

Generalized interaction aggregation operators in intuitionistic fuzzy multiplicative preference environment and their application to multicriteria decision-making

Harish Garg¹ 

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Abstract The main objective of this manuscript is to present a new preference relation called the intuitionistic fuzzy multiplicative preference relation. Under this, some series of new aggregation operators, by overcoming the shortcomings of some existing operators, have been defined. As most of the aggregation operators have been constructed under the intuitionistic fuzzy preference relation which deals with the conditions that the attribute values grades are symmetrical and uniformly distributed. In this manuscript, these assumptions have been relaxed by distributing the attribute grades to be asymmetrical around 1 and hence under it, some series of aggregation operators, namely intuitionistic fuzzy multiplicative interactive weighted, ordered weighted and hybrid weighted averaging operators have been proposed. Various desirable properties of these operators have also been discussed in details. A group decision-making method has been presented, based on the proposed operators, for ranking the different alternatives. A real example is taken to demonstrate the applicability and validity of the proposed methodology.

Keywords Multi-criteria decision making · Multiplicative preference relation · Intuitionistic numbers · Aggregation operators

1 Introduction

In many decision-making problems, it is difficult for a decision maker to give his assessments toward the object in crisp values due to ambiguity and incomplete information. To depict the decision-making problem mathematically, the preference relation is proposed which stores the preference information of the decision maker with respect to a set of alternatives or criteria in a matrix. There are mainly three sorts of preference relations, namely fuzzy preference relation (FPR) [21], intuitionistic fuzzy preference relation (IFPR) [1, 32] and multiplicative preference relation (MPR) [22]. In FPR, an analysis has been conducted in which decision-makers evaluate the object by describing their intensities of preferences only, but can't depict the degrees of non-preferences. But however, in some circumstances, it is difficult for the decision-maker to determine an exact membership function for a set. To handle this situation, an IFPR has been defined as $R = (\alpha_{ij})_{n \times n}$ with the condition that $\alpha_{ij} + \alpha_{ji} = 1$ and $0 \leq \alpha_{ij} \leq 1$, where α_{ij} indicates the alternative x_i is preferred to x_j . Under this environment, various researchers have addressed the problem of the decision-making by using different kinds of aggregation operators [34]. For instance, [31, 33] presented a weighted average and geometric aggregation operator for the intuitionistic fuzzy numbers (IFNs). [25] presented an aggregation operator by using Einstein norm operations. [12] presented a decision-making method based on the averaging aggregation operations. [2] presented a generalized intuitionistic fuzzy interactive geometric aggregation operator using Einstein t-norm and t-conorm operations. [38] presented a hybrid weighted averaging and geometric aggregation operator based on the Einstein norm operations for IFNs. [8] proposed some series of interactive aggregation

✉ Harish Garg
harishg58iitr@gmail.com
<http://sites.google.com/site/harishg58iitr/>

¹ School of Mathematics, Thapar University Patiala-147004, Punjab, India

operators for IFNs. [4], further, presented a generalized improved score function to rank the different interval-valued IFNs (IVIFNs). Apart from these, various authors [5–7, 11, 15, 18, 19, 28, 35] have addressed the problems of the decision-making in the IFPR environment. All these theories have been investigated under the IFPR environment which are valid only for those grades who are distributed uniformly and symmetrical. But due to the large complexities of the system day-to-day, it is difficult for the system analyst to represent the data in a symmetrical form and hence, IFPR theory may have unable to get the correct option to the decision-maker to opt for the system representation.

To resolve this issue, [22] presented a different scale named as 1–9 scale, where the decision maker gives their preferences on the scale of 1/9–9 instead of 0–1 to represent the data. However, it can only represent the degree of acceptance of the alternative but is unable to represent the degree of rejection. To cope with this situation, the intuitionistic multiplicative preference relation (IMPR) [29] has been proposed in which the representation of each alternative is considered as a pair of acceptance and rejection degrees and hence, their corresponding preference relation is called IMPR, denoted by $S = (\alpha_{ij})_{m \times n}$ with the condition that $\alpha_{ij}\alpha_{ji} = 1$ and $\frac{1}{9} \leq \alpha_{ij} \leq 9$ where α_{ij} represents the asymmetrical distribution degree of the alternative x_i w.r.t. x_j . From these, it has been concluded that IMPR describes the characteristics of the alternative in a better way than IFPR. Under this preference environment, [29] introduced an aggregation operator for different intuitionistic multiplicative sets (IMSs) in the decision making process while [36] extended it to an interval-valued IMS environment. [37] introduced the aggregation operator under the triangular intuitionistic multiplicative numbers. [3] developed some improved weighted geometric aggregation operators to solve the decision making problems. [30] introduced the aggregation operator based on the Choquet integral and applied them to the group decision-making problems. [37] introduced an aggregation operator under the triangular intuitionistic multiplicative numbers. [14] proposed a method for ranking the IMSs based on distance measures. Recently, Garg [9] presented the distance and similarity measures for intuitionistic multiplicative preference relation and applied them to solve the decision-making problems. Apart from these, various authors [16, 17, 27] have addressed the problems of the decision-making in the IMPR environment.

Further, during the decision-making process, the aggregation operators are very important tools to solve the problems of information fusion. But from the literature, it has been observed that few operators come into the existence in the IMPR environment which are carried out based on the traditional operational rules of intuitionistic

multiplicative numbers (IMNs). However, these operational rules have some drawbacks. For instance, there is no impact of changing the membership or non-membership degrees on the aggregated degree by the existing aggregation operators. In addition, we know that the membership degree and non-membership degree of aggregated result are obtained independently from the membership degrees and non-membership degrees of n IMNs, respectively. Obviously, this is also unreasonable. Therefore, their corresponding aggregated value is unable to give the correct information about the decision practices during the decision-making process. Hence, there is a need to improve the operational laws between the pairs of IMNs and hence their corresponding aggregation operators.

Thus, in the view of the fact that the IMPR can express the uncertain and fuzzy decision process more precisely and objectively it is a useful tool for solving the decision-making problems under the uncertain environment. By taking the advantages of these, the main objective of this manuscript is to present the new operational law by considering the interaction between the pairs of the membership degrees on the IMPR. Based on these laws, some series of aggregation operators, namely intuitionistic fuzzy multiplicative weighted, ordered weighted and hybrid weighted averaging have been proposed. Some desirable properties of these operators are also investigated. Furthermore, these operators have been extended to its generalized version and named as generalized intuitionistic fuzzy multiplicative interactive weighted, ordered weighted and hybrid weighted averaging operators. Finally, decision-making methods based on these operators have been presented and an illustrative example has been taken for demonstrating the approach.

2 Intuitionistic multiplicative preference relations

In this section, some basic concepts of intuitionistic multiplicative set (IMS) and intuitionistic multiplicative preference relation have been presented.

Definition 1 An IMS D on the universal set X is defined as [29]

$$D = \{ \langle x, \mu(x), \nu(x) \rangle \mid x \in X \} \quad (1)$$

where $\mu, \nu : X \rightarrow [1/9, 9]$ represents, respectively, the degrees of membership and non-membership such that $\frac{1}{9} \leq \mu(x), \nu(x) \leq 9, \mu(x)\nu(x) \leq 1, \forall x \in X$. On the other hand, the degree of indeterminacy is given by $\tau(x) = 1/\mu(x)\nu(x)$. The pairs of the membership degrees is called as intuitionistic multiplicative number (IMN) and denoted by $\alpha = \langle \mu, \nu \rangle$.

Definition 2 [29] A score function S of an IMN α can be represented as $S(\alpha) = \frac{\mu}{\nu}$ and an accuracy function H is represented as $H(\alpha) = \mu \cdot \nu$. Based on these, an order relation between two IMNs α and β is defined as: (i) $S(\alpha) < S(\beta)$ then α is smaller than β ; (ii) if $S(\alpha) = S(\beta)$ then (a) $H(\alpha) < H(\beta)$ which implies that α is smaller than β ; (b) $H(\alpha) = H(\beta)$ then $\alpha \sim \beta$.

Definition 3 For three IMNs $\alpha = \langle \mu, \nu \rangle$, $\alpha_1 = \langle \mu_1, \nu_1 \rangle$ and $\alpha_2 = \langle \mu_2, \nu_2 \rangle$, $\lambda > 0$ be a real number, the basic operational law between them have been defined as follows [29]

$$\begin{aligned} \text{(i)} \quad \alpha_1 \oplus \alpha_2 &= \left\langle \frac{(1+2\mu_1)(1+2\mu_2)-1}{2}, \frac{2\nu_1\nu_2}{(2+\nu_1)(2+\nu_2)-\nu_1\nu_2} \right\rangle \\ \text{(ii)} \quad \alpha_1 \otimes \alpha_2 &= \left\langle \frac{2\mu_1\mu_2}{(2+\mu_1)(2+\mu_2)-\mu_1\mu_2}, \frac{(1+2\nu_1)(1+2\nu_2)-1}{2} \right\rangle \\ \text{(iii)} \quad \lambda\alpha &= \left\langle \frac{(1+2\mu)^\lambda-1}{2}, \frac{2\nu^\lambda}{(2+\nu)^\lambda-\nu^\lambda} \right\rangle \\ \text{(iv)} \quad \alpha^\lambda &= \left\langle \frac{2\mu^\lambda}{(2+\mu)^\lambda-\mu^\lambda}, \frac{(1+2\nu)^\lambda-1}{2} \right\rangle \end{aligned}$$

Definition 4 [13] Let $X = \{x_1, x_2, \dots, x_n\}$ be a finite set and $D_i = \{x_i, \langle \mu(x_i), \nu(x_i), \tau(x_i) \rangle \mid x_i \in X\}$, ($i = 1, 2, \dots, n$) be an IMS then the corresponding IFS $G_i = \{x_i, \langle \rho(x_i), \xi(x_i), \pi(x_i) \rangle \mid x_i \in X\}$, ($i = 1, 2, \dots, n$) associated with D_i is given as follows:

$$\rho(x_i) = \frac{1}{2}(1 + \log_9 \mu(x_i)) ; \xi(x_i) = \frac{1}{2}(1 + \log_9 \nu(x_i)) ; \pi(x_i) = \frac{1}{2}(\log_9 \tau(x_i))$$

Definition 5 [14] Let $A = \{x_i, \langle \mu_A(x_i), \nu_A(x_i), \tau_A(x_i) \rangle \mid x_i \in X\}$ ($i = 1, 2, \dots, n$) be an IMS, then based on the weighted Manhattan distance,

(i) the extended weighted distance function is given as

$$Ep_w(A) = \frac{1 - \sum_{i=1}^n \omega_i \log_9 \nu_A(x_i)}{2 - \sum_{i=1}^n \omega_i (\log_9 \mu_A(x_i) + \log_9 \nu_A(x_i))} \tag{2}$$

(ii) the extended weighted accuracy function is given as

$$Eq_w(A) = 1 - \frac{1}{2} \sum_{i=1}^n \omega_i \log_9 \tau_A(x_i) \tag{3}$$

Definition 6 For a collection of IMNs $\alpha_i = \langle \mu_i, \nu_i \rangle$, ($i = 1, 2, \dots, n$), the intuitionistic multiplicative weighted aggregation (IMWA) operator is defined as [29]

$$\begin{aligned} \text{IMWA}(\alpha_1, \alpha_2, \dots, \alpha_n) &= \bigoplus_{i=1}^n \omega_i \alpha_i \\ &= \left\langle \frac{\prod_{i=1}^n (1 + 2\mu_i)^{\omega_i} - 1}{2}, \frac{2 \prod_{i=1}^n \nu_i^{\omega_i}}{\prod_{i=1}^n (2 + \nu_i)^{\omega_i} - \prod_{i=1}^n \nu_i^{\omega_i}} \right\rangle \end{aligned} \tag{4}$$

where $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$ be the weight vector of α_i ($i = 1, 2, \dots, n$) such that $\omega_i \geq 0$ and $\sum_{i=1}^n \omega_i = 1$.

Especially, if $\omega = (1/n, 1/n, \dots, 1/n)^T$, then the IMWA operator reduces to the intuitionistic multiplicative averaging operator.

It has been observed from the above defined IMWA operator that they have some sort of deficiencies during the decision-making process and hence it does not provide the sufficient information to the decision-maker during the phase of the aggregation process. These shortcomings have been highlighted as follows:

(i) **Degree of non-membership values becomes independent on the change of the other degree of membership values:** It has been observed from (4) that if we change the degree of membership of any IMN α , then their corresponding impact on the degree of non-membership becomes independent. In other words, we can say that the aggregated degrees obtained by IMWA operator become independent of each other. This shortcoming has been illustrated with the following example.

Example 1 Let $\alpha_1 = \langle 1/3, 1/4 \rangle$, $\alpha_2 = \langle 2, 1/6 \rangle$, $\alpha_3 = \langle 4, 1/7 \rangle$ and $\alpha_4 = \langle 1/9, 5 \rangle$ be four IMNs. If we replace only the degree of membership function of these α_i with a new one, then we get new IMNs γ_i ($i = 1, 2, 3, 4$) as $\gamma_1 = \langle 1/2, 1/4 \rangle$, $\gamma_2 = \langle 4, 1/6 \rangle$, $\gamma_3 = \langle 5, 1/7 \rangle$ and $\gamma_4 = \langle 1/6, 5 \rangle$. Let $\omega = (0.1, 0.4, 0.2, 0.3)^T$ be the weight vector of IMNs α_i and γ_i . Then by utilizing (4) we get $\text{IMWA}(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \langle 1.1510, \mathbf{0.3566} \rangle$ and $\text{IMWA}(\gamma_1, \gamma_2, \gamma_3, \gamma_4) = \langle 1.7726, \mathbf{0.3566} \rangle$. Therefore, it is clearly seen that the degree of non-membership value remain same i.e., 0.3566 in both the cases. Thus, the impact of changing μ_i ($i = 1, 2, 3, 4$) is completely independent on the degree of non-membership of IMWA operator. Similarly, we can see the impact of changing ν_i ($i = 1, 2, 3, 4$) on the aggregated degree of membership of IMWA operators and found that they are independent of each other during the aggregation phase of the operation.

(ii) **There is no proper interaction between the pairs of the degrees of IMNs:** It has been analyzed from the existing operators that there is not a proper interaction between the degrees of membership functions while defining their operational laws. For instance,

Example 2 Let $A = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ be the collection of IMNs where $\alpha_1 = \langle 0.3333, 2.000 \rangle$, $\alpha_2 = \langle 0.1429,$

3.000), $\alpha_3 = \langle 4.000, 0.2000 \rangle$ and $\alpha_4 = \langle 6.000, 0.1429 \rangle$ be four IMNs and $\omega = (0.2, 0.3, 0.4, 0.1)^T$ is the standardized weight vector of these numbers. Then by using the IMWA operator, the aggregate IMN becomes $\langle 1.4491, 0.5283 \rangle$. On the other hand, if we take another collections of the IMNs $B = (\beta_1, \beta_2, \beta_3, \beta_4)$ where $\beta_1 = \langle 0.2500, 3.000 \rangle$, $\beta_2 = \langle 0.1667, 5.000 \rangle$, $\beta_3 = \langle 7.0000, 0.1111 \rangle$ and $\beta_4 = \langle 4.0626, 0.1542 \rangle$ corresponding to same weight set then we get $\langle 1.4491, 0.5283 \rangle$ as the aggregated IMN by IMWA operator. Therefore, it has been seen that the aggregated IMN is same and hence it is unable to distinguish between the ranking of these IMNs and hence gives an insufficient information about the preference of the alternatives.

Therefore, there is a need to pay more attention on it in order to rank the different IMNs. Furthermore, the uncertainties degree has not been considered during the formulation, so the indeterminacy information has not been completely extracted. Therefore, in order to handle it, an improved operational law has been proposed in the next section by sufficiently considering the indeterminacy information of IMNs.

3 Proposed improved aggregation operators

The above shortcoming has been overcome by defining some new operational laws on IMNs, which has been defined by considering the degree of the hesitation between the membership functions, as follows:

Definition 7 Let $\alpha_1 = \langle \mu_1, \nu_1 \rangle$, $\alpha_2 = \langle \mu_2, \nu_2 \rangle$ and $\alpha = \langle \mu, \nu \rangle$ be three IMNs and $\lambda > 0$ be a real number, then the new operations on these IMNs are defined as follows:

- (i) $\alpha_1 \oplus \alpha_2 = \left\langle \frac{(1+2\mu_1)(1+2\mu_2)-1}{2}, \frac{2\{1-(1-\mu_1\nu_1)(1-\mu_2\nu_2)\}}{(1+2\mu_1)(1+2\mu_2)-1} \right\rangle$
- 1. $\alpha_1 \otimes \alpha_2 = \left\langle \frac{2\{1-(1-\mu_1\nu_1)(1-\mu_2\nu_2)\}}{(1+2\nu_1)(1+2\nu_2)-1}, \frac{(1+2\nu_1)(1+2\nu_2)-1}{2} \right\rangle$
- (ii) $\lambda\alpha = \left\langle \frac{(1+2\mu)^\lambda - 1}{2}, \frac{2\{1-(1-\mu\nu)^\lambda\}}{(1+2\mu)^\lambda - 1} \right\rangle$
- (iii) $\alpha^\lambda = \left\langle \frac{2\{1-(1-\mu\nu)^\lambda\}}{(1+2\nu)^\lambda - 1}, \frac{(1+2\nu)^\lambda - 1}{2} \right\rangle$

From $\alpha_1 \oplus \alpha_2$, it has been obtained that the membership function of $\alpha_1 \oplus \alpha_2$ does not contain the pair of μ_1, ν_2

$$\begin{aligned} IFMIWA(\alpha_1, \alpha_2) &= \omega_1\alpha_1 \oplus \omega_2\alpha_2 \\ &= \left\langle \frac{(1+2\mu_1)^{\omega_1} - 1}{2}, \frac{2\{1-(1-\mu_1\nu_1)^{\omega_1}\}}{(1+2\mu_1)^{\omega_1} - 1} \right\rangle \oplus \left\langle \frac{(1+2\mu_2)^{\omega_2} - 1}{2}, \frac{2\{1-(1-\mu_2\nu_2)^{\omega_2}\}}{(1+2\mu_2)^{\omega_2} - 1} \right\rangle \\ &= \left\langle \frac{\prod_{i=1}^2 (1+2\mu_i)^{\omega_i} - 1}{2}, \frac{2\left\{1 - \prod_{i=1}^2 (1-\mu_i\nu_i)^{\omega_i}\right\}}{\prod_{i=1}^2 (1+2\mu_i)^{\omega_i} - 1} \right\rangle \end{aligned}$$

and ν_1, μ_2 while the non-membership function contains $\mu_1 \cdot \nu_2$ and $\nu_1 \cdot \mu_2$. Thus, the influence of membership function is greater than the influence on non-membership function, which means that that attitude of decision maker is optimistic. Similarly, the geometric meaning of new multiplication operator $\alpha_1 \otimes \alpha_2$ has been obtained and found that influence of non-membership function is greater than that of membership functions. This is to say, the attitude of decision maker is pessimistic.

Theorem 1 Consider three IMNs $\alpha = \langle \mu, \nu \rangle$, $\alpha_1 = \langle \mu_1, \nu_1 \rangle$ and $\alpha_2 = \langle \mu_2, \nu_2 \rangle$ and a real $\lambda > 0$ then $\alpha_1 \oplus \alpha_2, \lambda\alpha, \alpha_1 \otimes \alpha_2$ and α^λ are also IMNs.

Proof Proof is straight forward, so we omit here. □

Let Ω be a collection of an IMN $\alpha_i = \langle \mu_i, \nu_i \rangle$, $i = 1, 2, \dots, n$ then, we have defined an aggregation operator as follows.

3.1 Weighted averaging operator

Definition 8 If IFMIWA : $\Omega^n \rightarrow \Omega$ be an intuitionistic fuzzy multiplicative interactive weighted averaging operator of dimension n that has an associated weight vector $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$, such that $\omega_i > 0, \sum_{i=1}^n \omega_i = 1$, then

$$IFMIWA(\alpha_1, \alpha_2, \dots, \alpha_n) = \omega_1\alpha_1 \oplus \omega_2\alpha_2 \oplus \dots \oplus \omega_n\alpha_n \quad (5)$$

Theorem 2 For an IMN $\alpha_i = \langle \mu_i, \nu_i \rangle$, ($i = 1, 2, \dots, n$), the aggregated value by using IFMIWA operator is still IMN and becomes

$$IFMIWA(\alpha_1, \alpha_2, \dots, \alpha_n) = \left\langle \frac{\prod_{i=1}^n (1+2\mu_i)^{\omega_i} - 1}{2}, \frac{2\left\{1 - \prod_{i=1}^n (1-\mu_i\nu_i)^{\omega_i}\right\}}{\prod_{i=1}^n (1+2\mu_i)^{\omega_i} - 1} \right\rangle \quad (6)$$

Proof We will prove this theorem by induction on n .

When $n = 2$, we have

$$\omega_1\alpha_1 = \left\langle \frac{(1+2\mu_1)^{\omega_1} - 1}{2}, \frac{2\{1-(1-\mu_1\nu_1)^{\omega_1}\}}{(1+2\mu_1)^{\omega_1} - 1} \right\rangle \quad (7)$$

$$\omega_2\alpha_2 = \left\langle \frac{(1+2\mu_2)^{\omega_2} - 1}{2}, \frac{2\{1-(1-\mu_2\nu_2)^{\omega_2}\}}{(1+2\mu_2)^{\omega_2} - 1} \right\rangle \quad (8)$$

Thus,

Hence, (6) holds for $n = 2$. Assume (6) holds for $n = k$, then by using the operational laws as defined in Definition 7, we have

$$\begin{aligned} \text{IFMIWA}(\alpha_1, \alpha_2, \dots, \alpha_{k+1}) &= \bigoplus_{i=1}^{k+1} \omega_i \alpha_i \\ &= \text{IFMIWA}(\alpha_1, \alpha_2, \dots, \alpha_k) \oplus \omega_{k+1} \alpha_{k+1} \\ &= \left\langle \frac{\prod_{i=1}^k (1 + 2\mu_i)^{\omega_i} - 1}{2}, \frac{2 \left\{ 1 - \prod_{i=1}^k (1 - \mu_i \nu_i)^{\omega_i} \right\}}{\prod_{i=1}^k (1 + 2\mu_i)^{\omega_i} - 1} \right\rangle \\ &\quad \oplus \left\langle \frac{(1 + 2\mu_{k+1})^{\omega_{k+1}} - 1}{2}, \frac{2 \{ 1 - (1 - \mu_{k+1} \nu_{k+1})^{\omega_{k+1}} \}}{(1 + 2\mu_{k+1})^{\omega_{k+1}} - 1} \right\rangle \\ &= \left\langle \frac{\prod_{i=1}^{k+1} (1 + 2\mu_i)^{\omega_i} - 1}{2}, \frac{2 \left\{ 1 - \prod_{i=1}^{k+1} (1 - \mu_i \nu_i)^{\omega_i} \right\}}{\prod_{i=1}^{k+1} (1 + 2\mu_i)^{\omega_i} - 1} \right\rangle \end{aligned}$$

Hence, result holds for $n = k + 1$. Therefore, by principal of mathematical induction, (6) holds for all positive integer n .

Finally, in order to show that the aggregated number is an IMN. For this, assume that $\text{IFMIWA}(\alpha_1, \alpha_2, \dots, \alpha_n) = \langle \mu_M, \nu_M \rangle$, where $\mu_M = \frac{\prod_{i=1}^n (1+2\mu_i)^{\omega_i} - 1}{2}$ and $\nu_M = \frac{2 \left\{ 1 - \prod_{i=1}^n (1 - \mu_i \nu_i)^{\omega_i} \right\}}{\prod_{i=1}^n (1+2\mu_i)^{\omega_i} - 1}$. Then, it is sufficient to show that $\frac{1}{9} \leq \mu_M, \nu_M \leq 9$ and $\mu_M \nu_M \leq 1$.

Since for each i , $\alpha_i = \langle \mu_i, \nu_i \rangle$ be an IMN, thus, we have $\frac{1}{9} \leq \mu_i, \nu_i \leq 9$ and $\mu_i \nu_i \leq 1$. It can be easily prove that $\frac{1}{9} \leq \frac{\prod_{i=1}^n (1+2\mu_i)^{\omega_i} - 1}{2} \leq 9, \frac{1}{9} \leq \frac{2 \left\{ 1 - \prod_{i=1}^n (1 - \mu_i \nu_i)^{\omega_i} \right\}}{\prod_{i=1}^n (1+2\mu_i)^{\omega_i} - 1} \leq 9$, i.e., $\frac{1}{9} \leq \mu_M, \nu_M \leq 9$. Finally, we have

$$\begin{aligned} \mu_M \nu_M &= \left(\frac{\prod_{i=1}^n (1+2\mu_i)^{\omega_i} - 1}{2} \right) \left(\frac{2 \left\{ 1 - \prod_{i=1}^n (1 - \mu_i \nu_i)^{\omega_i} \right\}}{\prod_{i=1}^n (1+2\mu_i)^{\omega_i} - 1} \right) \\ &= 1 - \prod_{i=1}^n (1 - \mu_i \nu_i)^{\omega_i} \\ &\leq 1 \end{aligned}$$

Hence, the aggregated number is again an IMN. \square

Next, based on the proposed operator, it has been analyzed that it successfully overcomes the shortcoming of the existing operators as described in above section.

- (i) **There is a significant effect of the change of degree of membership values on to the degree of non-membership values:** From the proposed operator, it has also been analyzed that if degree of membership functions has been changed then their corresponding non-membership degree of aggregated IMN changes. This has been illustrated and tested on the previous Example 1, where the existing operator has failed to justify it. Now, by applying the proposed IFMIWA operator on IMNs α_i 's given in Example 1, we get aggregated IMN as $\langle 1.1510, 0.3841 \rangle$. On the other hand, if we aggregate modified IMNs by proposed operator then we get $\langle 1.7726, 0.4010 \rangle$. Thus, it has been seen that the degree of non-membership changes from 0.3841 to 0.4010 when membership values of α_i changes. Hence, the change of membership degree will affect on the degree of non-membership.

- (ii) **Pairs of the membership functions have a proper interaction between them:** From the definition of the improved operations law and their corresponding operator, it has been seen that the non-membership degree contains the pairs of the membership and non-membership degrees i.e., $\mu_i \cdot \nu_i$ and hence there is a proper interaction between the membership functions during the aggregation operators. This can be illustrated by applying the proposed operator on Example 2 and we get $\text{IFMIWA}(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \langle 1.4491, 0.4902 \rangle$ and $\text{IFMIWA}(\beta_1, \beta_2, \beta_3, \beta_4) = \langle 1.4492, 0.5283 \rangle$. Thus, by using the score function, we get their respective values are 2.9559 and 2.7423,

and concluded that collection of A is better than the collection of B .

Lemma 1 [31] Let $\alpha_i = \langle \mu_i, \nu_i \rangle$, $\omega_i > 0$ for $i = 1, 2, \dots, n$ and $\sum_{i=1}^n \omega_i = 1$, then

$$\prod_{i=1}^n \alpha_i^{\omega_i} \leq \sum_{i=1}^n \omega_i \alpha_i$$

with equality holds if and only if $\alpha_1 = \alpha_2 = \dots = \alpha_n$.

Corollary 1 For a collection of IMN α_i , the proposed and the existing operators i.e., the IFMIWA and IMWA, satisfies the following inequality:

$$IFMIWA(\alpha_1, \alpha_2, \dots, \alpha_n) \leq IMWA(\alpha_1, \alpha_2, \dots, \alpha_n)$$

Proof Let $IFMIWA(\alpha_1, \alpha_2, \dots, \alpha_n) = \langle \mu_\alpha^p, \nu_\alpha^p \rangle = \alpha^p$ and $IMWA(\alpha_1, \alpha_2, \dots, \alpha_n) = \langle \mu_\alpha, \nu_\alpha \rangle = \alpha$, then we have $\mu_\alpha^p = \mu_\alpha$ and

$$\nu_\alpha^p = \frac{2 \left\{ 1 - \prod_{i=1}^n (1 - \mu_i \nu_i)^{\omega_i} \right\}}{\prod_{i=1}^n (1 + 2\mu_i)^{\omega_i} - 1} \geq \frac{2 \prod_{i=1}^n \nu_i^{\omega_i}}{\prod_{i=1}^n (\nu_i + 2)^{\omega_i} - \prod_{i=1}^n \nu_i^{\omega_i}} = \nu_\alpha$$

where equality holds if and only if $\mu_1 = \mu_2 = \dots = \mu_n$ and $\nu_1 = \nu_2 = \dots = \nu_n$. Therefore, $S(\alpha^p) = \frac{\mu_\alpha^p}{\nu_\alpha^p} \leq \frac{\mu_\alpha}{\nu_\alpha} = S(\alpha)$. If $S(\alpha^p) < S(\alpha)$ then by Definition 2, for every ω , we have $IFMIWA(\alpha_1, \alpha_2, \dots, \alpha_n) < IMWA(\alpha_1, \alpha_2, \dots, \alpha_n)$. On the other hand, if $S(\alpha^p) = S(\alpha)$ i.e., $\frac{\mu_\alpha^p}{\nu_\alpha^p} = \frac{\mu_\alpha}{\nu_\alpha}$ then by the condition $\mu_\alpha^p = \mu_\alpha$, we have $\nu_\alpha^p = \nu_\alpha$. So, by the accuracy function, we have $H(\alpha^p) = \mu_\alpha^p \nu_\alpha^p = \mu_\alpha \nu_\alpha = H(\alpha)$ and hence $IFMIWA(\alpha_1, \alpha_2, \dots, \alpha_n) = IMWA(\alpha_1, \alpha_2, \dots, \alpha_n)$. Therefore, we get

$$IFMIWA(\alpha_1, \alpha_2, \dots, \alpha_n) \leq IMWA(\alpha_1, \alpha_2, \dots, \alpha_n)$$

where that equality holds if and only if $\alpha_1 = \alpha_2 = \dots = \alpha_n$. \square

Thus, it has been concluded from the Corollary 1 that the proposed IFMIWA operator shows the decision maker's more optimistic attitude than the existing IMWA operator proposed by [29] in the phase of the aggregation process. This has been illustrated through an example given as below.

Example 3 Let $\alpha_1 = \langle 1/2, 1/3 \rangle$, $\alpha_2 = \langle 3, 1/5 \rangle$, $\alpha_3 = \langle 5, 1/7 \rangle$ and $\alpha_4 = \langle 1/6, 4 \rangle$ be four IMNs and $\omega = (0.2, 0.3, 0.1, 0.4)^T$ be their weight vectors, then

$$\begin{aligned} IFMIWA(\alpha_1, \alpha_2, \alpha_3, \alpha_4) &= \left\langle \frac{\prod_{i=1}^4 (1 + 2\mu_i)^{\omega_i} - 1}{2}, \frac{2 \left\{ 1 - \prod_{i=1}^4 (1 - \mu_i \nu_i)^{\omega_i} \right\}}{\prod_{i=1}^4 (1 + 2\mu_i)^{\omega_i} - 1} \right\rangle \\ &= \left\langle \frac{2.9366 - 1}{2}, \frac{2 \times (1 - 0.5836)}{2.9366 - 1} \right\rangle \\ &= \langle 0.9683, 0.6027 \rangle \\ IMWA(\alpha_1, \alpha_2, \alpha_3, \alpha_4) &= \left\langle \frac{\prod_{i=1}^4 (1 + 2\mu_i)^{\omega_i} - 1}{2}, \frac{2 \prod_{i=1}^4 \nu_i^{\omega_i}}{\prod_{i=1}^n (2 + \nu_i)^{\omega_i} - \prod_{i=1}^n \nu_i^{\omega_i}} \right\rangle \\ &= \left\langle \frac{2.9366 - 1}{2}, \frac{2 \times 0.7099}{3.3165 - 0.7099} \right\rangle \\ &= \langle 0.9683, 0.5447 \rangle \end{aligned}$$

Thus, by definition of score function, we conclude that

$$IFMIWA(\alpha_1, \alpha_2, \alpha_3, \alpha_4) < IMWA(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$$

Now, we have presented some properties of the IFMIWA operator for a collection of IMNs $\alpha_i = \langle \mu_i, \nu_i \rangle$, ($i = 1, 2, \dots, n$).

Property 1 Idempotency: If $\alpha_i = \alpha_0 = \langle \mu_0, \nu_0 \rangle$ for all i , then

$$IFMIWA(\alpha_1, \alpha_2, \dots, \alpha_n) = \alpha_0$$

Proof Since $\alpha_i = \alpha_0 = \langle \mu_0, \nu_0 \rangle$, ($i = 1, 2, \dots, n$) and $\sum_{i=1}^n \omega_i = 1$, so by Theorem 2, we have

$$\begin{aligned} IFMIWA(\alpha_1, \alpha_2, \dots, \alpha_n) &= \left\langle \frac{\prod_{i=1}^n (1 + 2\mu_0)^{\omega_i} - 1}{2}, \frac{2 \left\{ 1 - \prod_{i=1}^n (1 - \mu_0 \nu_0)^{\omega_i} \right\}}{\prod_{i=1}^n (1 + 2\mu_0)^{\omega_i} - 1} \right\rangle \\ &= \left\langle \frac{(1 + 2\mu_0)^{\sum_{i=1}^n \omega_i} - 1}{2}, \frac{2 \left\{ 1 - (1 - \mu_0 \nu_0)^{\sum_{i=1}^n \omega_i} \right\}}{(1 + 2\mu_0)^{\sum_{i=1}^n \omega_i} - 1} \right\rangle \\ &= \langle \mu_0, \nu_0 \rangle \\ &= \alpha_0 \end{aligned}$$

\square

Property 2 Boundedness: Let $\alpha^- = \left\langle \min_i \{\mu_i\}, \frac{\max_i \{\mu_i\} \max_i \{\nu_i\}}{\min_i \{\mu_i\}} \right\rangle$ and $\alpha^+ = \left\langle \max_i \{\mu_i\}, \frac{\min_i \{\mu_i\} \min_i \{\nu_i\}}{\max_i \{\mu_i\}} \right\rangle$ then

$$\alpha^- \leq IFMIWA(\alpha_1, \alpha_2, \dots, \alpha_n) \leq \alpha^+$$

Proof Let $IFMIWA(A_1, A_2, \dots, A_n) = \alpha = \langle \mu_\alpha, \nu_\alpha \rangle$. As $\min_i \{\mu_i\} \leq \mu_i \leq \max_i \{\mu_i\}$ then for weight vector ω , we have $\frac{\prod_{i=1}^n (1+2\min_i \{\mu_i\})^{\omega_i} - 1}{2} \leq \frac{\prod_{i=1}^n (1+2\mu_i)^{\omega_i} - 1}{2} \leq \frac{\prod_{i=1}^n (1+2\max_i \{\mu_i\})^{\omega_i} - 1}{2}$ which implies that $\min_i \{\mu_i\} \leq \mu_\alpha \leq \max_i \{\mu_i\}$. Also, $\min_i \{\nu_i\} \leq \nu_i \leq \max_i \{\nu_i\}$ and $\min_i \{\mu_i\} \min_i \{\nu_i\} \leq \mu_i \nu_i \leq \max_i \{\mu_i\} \max_i \{\nu_i\}$ which implies that

$$\begin{aligned} &\Rightarrow 1 - \max_i \{\mu_i\} \max_i \{\nu_i\} \leq 1 - \mu_i \nu_i \leq 1 - \min_i \{\mu_i\} \min_i \{\nu_i\} \\ &\Rightarrow \left(1 - \max_i \{\mu_i\} \max_i \{\nu_i\}\right)^{\sum_i \omega_i} \leq \prod_{i=1}^n (1 - \mu_i \nu_i)^{\omega_i} \leq \left(1 - \min_i \{\mu_i\} \min_i \{\nu_i\}\right)^{\sum_i \omega_i} \\ &\Rightarrow \min_i \{\mu_i\} \min_i \{\nu_i\} \leq 1 - \prod_{i=1}^n (1 - \mu_i \nu_i)^{\omega_i} \leq \max_i \{\mu_i\} \max_i \{\nu_i\} \\ &\Rightarrow \frac{\min_i \{\mu_i\} \min_i \{\nu_i\}}{\max_i \{\mu_i\}} \leq \frac{2(1 - \prod_{i=1}^n (1 - \mu_i \nu_i)^{\omega_i})}{\prod_{i=1}^n (1 + 2\mu_i)^{\omega_i} - 1} \leq \frac{\max_i \{\mu_i\} \max_i \{\nu_i\}}{\min_i \{\mu_i\}} \\ \text{i.e., } &\frac{\min_i \{\mu_i\} \min_i \{\nu_i\}}{\max_i \{\mu_i\}} \leq \nu_\alpha \leq \frac{\max_i \{\mu_i\} \max_i \{\nu_i\}}{\min_i \{\mu_i\}} \end{aligned}$$

Take $\alpha^- = \left\langle \min_i \{\mu_i\}, \frac{\max_i \{\mu_i\} \max_i \{\nu_i\}}{\min_i \{\mu_i\}} \right\rangle$ and $\alpha^+ = \left\langle \max_i \{\mu_i\}, \frac{\min_i \{\mu_i\} \min_i \{\nu_i\}}{\max_i \{\mu_i\}} \right\rangle$ and thus, by definition of score function, we get $S(\alpha) \leq S(\alpha^+)$ and $S(\alpha) \geq S(\alpha^-)$ and hence by order relation between two IMNs, we have $\alpha^- \leq IFMIWA(\alpha_1, \alpha_2, \dots, \alpha_n) \leq \alpha^+$. \square

Property 3 Monotonicity: If α_i and β_i , be two collections of IMNs such that $\alpha_i \leq \beta_i$ for all i , then

$$IFMIWA(\alpha_1, \alpha_2, \dots, \alpha_n) \leq IFMIWA(\beta_1, \beta_2, \dots, \beta_n)$$

Proof Proof of this property is similar to above, so we omit here. \square

Property 4 Shift-invariance: If $\beta = \langle \mu_\beta, \nu_\beta \rangle$ be another IMN, then

$$\begin{aligned} &IFMIWA(\alpha_1 \oplus \beta, \alpha_2 \oplus \beta, \dots, \alpha_n \oplus \beta) \\ &= IFMIWA(\alpha_1, \alpha_2, \dots, \alpha_n) \oplus \beta \end{aligned}$$

Proof See the proof in [Appendix](#). \square

Property 5 Homogeneity: If $\beta > 0$ be a real number, then $IFMIWA(\beta\alpha_1, \beta\alpha_2, \dots, \beta\alpha_n) = \beta IFMIWA(\alpha_1, \alpha_2, \dots, \alpha_n)$

Proof See the proof [Appendix](#). \square

Property 6 If $\alpha_i = \langle \mu_{\alpha_i}, \nu_{\alpha_i} \rangle$ and $\beta = \langle \mu_{\beta_i}, \nu_{\beta_i} \rangle (i = 1, 2, \dots, n)$ be two collections of IMNs then

$$\begin{aligned} &IFMIWA(\alpha_1 \oplus \beta_1, \alpha_2 \oplus \beta_2, \dots, \alpha_n \oplus \beta_n) \\ &= IFMIWA(\alpha_1, \alpha_2, \dots, \alpha_n) \oplus IFMIWA(\beta_1, \beta_2, \dots, \beta_n) \end{aligned}$$

Proof See the proof in [Appendix](#). \square

3.2 Ordered weighted averaging operator

In this section, we intend to take the idea of OWA into IFMIWA operator and propose a new operator which is defined as follows:

Definition 9 If $IFMIOWA : \Omega^n \rightarrow \Omega$ be an intuitionistic fuzzy multiplicative interactive ordered weighted averaging operator that has an associated weight $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$ such that $\omega_i > 0$ and $\sum_{i=1}^n \omega_i = 1$, then

$$IFMIOWA(\alpha_1, \alpha_2, \dots, \alpha_n) = \omega_1 \alpha_{\delta(1)} \oplus \dots \oplus \omega_n \alpha_{\delta(n)}$$

where $(\delta(1), \delta(2), \dots, \delta(n))$ be the permutation of $(1, 2, \dots, n)$ such that $\alpha_{\delta(i-1)} \geq \alpha_{\delta(i)}$ for $i = 2, 3, \dots, n$ and $\alpha_{\delta(i)}$ is the i^{th} largest of IMN α_i .

Theorem 3 Let $\alpha_i = \langle \mu_i, \nu_i \rangle, (i = 1, 2, \dots, n)$ be the collection of IMNs, then based on IFMIOWA operator, the aggregated IMN can be expressed as

$$IFMIOWA(\alpha_1, \alpha_2, \dots, \alpha_n) = \left\langle \frac{\prod_{i=1}^n (1+2\mu_{\delta(i)})^{\omega_i} - 1}{2}, \frac{2 \left\{ 1 - \frac{\prod_{i=1}^n (1-\mu_{\delta(i)} \nu_{\delta(i)})^{\omega_i} \right\}}{\prod_{i=1}^n (1+2\mu_{\delta(i)})^{\omega_i} - 1} \right\rangle \tag{9}$$

Proof The proof of this theorem is similar to that of Theorem 2 and hence it is omitted here. \square

Corollary 2 The IFMIOWA operator and IFMOWA operator have the following relation for a collection of IMNs $\alpha_i (i = 1, 2, \dots, n)$

$$IFMIOWA(\alpha_1, \alpha_2, \dots, \alpha_n) \leq IFMOWA(\alpha_1, \alpha_2, \dots, \alpha_n)$$

Proof Proof is similar to that of Corollary 1 and hence it is omitted here. \square

Furthermore, it has been observed that the IFMIOWA operator also satisfies the idempotent, boundedness, monotonicity, shift-invariance and homogeneity property.

Example 4 Let $\alpha_1 = \langle 1/3, 1/4 \rangle$, $\alpha_2 = \langle 2, 1/6 \rangle$, $\alpha_3 = \langle 4, 1/7 \rangle$ and $\alpha_4 = \langle 1/9, 5 \rangle$ be four IMNs and the IFMIOWA operator has an associated vector $\omega = (0.1, 0.4, 0.2, 0.3)^T$. Since $S(\alpha_3) > S(\alpha_2) > S(\alpha_1) > S(\alpha_4)$ and hence $\alpha_{\delta(1)} = \langle 4, 1/7 \rangle$, $\alpha_{\delta(2)} = \langle 2, 1/6 \rangle$, $\alpha_{\delta(3)} = \langle 1/3, 1/4 \rangle$ and $\alpha_{\delta(4)} = \langle 1/9, 5 \rangle$. Then, $\prod_{i=1}^4 (1 + 2\mu_{\delta(i)})^{\omega_i} = (1 + 2 \times 4)^{0.1} \times (1 + 2 \times 2)^{0.4} \times (1 + 2 \times 1/3)^{0.2} \times (1 + 2 \times 1/3)^{0.3} = 2.7895$, $\prod_{i=1}^4 (v_{\delta(i)})^{\omega_i} = (1/7)^{0.1} \times (1/6)^{0.4} \times (1/4)^{0.2} \times (5)^{0.3} = 0.4938$, $\prod_{i=1}^4 (2 + v_{\delta(i)})^{\omega_i} = (2 + 1/7)^{0.1} \times (2 + 1/6)^{0.4} \times (2 + 1/4)^{0.2} \times (2 + 5)^{0.3} = 3.1001$ and $\prod_{i=1}^4 (1 - \mu_{\delta(i)}v_{\delta(i)})^{\omega_i} = (1 - 4/7)^{0.1} \times (1 - 2/6)^{0.4} \times (1 - 1/12)^{0.2} \times (1 - 5/9)^{0.3} = 0.6019$. Therefore, $IFMIOWA(\alpha_1, \dots, \alpha_4) = \langle 0.8948, 0.4449 \rangle$ and $IMOWA(\alpha_1, \dots, \alpha_4) = \langle 0.8948, 0.3789 \rangle$ and hence it has been obtained that $IFMIOWA(\alpha_1, \dots, \alpha_4) < IMOWA(\alpha_1, \dots, \alpha_4)$.

3.3 Hybrid weighted averaging operator

In this section, by combining the advantage of both IFMIWA and IFMIOWA operators, we have defined a new hybrid averaging operator under the IMS environment as follows.

Definition 10 For a collection of IMNs, $\alpha_i = \langle \mu_i, v_i \rangle$, ($i = 1, 2, \dots, n$), if IFMIHWA : $\Omega^n \rightarrow \Omega$ be defined as

$$IFMIHWA(\alpha_1, \alpha_2, \dots, \alpha_n) = \omega_1 \dot{\alpha}_{\delta(1)} \oplus \dots \oplus \omega_n \dot{\alpha}_{\delta(n)}$$

where $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$ is the associated standardized weight vector of IFMIHWA operator satisfying $\omega_i > 0$ and $\sum_{i=1}^n \omega_i = 1$, $\dot{\alpha}_{\delta(i)}$ is the i^{th} largest of the weighted IMNs $\dot{\alpha}_i$ ($\dot{\alpha}_i = n w_i \alpha_i$, $i = 1, 2, \dots, n$), n is the number of IMNs and $w = (w_1, w_2, \dots, w_n)^T$ is the standard weight vector of α_i ($i = 1, 2, \dots, n$), then IFMIHWA is called as an intuitionistic fuzzy multiplicative interactive hybrid weighted averaging operator.

From the Definition 10, it has been concluded that

- (i) It firstly weights the IMNs α_i by the associated weights w_i and hence get the weighted IMNs $\dot{\alpha}_i = n w_i \alpha_i$ ($i = 1, 2, \dots, n$).

- (ii) It reorders the weighted arguments in descending order $(\dot{\alpha}_{\delta(1)}, \dot{\alpha}_{\delta(2)}, \dots, \dot{\alpha}_{\delta(n)})$, where $\dot{\alpha}_{\delta(i)}$ is the i^{th} largest of $\dot{\alpha}_i$ ($i = 1, 2, \dots, n$).
- (iii) It weights these ordered weighted IMNs $\dot{\alpha}_{\delta(i)}$ by the IFMIWA weights ω_i ($i = 1, 2, \dots, n$) and then aggregates all these values into a collective one.

Based on the proposed improved operational rules of the IMNs, we can derive the result shown in Theorem 4.

Theorem 4 For a collection of IMN $\alpha_i = \langle \mu_i, v_i \rangle$, ($i = 1, 2, \dots, n$), the aggregated value by IFMIHWA operator can be expressed as

$$IFMIHWA(\alpha_1, \alpha_2, \dots, \alpha_n) = \left\langle \frac{\prod_{i=1}^n (1 + 2\mu_{\delta(i)})^{\omega_i} - 1}{2}, \frac{2 \left\{ 1 - \prod_{i=1}^n (1 - \mu_{\delta(i)} v_{\delta(i)})^{\omega_i} \right\}}{\prod_{i=1}^n (1 + 2\mu_{\delta(i)})^{\omega_i} - 1} \right\rangle$$

Proof The proof is similar to Theorem 2, so it is omitted here. □

It can be easily proved that the IFMIHWA operator is also bounded, idempotent and monotonic.

Example 5 Let $\alpha_1 = \langle 1/3, 1/4 \rangle$, $\alpha_2 = \langle 2, 1/6 \rangle$, $\alpha_3 = \langle 4, 1/7 \rangle$ and $\alpha_4 = \langle 1/9, 5 \rangle$ be four IMNs and $w = (0.12, 0.27, 0.24, 0.31)^T$ be the standardized weight vector of the four IMNs, and $\omega = (0.1, 0.4, 0.2, 0.3)^T$ is the associated weighted vector of the IFMIHWA operator. Then, $\dot{\alpha}_i = n w_i \alpha_i$, ($i = 1, 2, 3, 4$) becomes $\dot{\alpha}_1 = \langle 0.1389, 0.2944 \rangle$, $\dot{\alpha}_2 = \langle 2.3435, 0.1513 \rangle$, $\dot{\alpha}_3 = \langle 3.6214, 0.1537 \rangle$ and $\dot{\alpha}_4 = \langle 0.1729, 4.0419 \rangle$. Thus $S(\dot{\alpha}_3) > S(\dot{\alpha}_2) > S(\dot{\alpha}_1) > S(\dot{\alpha}_4)$. Hence, $\dot{\alpha}_{\delta(1)} = \alpha_3$; $\dot{\alpha}_{\delta(2)} = \alpha_2$; $\dot{\alpha}_{\delta(3)} = \alpha_1$ and $\dot{\alpha}_{\delta(4)} = \alpha_4$. Therefore, based on these IMNs and by utilizing the IFMIHWA operator corresponding to weight vector ω , we get aggregated IMN as $\langle 0.9208, 0.5047 \rangle$. On the other hand, if we aggregate these IMNs by IMHWA operators [29] then the aggregated IMN becomes $\langle 0.9208, 0.4318 \rangle$.

Now, next, we have defined the generalized weighted, ordered and hybrid aggregation operators by using the proposed operational laws for a collection of an IMN $\alpha_i = \langle \mu_i, v_i \rangle$, $i = 1, 2, \dots, n$ and named as generalized intuitionistic fuzzy multiplicative interactive weighted averaging (GIFMIWA), generalized intuitionistic fuzzy multiplicative interactive ordered weighted averaging (GIFMIOWA) and generalized intuitionistic fuzzy multiplicative interactive hybrid weighted averaging (GIFMIHWA) operators.

Definition 11 Let $\alpha_i = \langle \mu_i, \nu_i \rangle$, ($i = 1, 2, \dots, n$) be a collection of IMNs. A GIFMIWA operator of dimension n is a mapping GIFMIWA : $\Omega^n \rightarrow \Omega$, that has an associated

weight vector $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$, such that $\omega_i > 0$, $\sum_{i=1}^n \omega_i = 1$. Furthermore,

$$\begin{aligned} \text{GIFMIWA}(\alpha_1, \alpha_2, \dots, \alpha_n) &= (\omega_1 \alpha_1^\lambda \oplus \omega_2 \alpha_2^\lambda \oplus \dots \oplus \omega_n \alpha_n^\lambda)^{1/\lambda} \\ &= \left\langle \frac{2 \left\{ \prod_{i=1}^n \{3 + (1 + 2\nu_i)^\lambda\}^{\omega_i} - \prod_{i=1}^n \{(1 + 2\nu_i)^\lambda - 1\}^{\omega_i}\right\}^{1/\lambda} \left[1 - \prod_{i=1}^n (1 - \mu_i \nu_i)^{\omega_i} \right]}{\left[\begin{aligned} &\left\{ \prod_{i=1}^n \{3 + (1 + 2\nu_i)^\lambda - 4(1 - \mu_i \nu_i)^\lambda\}^{\omega_i} - 4 \prod_{i=1}^n \{(1 + 2\nu_i)^\lambda - 1\}^{\omega_i} \prod_{i=1}^n (1 - \mu_i \nu_i)^{\lambda \omega_i} \right. \\ &\left. + 3 \prod_{i=1}^n \{(1 + 2\nu_i)^\lambda - 1\}^{\omega_i} \right\}^{1/\lambda} - \left\{ \prod_{i=1}^n \{3 + (1 + 2\nu_i)^\lambda\}^{\omega_i} - \prod_{i=1}^n \{(1 + 2\nu_i)^\lambda - 1\}^{\omega_i} \right\}^{1/\lambda} \end{aligned} \right]}{\left[\begin{aligned} &\left\{ \prod_{i=1}^n \{3 + (1 + 2\nu_i)^\lambda - 4(1 - \mu_i \nu_i)^\lambda\}^{\omega_i} - 4 \prod_{i=1}^n \{(1 + 2\nu_i)^\lambda - 1\}^{\omega_i} \prod_{i=1}^n (1 - \mu_i \nu_i)^{\lambda \omega_i} \right. \\ &\left. + 3 \prod_{i=1}^n \{(1 + 2\nu_i)^\lambda - 1\}^{\omega_i} \right\}^{1/\lambda} - \left\{ \prod_{i=1}^n \{3 + (1 + 2\nu_i)^\lambda\}^{\omega_i} - \prod_{i=1}^n \{(1 + 2\nu_i)^\lambda - 1\}^{\omega_i} \right\}^{1/\lambda} \end{aligned} \right]} \right\rangle \\ &\quad \left. \frac{2 \left\{ \prod_{i=1}^n \{3 + (1 + 2\nu_i)^\lambda\}^{\omega_i} - \prod_{i=1}^n \{(1 + 2\nu_i)^\lambda - 1\}^{\omega_i} \right\}^{1/\lambda}}{\dots} \right\rangle \end{aligned} \tag{10}$$

where λ is a real number greater than zero.

multiplicative averaging (GIFWMA) operator,

Especially,

$$\text{GIFWMA}(\alpha_1, \alpha_2, \dots, \alpha_n) = \left(\frac{1}{n} \bigoplus_{i=1}^n \alpha_i^\lambda \right)^{1/\lambda}$$

- (i) If $\lambda = 1$ then the GIFMIWA reduces to IFMIWA operator;
- (ii) If $\omega = (1/n, 1/n, \dots, 1/n)^T$ then GIFMIWA reduces to the generalized intuitionistic fuzzy weighted

Definition 12 For a collection of IMN α_i , $i = 1, 2, \dots, n$, a GIFMIOWA operator of dimension n is mapping GIFMIOWA : $\Omega^n \rightarrow \Omega$ and is defined as follows:

$$\begin{aligned} \text{GIFMIOWA}(\alpha_1, \alpha_2, \dots, \alpha_n) &= (\omega_1 \alpha_{\delta(1)}^\lambda \oplus \omega_2 \alpha_{\delta(2)}^\lambda \oplus \dots \oplus \omega_n \alpha_{\delta(n)}^\lambda)^{1/\lambda} \\ &= \left\langle \frac{2 \left\{ \prod_{i=1}^n \{3 + (1 + 2\nu_{\delta(i)})^\lambda\}^{\omega_i} - \prod_{i=1}^n \{(1 + 2\nu_{\delta(i)})^\lambda - 1\}^{\omega_i}\right\}^{1/\lambda} \left[1 - \prod_{i=1}^n (1 - \mu_{\delta(i)} \nu_{\delta(i)})^{\omega_i} \right]}{\left[\begin{aligned} &\left\{ \prod_{i=1}^n \{2 + (1 + 2\nu_{\delta(i)})^\lambda - 4(1 - \mu_{\delta(i)} \nu_{\delta(i)})^\lambda\}^{\omega_i} - 4 \prod_{i=1}^n \{(1 + 2\nu_{\delta(i)})^\lambda - 1\}^{\omega_i} \prod_{i=1}^n (1 - \mu_{\delta(i)} \nu_{\delta(i)})^{\lambda \omega_i} \right. \\ &\left. + 3 \prod_{i=1}^n \{(1 + 2\nu_{\delta(i)})^\lambda - 1\}^{\omega_i} \right\}^{1/\lambda} - \left\{ \prod_{i=1}^n \{3 + (1 + 2\nu_{\delta(i)})^\lambda\}^{\omega_i} - \prod_{i=1}^n \{(1 + 2\nu_{\delta(i)})^\lambda - 1\}^{\omega_i} \right\}^{1/\lambda} \end{aligned} \right]}{\left[\begin{aligned} &\left\{ \prod_{i=1}^n \{3 + (1 + 2\nu_{\delta(i)})^\lambda - 4(1 - \mu_{\delta(i)} \nu_{\delta(i)})^\lambda\}^{\omega_i} - 4 \prod_{i=1}^n \{(1 + 2\nu_{\delta(i)})^\lambda - 1\}^{\omega_i} \prod_{i=1}^n (1 - \mu_{\delta(i)} \nu_{\delta(i)})^{\lambda \omega_i} \right. \\ &\left. + 3 \prod_{i=1}^n \{(1 + 2\nu_{\delta(i)})^\lambda - 1\}^{\omega_i} \right\}^{1/\lambda} - \left\{ \prod_{i=1}^n \{3 + (1 + 2\nu_{\delta(i)})^\lambda\}^{\omega_i} - \prod_{i=1}^n \{(1 + 2\nu_{\delta(i)})^\lambda - 1\}^{\omega_i} \right\}^{1/\lambda} \end{aligned} \right]} \right\rangle \\ &\quad \left. \frac{2 \left\{ \prod_{i=1}^n \{3 + (1 + 2\nu_{\delta(i)})^\lambda\}^{\omega_i} - \prod_{i=1}^n \{(1 + 2\nu_{\delta(i)})^\lambda - 1\}^{\omega_i} \right\}^{1/\lambda}}{\dots} \right\rangle \end{aligned}$$

where λ is the real number greater than zero, $\delta : (1, 2, \dots, n) \rightarrow (1, 2, \dots, n)$ is the permutation mapping

and $\alpha_{\delta(i)}$ is the i^{th} largest of IMN α_i . When the parameter $\lambda = 1$, then it reduces to IFMIOWA operator.

Definition 13 For a collection of IMN $\alpha_i, i = 1, 2, \dots, n$, a GIFMIHWA operator of dimension n is mapping GIFMIHWA : $\Omega^n \rightarrow \Omega$ and is defined as follows:

$$\begin{aligned} \text{GIFMIHWA}(\alpha_1, \alpha_2, \dots, \alpha_n) &= \left(\omega_1 \dot{\alpha}_{\delta(1)}^\lambda \oplus \omega_2 \dot{\alpha}_{\delta(2)}^\lambda \oplus \dots \oplus \omega_n \dot{\alpha}_{\delta(n)}^\lambda \right)^{1/\lambda} \\ &= \left\langle \frac{2 \left\{ \prod_{i=1}^n \{3 + (1 + 2\dot{v}_{\delta(i)})^\lambda\}^{\omega_i} - \prod_{i=1}^n \{(1 + 2\dot{v}_{\delta(i)})^\lambda - 1\}^{\omega_i}\right\}^{1/\lambda} \left[1 - \prod_{i=1}^n (1 - \dot{\mu}_{\delta(i)} \dot{v}_{\delta(i)})^{\omega_i} \right]}{\left[\begin{aligned} &\left\{ \prod_{i=1}^n \{2 + (1 + 2\dot{v}_{\delta(i)})^\lambda - 4(1 - \dot{\mu}_{\delta(i)} \dot{v}_{\delta(i)})^\lambda\}^{\omega_i} - 4 \prod_{i=1}^n \{(1 + 2\dot{v}_{\delta(i)})^\lambda - 1\}^{\omega_i} \prod_{i=1}^n (1 - \dot{\mu}_{\delta(i)} \dot{v}_{\delta(i)})^{\lambda \omega_i} \right\}^{1/\lambda} \\ &+ 3 \prod_{i=1}^n \{(1 + 2\dot{v}_{\delta(i)})^\lambda - 1\}^{\omega_i} \left\}^{1/\lambda} - \left\{ \prod_{i=1}^n \{3 + (1 + 2\dot{v}_{\delta(i)})^\lambda\}^{\omega_i} - \prod_{i=1}^n \{(1 + 2\dot{v}_{\delta(i)})^\lambda - 1\}^{\omega_i} \right\}^{1/\lambda} \right]} \right. \\ &\left. \frac{\left[\begin{aligned} &\left\{ \prod_{i=1}^n \{2 + (1 + 2\dot{v}_{\delta(i)})^\lambda - 4(1 - \dot{\mu}_{\delta(i)} \dot{v}_{\delta(i)})^\lambda\}^{\omega_i} - 4 \prod_{i=1}^n \{(1 + 2\dot{v}_{\delta(i)})^\lambda - 1\}^{\omega_i} \prod_{i=1}^n (1 - \dot{\mu}_{\delta(i)} \dot{v}_{\delta(i)})^{\lambda \omega_i} \right\}^{1/\lambda} \\ &+ 3 \prod_{i=1}^n \{(1 + 2\dot{v}_{\delta(i)})^\lambda - 1\}^{\omega_i} \left\}^{1/\lambda} - \left\{ \prod_{i=1}^n \{3 + (1 + 2\dot{v}_{\delta(i)})^\lambda\}^{\omega_i} - \prod_{i=1}^n \{(1 + 2\dot{v}_{\delta(i)})^\lambda - 1\}^{\omega_i} \right\}^{1/\lambda} \right]}{2 \left\{ \prod_{i=1}^n \{3 + (1 + 2\dot{v}_{\delta(i)})^\lambda\}^{\omega_i} - \prod_{i=1}^n \{(1 + 2\dot{v}_{\delta(i)})^\lambda - 1\}^{\omega_i}\right\}^{1/\lambda}} \right] \right\rangle \end{aligned}$$

where $\dot{\alpha}_{\delta(i)}$ is the i^{th} largest of the weighted IMNs $\dot{\alpha}_i$ ($\dot{\alpha}_i = n w_i \alpha_i, i = 1, 2, \dots, n$), n is the number of IMNs and $w = (w_1, w_2, \dots, w_n)^T$ is the standard weight vector of α_i .

4 Group decision making approach based on the proposed operators with intuitionistic multiplicative preference relations

4.1 Proposed approach

Assume that a set of alternatives $X = (X_1, X_2, \dots, X_n)$ which are evaluated by the group of decision makers $D = (D^{(1)}, D^{(2)}, \dots, D^{(q)})$ who will receive the full responsibility for the whole process. These experts have evaluated each of the alternative and give their preferences in terms of IMNs $\alpha_{ij}^{(k)}$ and their corresponding decision matrix is denoted by $D^{(k)} = (\alpha_{ij}^{(k)})_{n \times n}$ where $\alpha_{ij}^{(k)} = \langle \mu_{ij}^{(k)}, \nu_{ij}^{(k)} \rangle$ represents the priority value of alternative X_i given by decision maker $D^{(k)}$ ($k = 1, 2, \dots, q$) such that $\frac{1}{9} \leq \mu_{ij}^{(k)}, \nu_{ij}^{(k)} \leq 9$ and $\mu_{ij}^{(k)} \nu_{ij}^{(k)} \leq 1$. Let $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$ is the corresponding weighting vector, satisfying $w_i > 0, \sum_{i=1}^n w_i = 1$. Then, in order to choose the best alternative, we forwarded the following steps of the proposed approach.

(Step 1:) Use of the GIFMIWA or GIFMIOWA or GIFMIHWA operators to aggregate the $(\alpha_{1j}^{(k)}, \alpha_{2j}^{(k)}, \dots, \alpha_{nj}^{(k)})$ and obtain the $\alpha_i^{(k)}$ ($k = 1, 2, \dots, q; i = 1, 2, \dots, n$).

- (Step 2:) Based on the IFMIWA or IFMIOWA or IFMIHWA operator to aggregate the $(\alpha_1^{(k)}, \alpha_2^{(k)}, \dots, \alpha_n^{(k)})$ to IMNs α_i which is the overall aggregated value of the alternative X_i ($i = 1, 2, \dots, n$).
- (Step 3:) Compute the score value of aggregated IMN α_i , denoted by $S(\alpha_i)$, in accordance with the score function of the IMN.
- (Step 4:) Rank the alternatives based on the values of $S(\alpha_i)$ and hence select the best alternative(s).
- (Step 5:) Do the sensitivity analysis on the parameter λ according to decision maker’s preference.

4.2 Numerical example

The above mentioned approach has been illustrated with a practical example of the DM which can be read as:

The Kedarnath valley, along with and other parts of the state of Uttarakhand in the north of India, was hit with unprecedented flash floods in July 2013. A large number of roads, which connect to the Kedarnath valley with other parts of Uttarakhand had been destroyed in this flood. In this context, Uttarakhand government has to take a considerable number of road building projects either to preserve the roads already built or to undertake the new roads. These projects have been carried out by a limited number of the well-established contractors, and the selection process has been on the basis of bid price alone. In recent years, increased project complexity, technical capability, higher performance, and safety and financial requirements

have been demanding the use of multi-attribute decision-making methods. For this, Uttarakhand government had been issued the notice in the newspapers, and the four officers ($D^{(1)}, D^{(2)}, D^{(3)}, D^{(4)}$) which takes the responsibility to select the best contractor out of the four possible alternatives, namely, Jaihind Road Builders Pvt. Ltd. (X_1), J.K. Construction (X_2), Buildquick Infrastructure Pvt. Ltd. (X_3), Relcon Intraprojects Ltd. (X_4) bid for these projects.

Then the objective of the Government is to choose the best contractor among them for the task. In order to fulfill it, let $\omega = (0.3, 0.2, 0.4, 0.1)^T$ be the weight vector corresponding to the four officers such that they have evaluated each candidate and gave their preferences in terms of IMNs and hence constructed the following four decision matrices $D^{(k)} = (\alpha_{ij}^{(k)})_{4 \times 4}$, ($k = 1, 2, 3, 4$) as shown below:

$$D^{(1)} = \begin{pmatrix} \langle 1, 1 \rangle & \langle 1/4, 3 \rangle & \langle 1/3, 1 \rangle & \langle 1/5, 1 \rangle \\ \langle 3, 1/4 \rangle & \langle 1, 1 \rangle & \langle 1/7, 2 \rangle & \langle 2/3, 1/3 \rangle \\ \langle 1, 1/3 \rangle & \langle 2, 1/7 \rangle & \langle 1, 1 \rangle & \langle 1/4, 1/7 \rangle \\ \langle 1, 1/5 \rangle & \langle 1/3, 2/3 \rangle & \langle 1/7, 1/4 \rangle & \langle 1, 1 \rangle \end{pmatrix}, D^{(2)} = \begin{pmatrix} \langle 1, 1 \rangle & \langle 1/7, 2 \rangle & \langle 2/3, 1/3 \rangle & \langle 1/4, 1 \rangle \\ \langle 2, 1/7 \rangle & \langle 1, 1 \rangle & \langle 1/8, 1 \rangle & \langle 2/5, 2 \rangle \\ \langle 1/3, 2/3 \rangle & \langle 1, 1/8 \rangle & \langle 1, 1 \rangle & \langle 1, 2/3 \rangle \\ \langle 1, 1/4 \rangle & \langle 2, 2/5 \rangle & \langle 2/3, 1 \rangle & \langle 1, 1 \rangle \end{pmatrix}$$

$$D^{(3)} = \begin{pmatrix} \langle 1, 1 \rangle & \langle 3, 1/5 \rangle & \langle 1/3, 1 \rangle & \langle 1/5, 2 \rangle \\ \langle 1/5, 3 \rangle & \langle 1, 1 \rangle & \langle 1/4, 3 \rangle & \langle 2/5, 1/2 \rangle \\ \langle 1, 1/3 \rangle & \langle 3, 1/4 \rangle & \langle 1, 1 \rangle & \langle 2/7, 3 \rangle \\ \langle 2, 1/5 \rangle & \langle 1/2, 2/5 \rangle & \langle 3, 2/7 \rangle & \langle 1, 1 \rangle \end{pmatrix}, D^{(4)} = \begin{pmatrix} \langle 1, 1 \rangle & \langle 1/4, 3 \rangle & \langle 3, 1/5 \rangle & \langle 1/4, 1 \rangle \\ \langle 3, 1/4 \rangle & \langle 1, 1 \rangle & \langle 1/4, 1/3 \rangle & \langle 2/5, 1/3 \rangle \\ \langle 1/5, 3 \rangle & \langle 1/3, 1/4 \rangle & \langle 1, 1 \rangle & \langle 3/5, 2/3 \rangle \\ \langle 1, 1/4 \rangle & \langle 1/3, 2/5 \rangle & \langle 2/3, 3/5 \rangle & \langle 1, 1 \rangle \end{pmatrix}$$

Here, in the first decision matrix, $D^{(1)}$, for example, the first preference is $\langle 1, 1 \rangle$ implies that when the first contractor X_1 compares with himself then the preference is $\langle 1, 1 \rangle$. On the other hand, the IMN $\langle 1/4, 3 \rangle$ indicates that the first officer argued that the degree of first contractor is priority to the second contractor is $1/4$ while at the same time, he thinks the degree of first contractor is not a priority to the second contractor is 3. Similarly, the other observations have their meaning. Now, for the sake of simplicity, we take $\lambda = 0.6$, the following steps are being executed for aggregating these

different preferences by using GIFMIWA and GIFMIHWA operators as follows:

4.2.1 By GIFMIWA operator

(Step 1:) To make use of GIFMIWA operator to aggregate $(\alpha_{1j}^{(k)}, \dots, \alpha_{4j}^{(k)})$ and hence obtain the aggregated IMNs $\alpha_i^{(k)}$, ($i, k = 1, 2, 3, 4$) and represented as a decision-matrix M as

$$M = \begin{pmatrix} \langle 0.2766, 3.6152 \rangle & \langle 0.5003, 1.9988 \rangle & \langle 0.6782, 1.4745 \rangle & \langle 0.1723, 5.8053 \rangle \\ \langle 0.3128, 3.1966 \rangle & \langle 0.3465, 2.8862 \rangle & \langle 0.4774, 2.0948 \rangle & \langle 0.6146, 1.6270 \rangle \\ \langle 0.5001, 1.9996 \rangle & \langle 0.2231, 4.4828 \rangle & \langle 0.8506, 1.1756 \rangle & \langle 1.1503, 0.8694 \rangle \\ \langle 0.8405, 1.1898 \rangle & \langle 0.5024, 1.9904 \rangle & \langle 0.3405, 2.9366 \rangle & \langle 0.3418, 2.9260 \rangle \end{pmatrix}$$

(Step 2:) Use the IFMIWA operator to aggregate the decision-making matrix M by using weight vector ω and get IMNs α_i as

$$\alpha_1 = \langle 0.4103, 2.4373 \rangle, \quad \alpha_2 = \langle 0.3445, 2.9026 \rangle,$$

$$\alpha_3 = \langle 0.6547, 1.5274 \rangle, \quad \alpha_4 = \langle 0.5611, 1.7824 \rangle$$

(Step 3:) The score values corresponding to these α_i are computed as

$$S(\alpha_1) = 0.1683, \quad S(\alpha_2) = 0.1187, \quad S(\alpha_3) = 0.4286, \quad S(\alpha_4) = 0.3148$$

(Step 4:) Therefore, based on these score values, it has been concluded that $S(\alpha_3) > S(\alpha_4) > S(\alpha_2) > S(\alpha_1)$. Thus, the ordering of the four contractors is $X_3 > X_4 > X_1 > X_2$ and hence the best contractor is X_3 .

4.2.2 By GIFMIHWA operator

(Step 1:) To make use of GIFMIHWA operator to aggregate $(\alpha_{1j}^{(k)}, \dots, \alpha_{4j}^{(k)})$ and hence obtain the aggregated IMNs $\alpha_i^{(k)}$ and their corresponding results are summarized as

$$M = \begin{pmatrix} \langle 0.2299, 4.3496 \rangle & \langle 0.6752, 1.4810 \rangle & \langle 0.6491, 1.5405 \rangle & \langle 0.1755, 5.6981 \rangle \\ \langle 0.2689, 3.7188 \rangle & \langle 0.3371, 2.9664 \rangle & \langle 0.4721, 2.1181 \rangle & \langle 0.7139, 1.4008 \rangle \\ \langle 0.5462, 1.8309 \rangle & \langle 0.3050, 3.2782 \rangle & \langle 0.9036, 1.1066 \rangle & \langle 1.1784, 0.8486 \rangle \\ \langle 0.8117, 1.2320 \rangle & \langle 0.3906, 2.5601 \rangle & \langle 0.3652, 2.7380 \rangle & \langle 0.2964, 3.3735 \rangle \end{pmatrix}$$

(Step 2:) Use the IFMIHWA operator to aggregate the decision matrix M and hence get the overall aggregated value α_i as

$$\alpha_1 = \langle 0.4422, 2.2612 \rangle, \quad \alpha_2 = \langle 0.4088, 2.4464 \rangle, \\ \alpha_3 = \langle 0.7088, 1.4108 \rangle, \quad \alpha_4 = \langle 0.6314, 1.5839 \rangle$$

(Step 3:) The score values corresponding to these $\alpha_i (i = 1, 2, 3, 4)$ are computed as

$$S(\alpha_1) = 0.1956, \quad S(\alpha_2) = 0.1671, \quad S(\alpha_3) \\ = 0.5024, \quad S(\alpha_4) = 0.3986$$

(Step 4:) Thus, the ordering of the four contractors is $X_3 \succ X_4 \succ X_1 \succ X_2$ and therefore, the best contractor for the job is X_3 .

On the other hand, if we apply IMHWA operator to aggregated the IMNs then we get the overall IMN as

$$\alpha_1 = \langle 0.4680, 1.2274 \rangle, \quad \alpha_2 = \langle 0.5491, 0.7389 \rangle, \\ \alpha_3 = \langle 0.6345, 0.6633 \rangle, \quad \alpha_4 = \langle 0.7309, 0.5885 \rangle$$

and hence by using score function of these numbers, we get the ordering of these four contractors as $X_4 \succ X_3 \succ X_1 \succ X_2$.

4.3 Comparative study

In this section, we have compared the performance of the proposed operator based decision-making approach under the IMS as well as IFS environments.

4.3.1 Comparison of results with those obtained with IMS environment

If we compare the proposed approach results with some existing measures [9, 14, 29] under IMS environment by taking weight vector ω then we get

- (i) If we use [29] method to aggregate all preference values as defined in (4) then we obtain the averaged IMNs $\alpha_i (i = 1, 2, 3, 4)$ of the candidate $X_i (i = 1, 2, 3, 4)$ as $\alpha_1 = \langle 0.6414, 0.8222 \rangle, \alpha_2 = \langle 0.5496, 0.8971 \rangle, \alpha_3 = \langle 0.9506, 0.5202 \rangle$ and $\alpha_4 = \langle 0.9714, 0.3573 \rangle$. Thus, the score values of these $\alpha_i (i = 1, 2, 3, 4)$ are $S(\alpha_1) = 0.7801, S(\alpha_2) = 0.6127, S(\alpha_3) = 1.8274$ and $S(\alpha_4) = 2.7188$ and hence $\alpha_4 \succ \alpha_3 \succ \alpha_1 \succ \alpha_2$ by which we can get the ranking $X_4 \succ X_3 \succ X_1 \succ X_2$. Therefore, the best contractor for the required job is X_4 .

- (ii) If we use (2) as proposed by Jiang et al. [14] under IMS environment, then we get $Ep(X_1) = 0.3054, Ep(X_2) = 0.2940, Ep(X_3) = 0.3729$ and $Ep(X_4) = 0.3343$. Thus $Ep(X_3) > Ep(X_4) > Ep(X_1) > Ep(X_2)$, by which we get the ranking order as $X_3 \succ X_4 \succ X_1 \succ X_2$. Thus, the best contractor for the required post is X_3 .
- (iii) If we use weighted hamming distance measure, denoted by $d(\cdot)$ as proposed by Garg [9] to the considered data for finding the best alternative with respect to ideal alternative $X^* = \langle 9, 1/9 \rangle$ then we get $d(X_1, X^*) = 0.7347, d(X_2, X^*) = 0.6962, d(X_3, X^*) = 0.6260$ and $d(X_4, X^*) = 0.6714$. Thus, based on these measure values, we get $X_3 \succ X_4 \succ X_2 \succ X_1$ and hence best contractor for the job is X_3 .

4.3.2 Comparison of results with those obtained with IFS environment

In order to compare the proposed approach with the results obtained through IFS environment. For it, firstly Definition 4 has been used for converting the preferences of the alternatives from IMS to IFS and then existing measures [2, 4–6, 10, 19, 20, 23, 24, 31, 33, 35] have been utilized for finding the most suitable alternative(s). The results corresponding to these existing approaches are summarized in Table 1. From this table, it has been concluded that the best alternative obtained by the proposed approach coincides with these existing studies. Therefore, the considered approach can be taken as an alternative way to solve these types of problem in a more profitable way.

4.4 Effect of λ on the ranking of the candidate

In order to analyze the effect of the parameter λ on the final ranking of the contractors, an investigation has been done, in which λ varies from 0 to 15, by using the proposed approach. The overall score values of each contractor corresponding to GIFMIWA, GIFMIOWA, and GIFMIHWA operators are represented in Table 2 along with their ranking order. From this table, it has been seen that for different values λ , the decision-maker have different ranking related to the contractor, which shows to the decision makers that they can choose the best values according to their preferences. Furthermore, the results corresponding to $\lambda = 1$ are equivalent to the results as obtained by IFMIWA operator, which represents that the decision maker’s attitude toward the analysis is neutral. Thus, the management meaning of λ is that the decision makers’ different preference had effects on the overall values of alternative, which leads to the different optimal alternative.

Table 1 Comparative study

	Method	Overall value of the contractor				Ranking order
		X_1	X_2	X_3	X_4	
[31]	Averaging operator	-0.3728	-0.4080	-0.2314	-0.3188	$X_3 > X_4 > X_1 > X_2$
[33]	Geometric operator	-0.4392	-0.4211	-0.3077	-0.3537	$X_3 > X_4 > X_2 > X_1$
[35]	Improved accuracy function	0.3136	0.2960	0.3843	0.3406	$X_3 > X_4 > X_1 > X_2$
[4]	Improved score function	0.3379	0.2973	0.3937	0.3346	$X_3 > X_1 > X_4 > X_2$
[24]	Einstein averaging operator	-0.3752	-0.4085	-0.2341	-0.3203	$X_3 > X_4 > X_1 > X_2$
[2]	Einstein Geometric operator	-0.4383	-0.4209	-0.3062	-0.3530	$X_3 > X_4 > X_2 > X_1$
[19]	novel accuracy function	-0.1576	-0.1996	0.0052	-0.0942	$X_3 > X_4 > X_1 > X_2$
[6]	Improved accuracy function	-0.5421	-0.5542	-0.5219	-0.5436	$X_3 > X_1 > X_4 > X_2$
[23]	Accuracy score function	0.3136	0.2960	0.3843	0.3406	$X_3 > X_4 > X_1 > X_2$
[20]	non-hesistance score function	0.3989	0.3918	0.4318	0.4107	$X_3 > X_4 > X_1 > X_2$
[5]	Pythagorean Einstein averaging operator	-0.3261	-0.3610	-0.1896	-0.2743	$X_3 > X_4 > X_1 > X_2$
[10]	Improved aggregation operator	-0.3817	-0.4101	-0.2421	-0.3249	$X_3 > X_4 > X_1 > X_2$

Table 2 Effect of λ on the score values and ranking of the alternatives

	Score values ($\lambda = 0.5$)			Score values ($\lambda = 0.7$)			Score values ($\lambda = 1$)		
	GIFMIWA	GIFMIOWA	GIFMIHWA	GIFMIWA	GIFMIOWA	GIFMIHWA	GIFMIWA	GIFMIOWA	GIFMIHWA
X_1	0.1087	0.0794	0.1172	0.2347	0.1944	0.2884	0.4520	0.4315	0.6333
X_2	0.0723	0.2691	0.1078	0.1724	0.5027	0.2327	0.3566	0.8624	0.4459
X_3	0.3074	0.1939	0.3196	0.5548	0.4337	0.7133	0.9380	0.8800	1.4583
X_4	0.1866	0.2780	0.2254	0.4848	0.5454	0.6181	1.0182	1.0015	1.4888
Ranking	(3412)	(4231)	(3412)	(3412)	(4231)	(3412)	(4312)	(4321)	(4312)
	Score values ($\lambda = 2$)			Score values ($\lambda = 2.5$)			Score values ($\lambda = 3.5$)		
	GIFMIWA	GIFMIOWA	GIFMIHWA	GIFMIWA	GIFMIOWA	GIFMIHWA	GIFMIWA	GIFMIOWA	GIFMIHWA
X_1	1.1689	1.3441	2.1438	1.5181	1.8064	3.0422	2.2627	2.7604	5.1543
X_2	1.0738	2.0043	1.1663	1.4950	2.6087	1.5392	2.4721	3.9579	2.4689
X_3	2.3144	2.3702	4.4177	2.7204	3.1825	5.9533	3.9482	4.9380	9.2221
X_4	3.0337	2.4730	5.8349	3.9653	3.1320	8.3845	5.6520	4.4371	14.464
Ranking	(4312)	(4321)	(4312)	(4312)	(3421)	(4312)	(4321)	(3421)	(4312)
	Score values ($\lambda = 4.5$)			Score values ($\lambda = 6.5$)			Score values ($\lambda = 10$)		
	GIFMIWA	GIFMIOWA	GIFMIHWA	GIFMIWA	GIFMIOWA	GIFMIHWA	GIFMIWA	GIFMIOWA	GIFMIHWA
X_1	3.0959	3.7647	7.7020	4.9006	5.7811	13.9369	7.7791	8.7588	25.7568
X_2	3.5733	5.3214	3.3498	5.8725	7.9936	5.0309	9.3619	12.021	7.7236
X_3	5.2476	6.8553	12.521	7.9055	10.379	14.8428	12.4081	16.203	29.7224
X_4	7.1311	5.6228	16.503	9.5222	7.7012	42.0610	12.3274	10.420	90.9131
Ranking	(4321)	(3421)	(4312)	(4321)	(3241)	(4312)	(3421)	(3241)	(4312)
	Score values ($\lambda = 12$)			Score values ($\lambda = 13.5$)			Score values ($\lambda = 15$)		
	GIFMIWA	GIFMIOWA	GIFMIHWA	GIFMIWA	GIFMIOWA	GIFMIHWA	GIFMIWA	GIFMIOWA	GIFMIHWA
X_1	9.1290	10.1015	30.4122	7.0602	6.7222	14.7973	10.8162	11.7525	45.4136
X_2	11.5098	13.9585	8.9566	8.8449	11.8429	9.7440	14.1005	16.5788	10.4319
X_3	14.8443	19.1506	34.9783	12.736	15.1107	38.4755	18.2474	23.0462	41.7395
X_4	13.4338	11.6103	125.8473	8.3977	8.5921	154.9157	14.7061	12.9910	186.1265
Ranking	(3421)	(3241)	(4312)	(3241)	(3241)	(4312)	(3421)	(3241)	(4132)

4.5 Validity test of the proposed approach

Since practically it is not possible to determine which one is the best suitable alternative for a given decision problem, [26] established following testing criteria to evaluate the validity of MCDM methods.

Test criterion 1: An effective MCDM method should not change the indication of the best alternative on replacing a non-optimal alternative by another worse alternative without changing the relative importance of each decision criteria.

Test criterion 2: An effective MCDM method should follow transitive property.

Test criterion 3: When a MCDM problem is decomposed into smaller problems and same MCDM method is applied

on smaller problems to rank the alternatives, combined ranking of the alternatives should be identical to the original ranking of un-decomposed problem.

The validity of the proposed aggregation operators based MCDM method is tested using these test criterions.

4.5.1 Validity test of proposed approach using test criterion 1

In order to test the validity of the proposed approach under test criterion 1, the following decision matrix M' is obtained by interchanging the membership and non-membership grades of the alternative X_2 (non-optimal alternative) and X_1 (worse alternative) as

$$M' = \begin{pmatrix} \langle 0.8412, 1.1888 \rangle & \langle 0.5003, 1.9988 \rangle & \langle 0.6782, 1.4745 \rangle & \langle 0.1723, 5.8053 \rangle \\ \langle 0.4677, 2.1379 \rangle & \langle 0.3465, 2.8862 \rangle & \langle 0.4774, 2.0948 \rangle & \langle 0.6146, 1.6270 \rangle \\ \langle 0.5343, 1.8717 \rangle & \langle 0.2231, 4.4828 \rangle & \langle 0.8506, 1.1756 \rangle & \langle 1.1503, 0.8694 \rangle \\ \langle 0.5457, 1.8325 \rangle & \langle 0.5024, 1.9904 \rangle & \langle 0.3405, 2.9366 \rangle & \langle 0.3418, 2.9260 \rangle \end{pmatrix}$$

Since, the relative importance of the criteria remain unchanged in the modified problem, so the proposed IFMIWA operator has been implemented to find the best alternative and hence the weighted IMN of each contractor is obtained as $\alpha_1 = \langle 0.6022, 1.6602 \rangle$, $\alpha_2 = \langle 0.3445, 2.9026 \rangle$, $\alpha_3 = \langle 0.6547, 1.5247 \rangle$ and $\alpha_4 = \langle 0.5611, 1.7824 \rangle$. Thus, the score values of each contractor are computed as 0.3626, 0.1187, 0.4286 and 0.3148 respectively and hence ranking order of the alternatives are is $X_3 > X_1 > X_4 > X_2$. Since the indication of the best alternative is again X_3 which is same as that of the original decision-making problem, therefore it is confirmed that the proposed method does not change the indication of the best alternative when a non-optimal alternative is replaced by another worse alternative. Hence the proposed approach is valid under *test criterion 1* established by [26].

4.5.2 Validity test of proposed approach using test criterion 2 and test criterion 3

In order to test the validity of proposed method using test criterion 2 and test criterion 3, original decision-making problem is decomposed into a set of smaller MCDM problems $\{X_1, X_2, X_4\}$, $\{X_1, X_3, X_4\}$ and $\{X_2, X_3, X_4\}$. Following the steps of proposed methods, ranking $X_4 > X_1 > X_2$, $X_3 > X_4 > X_1$ and $X_3 > X_4 > X_2$ respectively

are obtained for these smaller subproblems. If ranking of the alternatives of sub-problems are combined together, the final ranking $X_3 > X_4 > X_1 > X_2$ is obtained which is identical to the ranking of un-decomposed MCDM problem and exhibits transitive property. Hence the proposed method is valid under *test criterion 2 and test criterion 3* established by [26].

5 Conclusion

In the present paper, an effective technique to aggregate the preferences of the decision-maker has been presented under the IMS environment where the rating is measured on the scale of 1/9-9 rather than 0-1. For it, firstly some new operational laws for aggregating the different IMNs have been presented by overcoming the shortcoming of the existing operations. Based on these laws, some series of the aggregation operators such as IFMIWA, IFMIOWA, IFMIHWA, GIFMIWA, GIFMIOWA, and GIFMIHWA have forwarded during the phase of the aggregation process. Some of its desirable properties have also been investigated in details. From the study, it has been observed that the existing IMWA operator [29] can be derived from the proposed operator under some special cases. Further, we proposed the new method based on the proposed operators to solve the decision-making problems in practical applications. Finally, we proved the effectiveness and feasibility of proposed methods by some practical examples. In addition,

we illustrate the advantages of the new method by comparing with the existing methods. In the further research, we will apply improved operation rules of IMNs to more aggregation operators such as Bonferroni mean operator, Heronian mean operator. In addition, we will apply these methods to solve the real decision making problems such as pattern recognition, supply chain management, cluster analysis.

Appendix

Proof Property 4 As $\alpha_i, \beta \in \text{IMNs}$, so

$$\alpha_i \oplus \beta = \left\langle \frac{(1 + 2\mu_i)(1 + 2\mu_\beta) - 1}{2}, \frac{2 \{1 - (1 - \mu_i v_i)(1 - \mu_\beta v_\beta)\}}{(1 + 2\mu_i)(1 + 2\mu_\beta) - 1} \right\rangle$$

Therefore,

$$\begin{aligned} \text{IFMIWA}(\alpha_1 \oplus \beta, \alpha_2 \oplus \beta, \dots, \alpha_n \oplus \beta) &= \left\langle \frac{\prod_{i=1}^n \{(1 + 2\mu_1)(1 + 2\mu_\beta)\}^{\omega_i} - 1}{2}, \frac{2 \left\{1 - \prod_{i=1}^n \{(1 - \mu_i v_i)(1 - \mu_\beta v_\beta)\}^{\omega_i}\right\}}{\prod_{i=1}^n \{(1 + 2\mu_i)(1 + 2\mu_\beta)\}^{\omega_i} - 1} \right\rangle \\ &= \left\langle \frac{\prod_{i=1}^n (1 + 2\mu_i)^{\omega_i} (1 + 2\mu_\beta) - 1}{2}, \frac{2 \left\{1 - \prod_{i=1}^n (1 - \mu_i v_i)^{\omega_i} (1 - \mu_\beta v_\beta)\right\}}{\prod_{i=1}^n (1 + 2\mu_i)^{\omega_i} (1 + 2\mu_\beta) - 1} \right\rangle \\ &= \left\langle \frac{\prod_{i=1}^n (1 + 2\mu_i)^{\omega_i} - 1}{2}, \frac{2 \left\{1 - \prod_{i=1}^n (1 - \mu_i v_i)^{\omega_i}\right\}}{\prod_{i=1}^n (1 + 2\mu_i)^{\omega_i} - 1} \right\rangle \oplus \langle \mu_\beta, v_\beta \rangle \\ &= \text{IFMIWA}(\alpha_1, \alpha_2, \dots, \alpha_n) \oplus \beta \end{aligned}$$

Hence, $\text{IFMIWA}(\alpha_1 \oplus \beta, \alpha_2 \oplus \beta, \dots, \alpha_n \oplus \beta) = \text{IFMIWA}(\alpha_1, \alpha_2, \dots, \alpha_n) \oplus \beta$.

Proof of Property 6: Since $\alpha_i = \langle \mu_i, v_i \rangle \in \text{IMNs}$ for $i = 1, 2, \dots, n$. Therefore, for $\beta > 0$, we have

$$\beta \alpha_i = \left\langle \frac{(1 + 2\mu_i)^\beta - 1}{2}, \frac{2 \{1 - (1 - \mu_i v_i)^\beta\}}{(1 + 2\mu_i)^\beta - 1} \right\rangle$$

Therefore,

$$\begin{aligned} \text{IFMIWA}(\beta \alpha_1, \beta \alpha_2, \dots, \beta \alpha_n) &= \left\langle \frac{\prod_{i=1}^n \left(1 + 2 \frac{(1 + 2\mu_i)^\beta - 1}{2}\right)^{\omega_i} - 1}{2}, \frac{2 \left\{1 - \prod_{i=1}^n (1 - \mu_i v_i)^{\beta \omega_i}\right\}}{\prod_{i=1}^n \left(1 + 2 \frac{(1 + 2\mu_i)^\beta - 1}{2}\right)^{\omega_i} - 1} \right\rangle \\ &= \left\langle \frac{\prod_{i=1}^n ((1 + 2\mu_i)^\beta)^{\omega_i} - 1}{2}, \frac{2 \left\{1 - \prod_{i=1}^n (1 - \mu_i v_i)^{\beta \omega_i}\right\}}{\prod_{i=1}^n ((1 + 2\mu_i)^\beta)^{\omega_i} - 1} \right\rangle \\ &= \left\langle \frac{\left(\prod_{i=1}^n (1 + 2\mu_i)^{\omega_i}\right)^\beta - 1}{2}, \frac{2 \left\{1 - \left(\prod_{i=1}^n (1 - \mu_i v_i)^{\omega_i}\right)^\beta\right\}}{\left(\prod_{i=1}^n (1 + 2\mu_i)^{\omega_i}\right)^\beta - 1} \right\rangle \\ &= \beta \left\langle \frac{\prod_{i=1}^n (1 + 2\mu_i)^{\omega_i} - 1}{2}, \frac{2 \left\{1 - \prod_{i=1}^n (1 - \mu_i v_i)^{\omega_i}\right\}}{\prod_{i=1}^n (1 + 2\mu_i)^{\omega_i} - 1} \right\rangle \\ &= \beta \text{IFMIWA}(\alpha_1, \alpha_2, \dots, \alpha_n) \end{aligned}$$

Hence, $\text{IFMIWA}(\beta\alpha_1, \beta\alpha_2, \dots, \beta\alpha_n) = \text{Proof of Property 6: As } \alpha_i = \langle \mu_{\alpha_i}, \nu_{\alpha_i} \rangle \text{ and } \beta = \langle \mu_{\beta_i}, \nu_{\beta_i} \rangle (i = 1, 2, \dots, n) \text{ be two collections of IMNs, then}$
 $\beta\text{IFMIWA}(\alpha_1, \alpha_2, \dots, \alpha_n)$

$$\alpha_i \oplus \beta_i = \left\langle \frac{(1 + 2\mu_{\alpha_i})(1 + 2\mu_{\beta_i}) - 1}{2}, \frac{2[1 - (1 - \mu_{\alpha_i}\nu_{\alpha_i})(1 - \mu_{\beta_i}\nu_{\beta_i})]}{(1 + 2\mu_{\alpha_i})(1 + 2\mu_{\beta_i}) - 1} \right\rangle$$

Therefore,

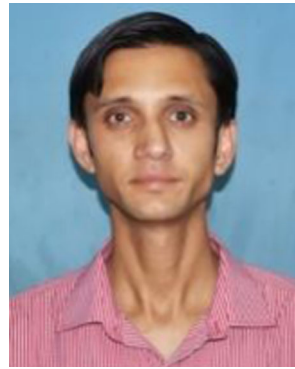
$$\begin{aligned} & \text{IFMIWA}(\alpha_1 \oplus \beta_1, \alpha_2 \oplus \beta_2, \dots, \alpha_n \oplus \beta_n) \\ &= \left\langle \frac{\prod_{i=1}^n \left(1 + 2\frac{(1+2\mu_{\alpha_i})(1+2\mu_{\beta_i})-1}{2}\right)^{\omega_i} - 1}{2}, \frac{2\left[1 - \prod_{i=1}^n \{(1 - \mu_{\alpha_i}\nu_{\alpha_i})(1 - \mu_{\beta_i}\nu_{\beta_i})\}^{\omega_i}\right]}{\prod_{i=1}^n \left(1 + 2\frac{(1+2\mu_{\alpha_i})(1+2\mu_{\beta_i})-1}{2}\right)^{\omega_i} - 1} \right\rangle \\ &= \left\langle \frac{\prod_{i=1}^n \{(1 + 2\mu_{\alpha_i})(1 + 2\mu_{\beta_i})\}^{\omega_i} - 1}{2}, \frac{2\left[1 - \prod_{i=1}^n \{(1 - \mu_{\alpha_i}\nu_{\alpha_i})(1 - \mu_{\beta_i}\nu_{\beta_i})\}^{\omega_i}\right]}{\prod_{i=1}^n \{(1 + 2\mu_{\alpha_i})(1 + 2\mu_{\beta_i})\}^{\omega_i} - 1} \right\rangle \\ &= \left\langle \frac{\prod_{i=1}^n (1 + 2\mu_{\alpha_i})^{\omega_i} \prod_{i=1}^n (1 + 2\mu_{\beta_i})^{\omega_i} - 1}{2}, \frac{2\left[1 - \prod_{i=1}^n (1 - \mu_{\alpha_i}\nu_{\alpha_i})^{\omega_i} \prod_{i=1}^n (1 - \mu_{\beta_i}\nu_{\beta_i})^{\omega_i}\right]}{\prod_{i=1}^n (1 + 2\mu_{\alpha_i})^{\omega_i} \prod_{i=1}^n (1 + 2\mu_{\beta_i})^{\omega_i} - 1} \right\rangle \\ &= \left\langle \frac{\prod_{i=1}^n (1 + 2\mu_{\alpha_i})^{\omega_i} - 1}{2}, \frac{2\left[1 - \prod_{i=1}^n (1 - \mu_{\alpha_i}\nu_{\alpha_i})^{\omega_i}\right]}{\prod_{i=1}^n (1 + 2\mu_{\alpha_i})^{\omega_i} - 1} \right\rangle \oplus \left\langle \frac{\prod_{i=1}^n (1 + 2\mu_{\beta_i})^{\omega_i} - 1}{2}, \frac{2\left[1 - \prod_{i=1}^n (1 - \mu_{\beta_i}\nu_{\beta_i})^{\omega_i}\right]}{\prod_{i=1}^n (1 + 2\mu_{\beta_i})^{\omega_i} - 1} \right\rangle \\ &= \text{IFMIWA}(\alpha_1, \alpha_2, \dots, \alpha_n) \oplus \text{IFMIWA}(\beta_1, \beta_2, \dots, \beta_n) \end{aligned}$$

Hence, $\text{IFMIWA}(\alpha_1 \oplus \beta_1, \dots, \alpha_n \oplus \beta_n) = \text{IFMIWA}(\alpha_1, \dots, \alpha_n) \oplus \text{IFMIWA}(\beta_1, \dots, \beta_n)$

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Dr. Harish Garg associated with the School of Mathematics at Thapar University Patiala, India as an Assistant Professor. Prior to joining this University, Dr. Garg received a PhD in Applied Mathematics with specialization of Reliability theory and soft computing techniques, from Indian Institute of Technology Roorkee, India in 2013 and an MSc in Mathematics from Punjabi University, Patiala, India in 2008. His research interest are in the field of Computational

Intelligence, Multi-criteria decision making problems, Reliability theory, Optimization techniques, various nature inspired algorithms (e.g. genetic algorithms, swarm optimization), fuzzy and intuitionistic fuzzy set theory, Expert Systems. Application areas include wide range of industrial and structural engineering design problems. Garg has authored/co-authored over 105 technical papers published in refereed International Journals including *Applied Intelligence*, *Applied Soft Computing*, *Experts Systems with Applications*, *Journal of Intelligent and Fuzzy Systems*, *Journal of Manufacturing Systems*, *Applied Mathematics & Computations*, *ISA Transactions*, *Computer and Industrial Engineering*, *Computer and Operations Research*, *International Journal of Uncertainty, Fuzziness and Knowledge-based Systems*, *Journal of Industrial and Management Optimization*, *International Journal of Intelligent Systems*, *Journal of Multiple Logic and Soft Computing*, *Japan Journal of Industrial and Applied Mathematics*, and many more. He has published seven book chapters also. He also serves as an Editorial Board member of various International Journals. He is listed in *International Who's Who of Professionals*, *Marquis Who's Who in the World*, and *Marquis Who's Who in Science and Engineering*. In the year 2016, Dr. Garg was awarded an awarded outstanding reviewer for the journal *Applied Soft Computing*, Elsevier. His google citations are over 1100. For more details, visit <http://sites.google.com/site/harishg58iitr/>.