needs to add extra values into shorter IVHFEs to determine the weights of the DMs.

Notably, new method is based on multiplicative consistency and consensus analysis that can guarantee the logical ranking and reflect the agreement level of individual opinions. Furthermore, new method determines the weighting information without any extra subjective information. To show the advantages of the new method clearly, it can be summarized as follows:

(i) It is based on the consistency analysis that neither causes information loss nor disregards any information;

Table 7 IVHFPR $\tilde{H}^3$offered by the DM e_3

	Gree	Haier	Hisense	Midea
Gree	{[0.5, 0.5]}	x	{[0.7, 0.8]}	{[0.8, 0.9]}
Haier	x	{[0.5, 0.5]}	{[0.3, 0.4] , [0.45, 0.55] , [0.6, 0.65]}	x
Hisense	{[0.2, 0.3]}	{[0.35, 0.4] , [0.45, 0.55] , [0.6, 0.7]}	{[0.5, 0.5]}	{[0.4, 0.6]}
Midea	{[0.1, 0.2]}	x	{[0.4, 0.6]}	{[0.5, 0.5]}

Table 8 Determined missing IVHFEs

Missing IVHFEs	Determined values
$\tilde{h}^1_{23}$	{[0.5000, 0.7083] , [0.3333, 0.3864] , [0.1429, 0.6539]}
$\tilde{h}^1_{14}$	{[0.8448.0.8571] , [0.8182, 0.9000] , [0.7500, 0.9310] , [0.9231, 0.9546]}
$\tilde{h}^1_{34}$	{[0.4900, 0.8000] , [0.6136, 0.7500] , [0.6667, 0.7044] , [0.6793, 0.9333]}
$\tilde{h}^2_{12}$	{[0.4498, 0.7017] , [0.6732, 0.9310] , [0.8290, 0.8516]}
$\tilde{h}^2_{24}$	{[0.7093, 0.7382] , [0.4000, 0.5422] , [0.2983, 0.6500]}
$\tilde{h}^3_{12}$	{[0.8506, 0.9000] , [0.7489, 0.8308] , [0.6806, 0.7114]}
$\tilde{h}^3_{24}$	{[0.4000, 0.4120] , [0.5500, 0.5620] , [0.5781, 0.7613]}

Table 9 Multiplicatively consistent QIFPRs obtained from the complete IVHFPR $\tilde{H}^1$ and the consistency probability distribution

Multiplicatively consistent QIFPRs	Multiplicatively consistent QIFPRs
$\bar{C}_1^1=\begin{pmatrix}[0.5000, 0.5000] & [0.7000, 0.6000] & [0.8500, 0.6000] & [0.8448, 0.8571]\\ [0.3000, 0.4000] & [0.5000, 0.5000] & [0.7083, 0.5000] & [0.7000, 0.8000]\\ [0.1500, 0.4000] & [0.2917, 0.5000] & [0.5000, 0.5000] & [0.4900, 0.8000]\\ [0.1552, 0.1429] & [0.3000, 0.2000] & [0.5100, 0.2000] & [0.5000, 0.5000]\end{pmatrix}$	$\bar{C}_2^1=\begin{pmatrix}[0.5000, 0.5000] & [0.7500, 0.9000] & [0.6000, 0.8500] & [0.7500, 0.9310]\\ [0.2500, 0.1000] & [0.5000, 0.5000] & [0.3333, 0.3864] & [0.5000, 0.6000]\\ [0.4000, 0.1500] & [0.6667, 0.6136] & [0.5000, 0.5000] & [0.6667, 0.7043]\\ [0.2500, 0.0690] & [0.5000, 0.4000] & [0.3333, 0.2957] & [0.5000, 0.5000]\end{pmatrix}$
$\bar{C}_3^1=\begin{pmatrix}[0.5000, 0.5000] & [0.6981, 0.6150] & [0.8500, 0.6000] & [0.9008, 0.8087]\\ [0.3019, 0.3850] & [0.5000, 0.5000] & [0.7102, 0.4843] & [0.7971, 0.7257]\\ [0.1500, 0.4000] & [0.2898, 0.5157] & [0.5000, 0.5000] & [0.6158, 0.7380]\\ [0.0992, 0.1913] & [0.2029, 0.2743] & [0.3842, 0.2620] & [0.5000, 0.5000]\end{pmatrix}$	$\bar{C}_4^1=\begin{pmatrix}[0.5000, 0.5000] & [0.7500, 0.9000] & [0.8500, 0.6001] & [0.9231, 0.9546]\\ [0.2500, 0.1000] & [0.5000, 0.5000] & [0.6539, 0.1429] & [0.8000, 0.7001]\\ [0.1500, 0.3999] & [0.3461, 0.8571] & [0.5000, 0.5000] & [0.6793, 0.9333]\\ [0.0769, 0.0454] & [0.2000, 0.2999] & [0.3207, 0.0667] & [0.5000, 0.5000]\end{pmatrix}$
The consistency probability distribution	$P_1=\{5/15, 5/15, 2/15, 3/15\}$

Table 10 Multiplicatively consistent QIFPRs obtained from the complete IVHFPR $\tilde{H}^2$ and the consistency probability distribution

Multiplicative consistent QIFPRs	Multiplicative consistent QIFPRs
$\bar{C}_1^2=\begin{pmatrix}[0.5000, 0.5000] & [0.7225, 0.4249] & [0.6576, 0.5535] & [0.8640, 0.6756]\\ [0.2775, 0.5751] & [0.5000, 0.5000] & [0.4246, 0.6266] & [0.7093, 0.7382]\\ [0.3424, 0.4465] & [0.5754, 0.3734] & [0.5000, 0.5000] & [0.7678, 0.6269]\\ [0.1360, 0.3244] & [0.2907, 0.2618] & [0.2322, 0.3731] & [0.5000, 0.5000]\end{pmatrix}$	$\bar{C}_2^2=\begin{pmatrix}[0.5000, 0.5000] & [0.9310, 0.6732] & [0.9000, 0.7928] & [0.9000, 0.7093]\\ [0.0690, 0.3268] & [0.5000, 0.5000] & [0.4000, 0.6500] & [0.4000, 0.5422]\\ [0.1000, 0.2072] & [0.6000, 0.3500] & [0.5000, 0.5000] & [0.5000, 0.3894]\\ [0.1000, 0.2907] & [0.6000, 0.4578] & [0.5000, 0.6106] & [0.5000, 0.5000]\end{pmatrix}$
$\bar{C}_3^2=\begin{pmatrix}[0.5000, 0.5000] & [0.8290, 0.8516] & [0.9000, 0.7928] & [0.9000, 0.7093]\\ [0.1710, 0.1484] & [0.5000, 0.5000] & [0.6500, 0.4000] & [0.6500, 0.2983]\\ [0.1000, 0.2072] & [0.3500, 0.6000] & [0.5000, 0.5000] & [0.5000, 0.3894]\\ [0.1000, 0.2907] & [0.3500, 0.7017] & [0.5000, 0.6106] & [0.5000, 0.5000]\end{pmatrix}$	$\bar{C}_5^2=\begin{pmatrix}[0.5000, 0.5000] & [0.8290, 0.8516] & [0.9000, 0.7928] & [0.9000, 0.7093]\\ [0.1710, 0.1484] & [0.5000, 0.5000] & [0.6500, 0.4000] & [0.6500, 0.2983]\\ [0.1000, 0.2072] & [0.3500, 0.6000] & [0.5000, 0.5000] & [0.5000, 0.3894]\\ [0.1000, 0.2907] & [0.3500, 0.7017] & [0.5000, 0.6106] & [0.5000, 0.5000]\end{pmatrix}$
$\bar{C}_4^2=\begin{pmatrix}[0.5000, 0.5000] & [0.7017, 0.4498] & [0.6106, 0.6029] & [0.8516, 0.6974]\\ [0.2983, 0.5502] & [0.5000, 0.5000] & [0.4000, 0.6500] & [0.7093, 0.7382]\\ [0.3894, 0.3971] & [0.6000, 0.3500] & [0.5000, 0.5000] & [0.7854, 0.6029]\\ [0.1484, 0.3026] & [0.2907, 0.2618] & [0.2146, 0.3971] & [0.5000, 0.5000]\end{pmatrix}$	$\bar{C}_6^2=\begin{pmatrix}[0.5000, 0.5000] & [0.8290, 0.8516] & [0.8905, 0.7366] & [0.9088, 0.7694]\\ [0.1710, 0.1484] & [0.5000, 0.5000] & [0.6266, 0.3277] & [0.6727, 0.3677]\\ [0.1095, 0.2634] & [0.3734, 0.6723] & [0.5000, 0.5000] & [0.5505, 0.5440]\\ [0.0912, 0.2306] & [0.3273, 0.6323] & [0.4495, 0.4560] & [0.5000, 0.5000]\end{pmatrix}$
The consistency probability distribution	$P_2=\{1/10, 2/10, 2/10, 2/10, 2/10, 1/10\}$

Table 11 Multiplicatively consistent QIFPRs obtained from the complete IVHFPR $\tilde{H}^3$ and the consistency probability distribution

Multiplicative consistent QIFPRs	Multiplicative consistent QIFPRs
$\bar{C}_1^3 = \begin{pmatrix} [0.5000, 0.5000] & [0.8989, 0.8508] & [0.8571, 0.7093] & [0.8587, 0.7925] \\ [0.1011, 0.1492] & [0.5000, 0.5000] & [0.4030, 0.2996] & [0.4060, 0.4010] \\ [0.1429, 0.2907] & [0.5970, 0.7004] & [0.5000, 0.5000] & [0.5031, 0.6101] \\ [0.1413, 0.2075] & [0.5940, 0.5990] & [0.4969, 0.3899] & [0.5000, 0.5000] \end{pmatrix}$	$\bar{C}_2^3 = \begin{pmatrix} [0.5000, 0.5000] & [0.8291, 0.7512] & [0.8571, 0.7093] & [0.8586, 0.7908] \\ [0.1709, 0.2488] & [0.5000, 0.5000] & [0.5530, 0.4470] & [0.5560, 0.5560] \\ [0.1429, 0.2907] & [0.4470, 0.5530] & [0.5000, 0.5000] & [0.5030, 0.6077] \\ [0.1414, 0.2092] & [0.4440, 0.4440] & [0.4970, 0.3923] & [0.5000, 0.5000] \end{pmatrix}$
$\bar{C}_3^3 = \begin{pmatrix} [0.5000, 0.5000] & [0.6806, 0.7114] & [0.7617, 0.8207] & [0.7449, 0.8872] \\ [0.3194, 0.2886] & [0.5000, 0.5000] & [0.6000, 0.6500] & [0.5781, 0.7613] \\ [0.2383, 0.1793] & [0.4000, 0.3500] & [0.5000, 0.5000] & [0.4774, 0.6320] \\ [0.2551, 0.1128] & [0.4219, 0.2387] & [0.5226, 0.3680] & [0.5000, 0.5000] \end{pmatrix}$	$\bar{C}_4^3 = \begin{pmatrix} [0.5000, 0.5000] & [0.7295, 0.8445] & [0.7676, 0.8159] & [0.7575, 0.8802] \\ [0.2705, 0.1555] & [0.5000, 0.5000] & [0.5505, 0.4494] & [0.5366, 0.5751] \\ [0.2324, 0.1841] & [0.4495, 0.5506] & [0.5000, 0.5000] & [0.4860, 0.6238] \\ [0.2425, 0.1198] & [0.4634, 0.4249] & [0.5140, 0.3762] & [0.5000, 0.5000] \end{pmatrix}$
The consistency probability distribution	$P_3 = \{3/9, 1/9, 3/9, 2/9\}$

(ii) It offers a method to obtain consistent IVHFPRs from inconsistent ones, which only applies the judgements offered by the DMs;

(iii) It can address incomplete IVHFPRs by only using the provided preferences;

(iv) It provides a consensus-based method to determine the weights of the DMs;

(v) It defines a distance measure-based consensus formula to complete the consensus levels among DMs and proposes an interactive method to improve the consensus degree.

6 Conclusions

Interval-valued hesitant fuzzy preference relations are powerful to denote the decision makers' uncertain and hesitant information, which is an extension of hesitant fuzzy preference relations. After reviewing previous researches, we find that there is no research about interval-valued hesitant fuzzy preference relations based on the consistency and consensus. To ensure their reasonable application, this paper continues to study decision making with interval-valued hesitant fuzzy preference relations. To do this, a multiplicative consistency concept based on interval fuzzy preference relations is presented. Unlike the previous consistency concepts for hesitant fuzzy preference relations, the new concept does not need to add values to interval-valued hesitant fuzzy elements or ignore any information. Then, several multiplicative consistency-based programming models are constructed to address inconsistent and incomplete interval-valued hesitant fuzzy preference relations. After that, group decision making is further researched and a group decision-making algorithm is developed. Finally, a practical decision-making problem is offered to show the application of the new method.

This paper focuses on the multiplicative consistency of interval-valued hesitant fuzzy preference relations, and we can similarly study the additive consistency. Furthermore, we can extend the new theoretical results to other types of preference relations, such as interval-valued multiplicative hesitant fuzzy preference relations and interval-valued linguistic hesitant fuzzy preference relations. Besides the theoretical aspect, we shall continue to research the application of the new method in some other fields, such as medical recommendation, evaluating double first-class universities in China, assessing online shopping platforms, and evaluating Chinese airlines.

Acknowledgments This work was supported by the National Natural Science Foundation of China (Nos. 71571192, and 71671188), the Innovation-Driven Project of Central South University (No. 2018CX039), the Fundamental Research Funds for the Central Universities of Central South University (No. 2018zzts094), the State

Key Program of National Natural Science of China (No. 71431006), and the Hunan Province Foundation for Distinguished Young Scholars of China (No. 2016JJ1024).

References

1. Atanassov KT (1986) Intuitionistic fuzzy sets. Fuzzy Sets Syst 20(1):87–96
2. Atanassov KT, Gargov G (1989) Interval valued intuitionistic fuzzy sets. Fuzzy Sets Syst 31(3):343–349
3. Chen N, Xu ZS, Xia MM (2013) Interval-valued hesitant preference relation relations and their applications to group decision making. Knowl-Based Syst 37:528–540
4. Chen N, Xu ZS, Xia MM (2013) Correlation coefficients of hesitant fuzzy sets and their applications to clustering analysis. Appl Math Modell 37:2197–2211
5. Gitinavard H, Mousavi SM, Vahdani B (2016) A new multi-criteria weighting and ranking model for group decision-making analysis based on interval-valued hesitant fuzzy sets to selection problems. Neural Comput Appl 27(6):1593–1605
6. Gonçalves CDF, Dias JAM, Machado VAC (2015) Multi-criteria decision methodology for selecting maintenance key performance indicators. Int J Manag Sci Eng Manag 10(3):215–223
7. Hao ZN, Xu ZS, Zhao H, Fujita H (2017) A dynamic weight determination approach based on the intuitionistic fuzzy Bayesian network and its application to emergency decision making, IEEE Trans Fuzzy Syst. https://doi.org/10.1109/TFUZZ.2017.2755001
8. Jin FF, Ni ZW, Chen HY, Li YP, Zhou LG (2016) Multiple attribute group decision making based on interval-valued hesitant fuzzy information measures. Comput Ind En 101:103–115
9. Liao HC, Si GS, Xu ZS, Fujita H (2018) Hesitant fuzzy linguistic preference utility set and its application in selection of fire rescue plans. Int J Environ Res Public Health 15(4):1–18
10. Meng FY, Tan CQ (2017) Distance measures for intuitionistic hesitant fuzzy linguistic sets. Econ Comput Econ Cyber Stud Res 51(4):207–224
11. Meng FY, Tang J, An QX, Chen XH (2017) Decision making with intuitionistic linguistic preference relations. Int Trans Oper Res. https://doi.org/10.1111/itor.12383
12. Meng FY, Zhang Q, Chen XH (2017) Fuzzy multichoice games with fuzzy characteristic functions. Group Decis Negot 26(3):262–276
13. Meng FY, Tan CQ, Chen XH (2017) Multiplicative consistency analysis for interval reciprocal preference relations: a comparative study. Omega 68:17–38
14. Meng FY, Chen XH (2014) An approach to interval-valued hesitant fuzzy multi-attribute decision making with incomplete weight information based on hybrid Shapley operators. Informatica 25(4):617–642
15. Meng FY, Wang C, Chen XH, Zhang Q (2016) Correlation coefficients of interval-valued hesitant fuzzy sets and their application based on Shapley function. Int J Intell Syst 31(1):17–43
16. Meng FY, Tang J, Fujita H (2019) Linguistic intuitionistic fuzzy preference relations and their application to multi-criteria decision making. Inform Fusion 46:77–90
17. Ölçer I, Odabaşi AY (2005) A new fuzzy multiple attributive group decision making methodology and its application to propulsion/ manoeuvring system selection problem. Eur J Oper Res 166(1):93–114
18. Pérez-Fernándeza R, Alonsob P, Bustincec H, Díazd I, Montesa S (2016) Applications of finite interval-valued hesitant fuzzy preference relations in group decision making. Inform Sci 326:89–101
19. Tanino T (1984) Fuzzy preference orderings in group decision making. Fuzzy Sets Syst 12(2):117–131
20. Tang J, An QX, Meng FY, Chen XH (2017) A natural method for ranking objects from hesitant fuzzy preference relations. Int J Inf Tech Decis Ma 16:1611–1646
21. Tang J, Meng FY (2017) Decision making with multiplicative hesitant fuzzy linguistic preference relations. Neural Comput Appl. https://doi.org/10.1007/s00521-017-3227-x
22. Torra V (2010) Hesitant fuzzy sets. Int J Intell Syst 25(6):529–539
23. Wang YM, Chin KS (2008) A linear goal programming priority method for fuzzy analytic hierarchy process and its applications in new product screening. Int J Approx Reason 49(2):451–465
24. Wang TC, Chen YH (2008) Applying fuzzy linguistic preference relations to the improvement of consistency of fuzzy AHP. Inform Sci 178(19):3755–3765
25. Wang H, Xu Z (2015) Some consistency measures of extended hesitant fuzzy linguistic preference relations. Inform Sci 297:316–331
26. Wu ZB, Xu JP (2016) Possibility distribution based approach for MAGDM with hesitant fuzzy linguistic information. IEEE Trans Cy 46(3):694–705
27. Wu J, Dai LF, Chiclana F, Fujita H, Herrera-Viedma E (2018) A minimum adjustment cost feedback mechanism based consensus model for group decision making under social network with distributed linguistic trust. Inform Fusion 41:232–242
28. Xu ZS (2001) A practical method for priority of interval number complementary judgment matrix. Oper Res Manag Sci 10(1):16–19
29. Xu ZS, Xia MM (2011) Distance and similarity measures for hesitant fuzzy sets. Inform Sci 181:2128–2138
30. Xia MM, Xu ZS (2011) Methods for fuzzy complementary preference relations based on multiplicative consistency. Comput Ind En 61(4):930–935
31. Xia MM, Xu ZS (2011) Hesitant fuzzy information aggregation in decision making. Int J Approx Reason 52(3):395–407
32. Xu YJ, Chen L, Rodríguez RM, Herrera F, Wang HM (2016) Deriving the priority weights from incomplete hesitant fuzzy preference relations in group decision making. Knowl-Based Syst 99:71–78
33. Yuan JH, Li CB, Xu FQ et al (2016) A group decision making approach in interval-valued intuitionistic hesitant fuzzy environment with confidence levels. J Intell Fuzzy Syst 31(3):1909–1919
34. Zadeh LA (1975) The concept of a linguistic variable and its application to approximate reasoning–Part I. Inform Sci 8(3):199–249
35. Zhang ZM, Wu C (2014) Deriving the priority weights from hesitant multiplicative preference relations in group decision making. Appl Soft Comput 25:107–117
36. Zhang ZM, Wang C (2014) A decision support model for group decision making with hesitant multiplicative preference relations. Inform Sci 68:136–166
37. Zhang ZM, Wang C (2014) On the use of multiplicative consistency in hesitant fuzzy linguistic preference relations. Knowl-Based Syst 72:13–27
38. Zhang ZM, Wang C, Tian XD (2015) A decision support model for group decision making with hesitant fuzzy preference relations. Knowl-Based Syst 86:77–101
39. Zhang ZM, Wang C, Tian XD (2015) Multi–criteria group decision making with incomplete hesitant fuzzy preference relations. Appl Soft Comput 36:1–23
40. Zhu B, Xu ZS (2014) Analytic hierarchy process-hesitant group decision making. Eur J Oper Res 239(3):794–801
41. Zhu B, Xu ZS (2014) Consistency measures for hesitant fuzzy linguistic preference relations. IEEE Trans Fuzzy Syst 22(1):35–45
42. Zhu B, Xu ZS, Xu JP (2014) Deriving a ranking from hesitant fuzzy preference relations under group decision making. IEEE Trans Cy 44(8):1328–1337