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Multi-attribute group decision making under probabilistic hesitant fuzzy environment with application to evaluate the transformation efficiency

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Abstract Hesitant fuzzy sets were introduced by Torra to efficiently address situations in which the membership degree of an element in a set is expressed by several different values. However, there is a large shortcoming in hesitant fuzzy sets—the serious loss of information. Therefore, in this paper, we improve upon hesitant fuzzy sets by implementing a probabilistic hesitant fuzzy set (PHFS). Then, we introduce some new basic operations on the probabilistic hesitant fuzzy elements (PHFEs) using the Frank t-conorm and t-norm. Based on these proposed operations, we further develop probabilistic hesitant fuzzy weighted arithmetic and geometric aggregation operators. The desired properties and the relationships among them are investigated in detail. In addition, an approach to multi-attribute group decision making (MAGDM) is investigated on the basis of the new operators. Finally, a numerical example of public company efficiency evaluation is provided to illustrate the application and validity of the proposed approach.

Keywords Multi-attribute group decision making · Probabilistic hesitant fuzzy set · Frank t-conorm and t-norm · Information aggregation operators

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1 Introduction

Fuzzy set theory, proposed by Zadeh [39], has been successfully applied in various fields. Due to rapid development of human beings, socioeconomic environments have become more and more complicated. Several extensions of the fuzzy sets have been developed for different applications, including socioeconomic applications, such as type-2 fuzzy sets [11, 19, 37], type-n fuzzy sets [11], interval-valued fuzzy sets [7, 24], fuzzy multisets [18], intuitionistic fuzzy sets (IFS) [1–4, 6, 14, 16, 20], interval-valued intuitionistic fuzzy sets [26, 35, 40], linguistic fuzzy sets [33, 34] and hesitant fuzzy sets (HFS) [5, 13, 17, 23, 25, 27].

The HFS is intended to take into account the possible membership degrees of x in set A, such as 0.2 and 0.3. It does not attempt to handle a margin of error or a possibility distribution, in contrast with from IFS and type-2 fuzzy sets. As stated by Torra [21–23], an HFS can be very useful for characterizing an assessment covering different opinions of experts instead of aggregating them to analyze multiple attribute group decision making (MAGDM) problems. In other words, all possible opinions should be considered. For example, when an assessment is profiled by {0.2, 0.3}, it cannot be replaced by any value limited to [0.2, 0.3] unless a decision maker specifies an aggregated value to represent the assessment through his or her preference [28–30, 38, 41].

Although an HFS can characterize possible membership degrees of x into the set A in a discrete way, it may lose some original information. Let us reconsider MAGDM problems with several experts. Suppose that a decision maker provides assessments of alternatives for each attribute for a MAGDM problem by considering the opinions of five experts. Given that an assessment {0.2, 0.3} is provided by the decision maker based on the opinions of the five experts, some important original information cannot be exposed. The assessment cannot differentiate the following three situations: (1) one expert supports 0.2 and the others favor 0.3; (2) three experts prefer 0.2 and the others prefer 0.3; and (3) three experts provide input of 0.2, one expert specifies 0.3, and the other one has no opinion. In other words, the hesitant fuzzy assessment is not sufficient to reveal original information about how the decision maker derives the decision based on the opinions of the five experts. In general, a combination of the relative importance of the five experts and the opinions of the five experts can fully indicate the original information concerning the hesitant fuzzy assessment. For example, on the condition that the decision maker uses the opinions of the five experts to provide the assessment and the five experts are equally important, two possible membership degrees should be assigned different probabilities in the above three situations. That is, the probabilities of 0.2 or 0.3 arising in real cases should be the indication of experts who support 0.2 or 0.3. Using this assumption, the probabilities of 0.2 and 0.3 for the above three conditions can be deduced, which are {0.2, 0.8}, {0.6, 0.4}, and {0.75, 0.25}, respectively.

In this paper, we extend the HFS to sufficiently indicate original information about assessments, henceforth referred to as the probabilistic hesitant fuzzy set (PHFS). Two dimensions are involved in a PHFS, i.e., the possible membership degree of x in set A and its probability of occurring. The original information regarding how to create hesitant fuzzy assessments can be profiled by the probabilities of possible membership degrees in the assessments. The probabilities guarantee the handling of each possible membership degree in the assessments, which is consistent with the original view of Torra [23] about the HFS. Basic operations of hesitant fuzzy assessments should also be extended in the context of a PHFS. Motivated by the idea that the framework of the t-conorm and t-norm can be used to unify various operations of hesitant fuzzy assessments [42], we define the generalized operations on PHFEs based on the Frank t-conorm and the t-norm and discuss their properties. The frameworks follow the operational principles of hesitant fuzzy information [32] and the independence principle of probability information [8]. These operations are then applied to develop weighted arithmetic and geometric aggregation operators on PHFEs. We then analyze the monotonicity of the arithmetic and geometric aggregation operators with respect to a parameter in the Frank t-conorm and the t-norm and reveal the relationship between the arithmetic and geometric aggregation operators. With a view to comparing two PHFEs, their score functions are designed via a combination of the mean and variance of possible membership degrees.

The rest of this paper is organized as follows. In Section 2, we review some basic concepts related to the PHFS. Section 3 introduces the PHFS, the basic Frank operations of PHFSs and score functions of PHFEs. In Section 4, based on these new Frank operational rules on PHFSs, we present some probabilistic hesitant fuzzy Frank aggregation operators. Section 5 develops an approach to MAGDM in the probabilistic hesitant fuzzy information environment. In Section 6, an efficiency evaluation problem is presented to verify the proposed method, and a sensitivity analysis is provided. Finally, some conclusions and future research possibilities are given in Section 7.

2 Preliminaries

This section is devoted to describing the basic definitions and notions of the hesitant fuzzy set (HFS). Then, the basics of probability theory and the concept of the Frank t-conorm and t-norm are reviewed.

2.1 Hesitant fuzzy set

In this subsection, we review the basic concepts of HFSs and their basic operations.

Definition 1 [23] Let X be a reference set, then an HFS on X is in terms of a function that when applied to X returns a subset of [0, 1].

To be easily understood, Xia and Xu [31] expressed the HFS by the mathematical symbol:

$$E = \{< x, h_E(x) > | x \in X\}, \tag{1}$$

where $h_E(x)$ is a set of different values in [0,1], representing the possible membership degrees of the element. For convenience, Xia and Xu [31] called $h = h_A(x)$ a hesitant fuzzy element (HFE).

The following comparison rules were also defined by Xia and Xu [31] for HFEs.

Definition 2 For an HFE h, $s(h) = \frac{1}{l}\sum_{\gamma \in h} \gamma$ is called the score function of h, where l is the number of elements in h. For two HFEs h_1 and h_2, if $s(h_1) = s(h_2)$, then $h_1 = h_2$; if $s(h_1) > s(h_2)$, then $h_1 > h_2$.

2.2 Basics of probability theory

In researching random events, we need to know what can occur and the probability of each event. The probability of a random event can be denoted by a value in [0, 1]. We use $p(A)$ to represent the possibility of event A.

Definition 3 [8] Let A and B be two random events, they are mutually independent if and only if $p(AB) = p(A)p(B)$. Generally, $A_1, A_2, \dots, A_n$ are called n mutually independent events ($n \geq 3$), if for any $k(1 < k \leq n), 1 \leq i_1 < \dots < i_k \leq n$, $p(A_{i_1}A_{i_2}\dots A_{i_k}) = p(A_{i_1})p(A_{i_2})\dots p(A_{i_k})$.

Definition 4 [8] Suppose in n repetitions of a random trial, event A happens n_A times. Then, the ratio $\frac{n_A}{n}$ is called the frequency of A, which is denoted by $f_n(A)$. $f_n(A)$ should satisfy the following properties:

(1) $0 \leq f_n(A) \leq 1$;
(2) $f_n(\Omega) = 1$;
(3) $A_1, A_2, \dots, A_n$ are two-two incompatible, if $f_n(\bigcup_{k=1}^{n} A_k) = \sum_{k=1}^{n} f_n(A_k)$.

2.3 T-conorm and t-norm

An important notion in fuzzy set theory is that of the t-norm and t-conorm that are used to define a generalized intersection and union of fuzzy sets [10].

Definition 5 [9, 15] A function T: $[0, 1]^2 \to [0, 1]$ is called a t-norm if it is commutative, associative, non-decreasing and $T(1, x) = x$, for all $x \in [0, 1]$.

Definition 6 [9, 35] A function S: $[0, 1]^2 \to [0, 1]$ is called a t-conorm if it is commutative, associative, non-decreasing and $S(0, x) = x$, for all $x \in [0, 1]$.

As stated by Xia et al. [32], a strictly decreasing function $g : [0, 1] \to [0, +\infty]$ such that $g(1) = 0$ can be used to create a strict Archimedean t-norm $T(xy) = g^1(g(x) + g(y))$ and its dual t-conorm $S(xy) = h^1(h(x) + h(y))$ such that $h(x) = g(1x)$. Different types of g were introduced by Xia et al. [32] to generate different Archimedean t-norms and t-conorms. The Frank t-conorm and t-norm are defined as follows.

Definition 7 Let $g(t) = \log(\frac{r-1}{r^t-1}), r > 1$, then $h(t) = g(1-t) = \log(\frac{r-1}{r^{1-t}-1}), r > 1$, and the Frank t-conorm and t-norm are defined as follows:

$$S_r^F(x, y) = 1 - \log_r(1 + \frac{(r^{1-x} - 1)(r^{1-y} - 1)}{r - 1}), \quad r > 1, \tag{2}$$

$$T_r^F(x, y) = \log_r(1 + \frac{(r^x - 1)(r^y - 1)}{r - 1}), \quad r > 1. \tag{3}$$

3 Probabilistic hesitant fuzzy sets (PHFSs)

In this section, we propose PHFSs to be an extension of HFSs. Some generalized operations of PHFEs are defined based on the Frank t-norm and t-conorm. Furthermore, score functions are designed for ranking PHFEs.

3.1 Probabilistic hesitant fuzzy set

Because the possibility of each value is not given when defining the membership as a set of possible values, we propose that a PHFS sufficiently describes original information as follows:

Definition 8 Given a fixed set $X = \{x_1, x_2, \dots, x_n\}$, a PHFS is defined as

$$E = \{< x_i, H_i(x) | x_i \in X >\}, \tag{4}$$

in which $H_i(x) = \left\{ \bigcup_{(h,p)\in H_i(x)} (h, p) \right\}$, h is a value in [0,1], denoting the membership degree of x_i. p is the corresponding probability of h, denoting the intensity of the membership degree. All the possible arrays (h, p) constitute a PHFS.

Definition 9 Given a PHFS expressed by $E = \{< x_i, H_i(x) | x_i \in X >\}$, we call $H = \{ \bigcup_{(h,p)\in H(x)} (h, p)\}$ a PHFE, where $0 \leq h \leq 1, 0 \leq p \leq 1$ and $\sum p = 1$.

Example 3.1 Suppose ten experts are invited to evaluate an industry index using fuzzy elements. Four of them assign the number 0.8, two experts are sure that 0.7 is suitable, two experts give the value 0.6, and the other two provide 0.5.

The evaluation of the index is normally represented by a hesitant fuzzy element {0.5, 0.6, 0.7, 0.8}. Notably, the

HFE only collects all of the possible values, and each value provided only means that it is a possible value, but its importance is unknown [36]. Remarkably, the importance of any value is considered to be the same as the others. Obviously, this does not conform to the original information.

However, the information used to perform the evaluation is clearly depicted by a PHFE $H = \{(0.5, 0.2), (0.6, 0.2), (0.7, 0.2), (0.8, 0.4)\}$. Each membership degree with its corresponding possibility correctly reveals the original decision-making information.

To analyze the possibility of the membership degree in the PHFSs, we suppose that p_i $(i = 1, 2, ..., n)$ satisfies the following assumption.

Let p_i $(i = 1, 2, ..., n)$ be the possibility of h_i $(i = 1, 2, ..., n)$ in (4). To analyze and integrate the PHFEs, it is required that

$$p(h_i h_j) = p(h_i)p(h_j), \forall i \neq j, i, j = 1, 2, ..., n, \tag{5}$$

and

$$p(h_1 h_2 ... h_k) = p(h_1)p(h_2)...p(h_k), (1 < k \leq n). \tag{6}$$

3.2 Probabilistic hesitant fuzzy operational laws

In this subsection, some operations on PHFEs based on the Frank t-norm and t-conorm are introduced. Some relations among these operations are then analyzed.

Definition 10 Let H, H_1 and H_2 be three PHFEs, then

(1)

$$\begin{aligned} H_1 \oplus H_2 &= \bigcup_{\gamma_1 \in H_1, \gamma_2 \in H_2, p_1 \in H_1, p_2 \in H_2} \{(S(\gamma_1, \gamma_2), p_1 p_2)\} \\ &= \bigcup_{\gamma_1 \in H_1, \gamma_2 \in H_2, p_1 \in H_1, p_2 \in H_2} \left\{(h^{-1}(h(\gamma_1) + h(\gamma_2)), p_1 p_2)\right\}; \end{aligned}$$

(2)

$$\begin{aligned} H_1 \otimes H_2 &= \bigcup_{\gamma_1 \in H_1, \gamma_2 \in H_2, p_1 \in H_1, p_2 \in H_2} \{(T(\gamma_1, \gamma_2), p_1 p_2)\} \\ &= \bigcup_{\gamma_1 \in H_1, \gamma_2 \in H_2, p_1 \in H_1, p_2 \in H_2} \left\{(g^{-1}(g(\gamma_1) + g(\gamma_2)), p_1 p_2)\right\}; \end{aligned}$$

(3)

$$\lambda H = \bigcup_{\gamma \in H, p \in H} \left\{h^{-1}(\lambda h(\gamma)), p\right\}, \lambda > 0;$$

(4)

$$H^\lambda = \bigcup_{\gamma \in H, p \in H} \left\{(g^{-1}(\lambda g(\gamma)), p)\right\}, \lambda > 0.$$

Based on the Frank t-conorm and t-norm, let $g(t) = \log(\frac{r-1}{r^t-1})$, $r > 1$; we then have

(1)

$$H^c = \bigcup_{\gamma \in H, p \in H} \{(1 - \gamma, p)\};$$

(2)

$$\lambda H = \bigcup_{\gamma \in H, p \in H} \left\{(1 - \log_r(1 + \frac{(r^{1-\gamma} - 1)^\lambda}{(r-1)^{\lambda-1}}), p)\right\},$$

$\lambda > 0, r > 1;$

(3)

$$H^\lambda = \bigcup_{\gamma \in H, p \in H} \left\{(\log_r(1 + \frac{(r^\gamma - 1)^\lambda}{(r-1)^{\lambda-1}}), p)\right\},$$

$\lambda > 0, r > 1;$

(4)

$$H_1 \oplus H_2 = \bigcup_{\gamma_1 \in H_1, \gamma_2 \in H_2, p_1 \in H_1, p_2 \in H_2} \left\{(1 - \log_r(1 + \frac{(r^{1-\gamma_1} - 1)(r^{1-\gamma_2} - 1)}{r-1}), p_1 p_2)\right\},$$

$r > 1;$

(5)

$$H_1 \otimes H_2 = \bigcup_{\gamma_1 \in H_1, \gamma_2 \in H_2, p_1 \in H_1, p_2 \in H_2} \left\{(\log_r(1 + \frac{(r^{\gamma_1} - 1)(r^{\gamma_2} - 1)}{r-1}), p_1 p_2)\right\},$$

$r > 1.$

Furthermore, the relationships among the above operational laws are given by Theorem 1.

Theorem 1 *Let H, H_1 and H_2 be three PHFEs. Then*

(1) $H_1 \oplus H_2 = H_2 \oplus H_1$;
(2) $H_1 \otimes H_2 = H_2 \otimes H_1$;
(3) $\lambda(H_1 \oplus H_2) = \lambda H_1 \oplus \lambda H_2$;
(4) $(H_1 \otimes H_2)^\lambda = H_1^\lambda \otimes H_2^\lambda$;
(5) $\lambda_1 H \oplus \lambda_2 H = (\lambda_1 + \lambda_2)H$;
(6) $H^{\lambda_1} \otimes H^{\lambda_2} = H^{\lambda_1 + \lambda_2}$.

Proof Properties (1) and (2) clearly hold.

(3)

$$\begin{aligned}
\lambda(H_1 \oplus H_2) &= \lambda \bigcup\nolimits_{\gamma_1 \in H_1, \gamma_2 \in H_2, p_1 \in H_1, p_2 \in H_2} \\
&\quad \left\{(h^{-1}(h(\gamma_1) + h(\gamma_2)), p_1 p_2)\right\} \\
&= \bigcup\nolimits_{\gamma_1 \in H_1, \gamma_2 \in H_2, p_1 \in H_1, p_2 \in H_2} \\
&\quad \left\{(h^{-1}(\lambda h(h^{-1}(h(\gamma_1) + h(\gamma_2)))), p_1 p_2)\right\}, \\
\lambda H_1 \oplus \lambda H_2 &= \bigcup\nolimits_{\gamma_1 \in H_1, p_1 \in H_1} \left\{(h^{-1}(\lambda h(\gamma_1)), p_1)\right\} \\
&\quad \oplus \bigcup\nolimits_{\gamma_2 \in H_2, p_2 \in H_2} \left\{(h^{-1}(\lambda h(\gamma_2)), p_2)\right\} \\
&= \bigcup\nolimits_{\gamma_1 \in H_1, p_1 \in H_1, \gamma_2 \in H_2, p_2 \in H_2} \\
&\quad \left\{(h^{-1}(h(h^{-1}(\lambda h(\gamma_1))) \right. \\
&\quad \left. + h(h^{-1}(\lambda h(\gamma_2)))), p_1 p_2)\right\} \\
&= \bigcup\nolimits_{\gamma_1 \in H_1, \gamma_2 \in H_2, p_1 \in H_1, p_2 \in H_2} \\
&\quad \left\{(h^{-1}(\lambda(h(\gamma_1) + h(\gamma_2))), p_1 p_2)\right\}
\end{aligned}$$

(4)

$$\begin{aligned}
(H_1 \otimes H_2)^\lambda &= \left[\bigcup\nolimits_{\gamma_1 \in H_1, \gamma_2 \in H_2, p_1 \in H_1, p_2 \in H_2} \right. \\
&\quad \left. \left\{(g^{-1}(g(\gamma_1) + g(\gamma_2)), p_1 p_2)\right\}\right]^\lambda \\
&= \bigcup\nolimits_{\gamma_1 \in H_1, \gamma_2 \in H_2, p_1 \in H_1, p_2 \in H_2} \\
&\quad \left\{(g^{-1}(\lambda g(g^{-1}(g(\gamma_1) + g(\gamma_2)))), p_1 p_2)\right\} \\
&= \bigcup\nolimits_{\gamma_1 \in H_1, \gamma_2 \in H_2, p_1 \in H_1, p_2 \in H_2} \\
&\quad \left\{(g^{-1}(\lambda(g(\gamma_1) + g(\gamma_2))), p_1 p_2)\right\} \\
H_1^\lambda \otimes H_2^\lambda &= \bigcup\nolimits_{\gamma_1 \in H_1, p_1 \in H_1} \left\{(g^{-1}(\lambda g(\gamma_1)), p_1)\right\} \\
&\quad \otimes \bigcup\nolimits_{\gamma_2 \in H_2, p_2 \in H_2} \left\{(g^{-1}(\lambda g(\gamma_2)), p_2)\right\} \\
&= \bigcup\nolimits_{\gamma_1 \in H_1, \gamma_2 \in H_2, p_1 \in H_1, p_2 \in H_2} \\
&\quad \left\{(g^{-1}(g(g^{-1}(\lambda g(\gamma_1))) \right. \\
&\quad \left. + g(g^{-1}(\lambda g(\gamma_2)))), p_1 p_2)\right\} \\
&= \bigcup\nolimits_{\gamma_1 \in H_1, \gamma_2 \in H_2, p_1 \in H_1, p_2 \in H_2} \\
&\quad \left\{(g^{-1}(\lambda(g(\gamma_1) + g(\gamma_2))), p_1 p_2)\right\}.
\end{aligned}$$

Similarity, (5) and (6) can be confirmed. □

3.3 Score functions for PHFE

As with hesitant fuzzy elements, score functions of PHFEs are available for ranking the assessments. In the following, we design score functions of BHF assessments which are profiled by BHFEs

3.3.1 Basic score functions

Motivated by the score functions for hesitant fuzzy elements proposed by Farhadinia [12], we extend them to the PHFS environment.

Definition 11 Let $H = \{(h_1, p_1), (h_2, p_2), ..., (h_n, p_n)\}$ be a PHFE, in which $\sum_{i=1}^{n} p_i = 1$. The following functions can be considered as score functions for the PHFE:

1. The arithmetic-mean score function:

$$S_{am}(H) = \sum\nolimits_{i=1}^{n} h_i p_i \tag{7}$$

2. The geometric-mean score function:

$$S_{gm}(H) = \prod_{i=1}^{n} h_i^{p_i} \tag{8}$$

3.3.2 Variance-arithmetic-mean score function

The aforementioned score functions provide a feasible method for comparing PHFEs. However, the score functions are not appropriate in some situations.

For example, suppose that there are two PHFEs {(0.2, 0.5), (0.4, 0.5)} and {(0.1, 0.5), (0.5, 0.5)}, which have the same score of 0.3 calculated using the arithmetic-mean score function. In other words, their mean cannot be used to estimate which one is better. From a statistical point of view, this problem may be due to not considering the variance of possible membership degrees, which measures the spread or dispersion of membership degrees. Thus, we develop the arithmetic-mean score function based on a measure of variance.

Definition 12 Given $H = \{(h_1, p_1), (h_2, p_2), ..., (h_n, p_n)\}$, its variance-arithmetic-mean score function is defined as

$$S(H) = \bar{h}(1 - \delta^2), \tag{9}$$

where $\bar{h} = \sum_{i=1}^{n} h_i p_i$, $\delta^2 = \sum_{i=1}^{n} (h_{\delta(i)} - \bar{h})^2 p_{\delta(i)}$, $0 < h_{\delta(1)} < h_{\delta(2)} < ... < h_{\delta(n)} \le 1$.

Theorem 2 *Let* $H = \{(h_1, p_1), (h_2, p_2), ..., (h_n, p_n)\}$ *be a PHFE, then the variance-arithmetic -mean score function denoted by* $S(H) = \bar{h}(1 - \delta^2)$ *satisfies:*

(1) $S(H) > h_{\delta(1)}$,
(2) $S(H) < h_{\delta(n)}$.

Proof First, we prove $S(H) > h_{\delta(1)}$. Since

$$\begin{aligned}
S(H) - h_{\delta(1)} &= \bar{h}(1 - \sum_{i=1}^{n}(h_{\delta(i)} - \bar{h})^2 p_{\delta(i)}) - h_{\delta(1)} \\
&= (\bar{h} - h_{\delta(1)}) - \bar{h}\sum_{i=1}^{n}(h_{\delta(i)} - \bar{h})^2 p_{\delta(i)} \\
&> (\bar{h} - h_{\delta(1)}) - \sum_{i=1}^{n}(h_{\delta(i)} - \bar{h})^2 p_{\delta(i)} \\
&= (\sum_{i=1}^{n} h_{\delta(i)} p_{\delta(i)} - h_{\delta(1)}) \\
&\quad - \sum_{i=1}^{n}(h_{\delta(i)} - \bar{h})^2 p_{\delta(i)} \\
&= \sum_{i=1}^{n}(h_{\delta(i)} - h_{\delta(1)}) p_{\delta(i)} \\
&\quad - \sum_{i=1}^{n}(h_{\delta(i)} - \bar{h})^2 p_{\delta(i)} \\
&= \sum_{i=1}^{n}[(h_{\delta(i)} - h_{\delta(1)}) - (h_{\delta(i)} - \bar{h})^2] p_{\delta(i)} \\
&> \sum_{i=1}^{n}[(h_{\delta(i)} - h_{\delta(1)})^2 - (h_{\delta(i)} - \bar{h})^2] p_{\delta(i)} \\
&= \sum_{i=1}^{n}(h_{\delta(i)}^2 + h_{\delta(1)}^2 - 2h_{\delta(i)}h_{\delta(1)} - h_{\delta(i)}^2 \\
&\quad - \bar{h}^2 + 2h_{\delta(i)}\bar{h}) p_{\delta(i)} \\
&= \sum_{i=1}^{n}(h_{\delta(1)}^2 - 2h_{\delta(i)}h_{\delta(1)} - \bar{h}^2 \\
&\quad + 2h_{\delta(i)}\bar{h}) p_{\delta(i)} \\
&= \sum_{i=1}^{n}(h_{\delta(1)} - \bar{h})(h_{\delta(1)} + \bar{h} - 2h_{\delta(i)}) p_{\delta(i)} \\
&= (h_{\delta(1)} - \bar{h})\sum_{i=1}^{n}(h_{\delta(1)} + \bar{h} - 2h_{\delta(i)}) p_{\delta(i)} \\
&= (h_{\delta(1)} - \bar{h})(\sum_{i=1}^{n} h_{\delta(1)} p_{\delta(i)} \\
&\quad + \sum_{i=1}^{n} \bar{h} p_{\delta(i)} - \sum_{i=1}^{n} 2h_{\delta(i)} p_{\delta(i)}) \\
&= (h_{\delta(1)} - \bar{h})(h_{\delta(1)} + \bar{h} - 2\bar{h}) \\
&= (h_{\delta(1)} - \bar{h})^2 > 0
\end{aligned}$$

Thus, $S(H) > h_{\delta(1)}$ is proven.

Since $S(H) \le \bar{h}$, $S(H) < h_{\delta(n)}$ clearly holds. □

4 Probabilistic hesitant fuzzy Frank aggregation operators and their relationship

In this section, we propose two new probabilistic hesitant fuzzy Frank aggregation operators, including a probabilistic hesitant fuzzy Frank weighted average (PHFFWA) operator and a probabilistic hesitant fuzzy Frank weighted geometric (PHFFWG) operator. Then, we discuss the monotonicity of the PHFFWA and PHFFWG operators with respect to r. We further analyze the relationships between the PHFFWA operators and the PHFFWG operators.

4.1 Probabilistic hesitant fuzzy Frank weighted average operator

Definition 13 Let $H_i(i = 1, 2, .., n)$ be a collection of PHFEs, $w = (w_1, w_2, ..., w_n)^T$ be the weight vector of $H_i(i = 1, 2, ..., n)$ with $w_i \in [0, 1], i = 1, 2, \cdots, n$ and $\sum_{i=1}^{n} w_i = 1$. Then, the probabilistic hesitant fuzzy Frank weighted average (PHFFWA) operator is defined as follows:

$$PHFFWA(H_1, H_2, ..., H_n) = w_1 H_1 \otimes w_2 H_2 \otimes \cdots \otimes w_n H_n. \tag{10}$$

Theorem 3 *Let $H_i(i = 1, 2, .., n)$ be a collection of PHFEs. Then, the aggregate value of the PHFFWA operator is also a PHFE, and*

$$PHFFWA(H_1, H_2, ..., H_n) = \bigcup_{\gamma_i \in H_i, p_i \in H_i} \left\{ (1 - \log_r(1 + \frac{r-1}{\prod_{i=1}^{n}(\frac{r-1}{r^{1-\gamma_i}-1})^{w_i}}), \prod_{i=1}^{n} p_i) \right\}. \tag{11}$$

Proof Equation (11) can be proved by mathematical induction on n as follows.

For n =2, we have:

$$\begin{aligned}
PHFFWA(H_1, H_2) &= w_1 H_1 \oplus w_2 H_2 \\
&= \bigcup_{\gamma_1 \in H_1, \gamma_2 \in H_2, p_1 \in H_1, p_2 \in H_2} \left\{ (h^{-1}(h(h^{-1}(w_1 h(\gamma_1))) + h(h^{-1}(w_2 h(\gamma_2)))), p_1 p_2 \right\} \\
&= \bigcup_{\gamma_1 \in H_1, \gamma_2 \in H_2, p_1 \in H_1, p_2 \in H_2} \left\{ (h^{-1}(w_1 h(\gamma_1) + w_2 h(\gamma_2)), p_1 p_2) \right\} \\
&= \bigcup_{\gamma_1 \in H_1, \gamma_2 \in H_2, p_1 \in H_1, p_2 \in H_2} \left\{ (1 - \log_r(1 + \frac{r-1}{(\frac{r-1}{r^{1-\gamma_1}-1})^{w_1}(\frac{r-1}{r^{1-\gamma_2}-1})^{w_2}}), p_1 p_2) \right\}.
\end{aligned}$$

Suppose (11) holds for $n = k$, that is

$$
\begin{aligned}
&PHFFWA(H_1, H_2, ..., H_k)\\
&= w_1 H_1 \oplus w_2 H_2 \oplus ... \oplus w_k H_k\\
&= \bigcup_{\gamma_i \in H_i, p_i \in H_i} \left\{ (h^{-1}(\sum_{i=1}^{k} w_i h(\gamma_i)), \prod_{i=1}^{k} p_i) \right\}\\
&= \bigcup_{\gamma_i \in H_i, p_i \in H_i} \left\{ (1 - \log_r(1 + \frac{r-1}{\prod_{i=1}^{k} (\frac{r-1}{r^{1-\gamma_i}-1})^{w_i}}), \prod_{i=1}^{k} p_i) \right\}.
\end{aligned}
$$

Then, when $n = k+1$, we have

$$
\begin{aligned}
&PHFFWA(H_1, H_2, ..., H_k, H_{k+1})\\
&= w_1 H_1 \oplus w_2 H_2 \oplus ... \oplus w_k H_k \oplus w_{k+1} H_{k+1}\\
&= \bigcup_{\gamma_i \in H_i, p_i \in H_i} \left\{ (h^{-1}(\sum_{i=1}^{k} w_i h(\gamma_i)), \prod_{i=1}^{k} p_i) \right\}\\
&\quad \oplus \bigcup_{\gamma_{k+1} \in H_{k+1}, p_{k+1} \in H_{k+1}} \left\{ (h^{-1}(w_{k+1} h(\gamma_{k+1}))), p_{k+1}) \right\}\\
&= \bigcup_{\gamma_i \in H_i, p_i \in H_i} \left\{ (h^{-1}(h(h^{-1}(\sum_{i=1}^{k} w_i h(\gamma_i))) + h(h^{-1}(w_{k+1} h(\gamma_{k+1})))), \prod_{i=1}^{k+1} p_i) \right\}\\
&= \bigcup_{\gamma_i \in H_i, p_i \in H_i} \left\{ (h^{-1}(\sum_{i=1}^{k} w_i h(\gamma_i)) + (w_{k+1} h(\gamma_{k+1}))), \prod_{i=1}^{k+1} p_i) \right\}\\
&= \bigcup_{\gamma_i \in H_i, p_i \in H_i} \left\{ (h^{-1}(\sum_{i=1}^{k+1} w_i h(\gamma_i)), \prod_{i=1}^{k+1} p_i) \right\}\\
&= \bigcup_{\gamma_i \in H_i, p_i \in H_i} \left\{ (1 - \log_r(1 + \frac{r-1}{\prod_{i=1}^{k+1} (\frac{r-1}{r^{1-\gamma_i}-1})^{w_i}}), \prod_{i=1}^{k+1} p_i) \right\},
\end{aligned}
$$

i.e., (11) holds for $n = k+1$. Thus, (11) holds for all n. □

Now, let us discuss a special situation of the PHFFWA operator with respect to r.

Corollary 1 *If $r \to 1$, then the PHFFWA operator reduces to the probabilistic hesitant fuzzy weighted average (PHFWA) operator, which is defined as follows:*

$$
\begin{aligned}
&PHFWA(H_1, H_2, ..., H_n)\\
&= \bigcup_{\gamma_i \in H_i, p_i \in H_i} \left\{ (1 - \prod_{i=1}^{n} (1 - \gamma_i)^{w_i}, \prod_{i=1}^{n} p_i) \right\}. \quad (12)
\end{aligned}
$$

Proof By Definition 7, we have $h(t) = g(1\text{-}t) = \log(\frac{r-1}{r^{1-t}-1}), r > 1, 0 < t < 1$, then

$$
\begin{aligned}
\lim_{r\to 1} h(t) &= \lim_{r\to 1} \log(\frac{r-1}{r^{1-t}-1})\\
&= \lim_{r\to 1} -\log(\frac{r^{1-t}-1}{r-1}) = -\log(1-t).
\end{aligned}
$$

Similar to the proof of Theorem 3, we have

$$
\begin{aligned}
&PHFWA(H_1, H_2, ..., H_n)\\
&= \overset{n}{\underset{i=1}{\oplus}} w_i H_i = \bigcup_{\gamma_i \in H_i, p_i \in H_i} \left\{ (h^{-1}(\sum_{i=1}^{n} w_i h(\gamma_i)), \prod_{i=1}^{n} p_i) \right\}\\
&= \bigcup_{\gamma_i \in H_i, p_i \in H_i} \left\{ (1 - \prod_{i=1}^{n} (1 - \gamma_i)^{w_i}, \prod_{i=1}^{n} p_i) \right\}.
\end{aligned}
$$

□

Corollary 2 *Let $H_i (i = 1, 2, .., n)$ be a collection of PHFEs, then the membership degree dimension of the PHFFWA operator is monotonically decreasing with respect to r.*

Proof We prove the monotonicity of $h(t)$ and $h^{-1}(t)$ with respect to the parameter r, respectively.

Since $h(t) = \log(\frac{r-1}{r^{1-t}-1}), r > 1, 0 < t < 1$, then we have

$$
\begin{aligned}
\frac{\partial h(t)}{\partial r} &= \frac{r^{1-t}-1}{r-1} . \frac{(r^{1-t}-1) - (1-t) r^{-t} (r-1)}{(r^{1-t}-1)^2}\\
&= \frac{(1-t) r^{-t} + t r^{1-t} - 1}{(r-1)(r^{1-t}-1)}.
\end{aligned}
$$

Let $f_1(r) = (1-t) r^{-t} + t r^{1-t} - 1, r > 1, 0 < t < 1$, then $f_1'(r) = t(1-t)(r-1) r^{-t-1} > 0$. It follows that $f_1(r)$ is a monotonically increasing function. Thus, $f_1(r) = (1-t) r^{-t} + t r^{1-t} - 1 > f_1(1) = 0$.

As $r - 1 > 0, r^{1-t} - 1 > 0$, therefore, $\frac{\partial h(t)}{\partial r} = \frac{(1-t) r^{-t} + t r^{1-t} - 1}{(r-1)(r^{1-t}-1)} > 0$. Thus, $h(t)$ is monotonically increasing with respect to the parameter r.

In addition, since $h^{-1}(t) = 1 - g^{-1}(t) = 1 - \frac{\log(\frac{r-1+e^t}{e^t})}{\log r}, r > 1, 0 < t < 1$, then

$$
\begin{aligned}
\frac{\partial g^{-1}(t)}{\partial r} &= \frac{\frac{\log r}{r-1+e^t} - \frac{\log(r-1+e^t)-t}{r}}{(\log r)^2}\\
&= \frac{r \log r - (r-1+e^t)\log(r-1+e^t) + t(r-1+e^t)}{r(r-1+e^t)(\log r)^2}.
\end{aligned}
$$

Let $f_2(r) = r\log r - (r-1+e^t)\log(r-1+e^t) + t(r-1+e^t), r > 1, 0 < t < 1$, then

$$f_2'(r) = \log r - \log(r-1+e^t) + t = \log(\frac{r}{r-1+e^t}) + \log e^t = \log(\frac{re^t}{r-1+e^t}).$$

Since $re^t - (r+1-e^t) = (r-1)(e^t-1) > 0$, then $f_2'(r) > 0$. It follows that $f_2(r)$ is a monotonically increasing function.

So, $f_2(r) = r\log r - (r-1+e^t)\log(r-1+e^t) + t(r-1+e^t) > f_2(1)$ =0.

While $r(r-1+e^t)(\log r)^2 > 0$, thus $\frac{\partial g^{-1}(t)}{\partial r} > 0$. Therefore, $h^{-1}(t) = 1 - g^{-1}(t)$ is a monotonically decreasing function with respect to r.

Because $h(t)$ is monotonically increasing and $h^{-1}(t)$ is monotonically decreasing with respect to r, we can deduce $h^{-1}(\sum_{i=1}^{n} w_i h(\gamma_i)) = 1 - \log_r(1 + \frac{r-1}{\prod_{i=1}^{n}(\frac{r-1}{r^{1-\gamma_i}-1})^{w_i}})$ is monotonically decreasing with respect to r, which completes the proof. □

4.2 Probabilistic hesitant fuzzy Frank weighted geometric operator

Definition 14 Let $H_i(i = 1, 2, .., n)$ be a collection of PHFEs, $w = (w_1, w_2, ..., w_n)^T$ be the weight vector of $H_i(i = 1, 2, ..., n)$ with $w_i \in [0, 1], i = 1, 2, \cdots, n$ and $\sum_{i=1}^{n} w_i = 1$. Then, the probabilistic hesitant fuzzy Frank weighted geometric (PHFFWG) operator is defined as follows:

$$PHFFWG(H_1, H_2, ..., H_n) = H_1^{w_1} \otimes H_2^{w_2} \otimes \cdots \otimes H_n^{w_n}. \quad (13)$$

Theorem 4 *Let $H_i(i = 1, 2, ..., n)$ be a collection of PHFEs, then the aggregated value by the PHFFWAG operator is also a PHFE, and*

$$PHFFWG(H_1, H_2, ..., H_n) = \bigcup_{\gamma_i \in H_i, p_i \in H_i} \left\{ (\log_r(1 + \frac{r-1}{\prod_{i=1}^{n}(\frac{r-1}{r^{\gamma_i}-1})^{w_i}}), \prod_{i=1}^{n} p_i) \right\}. \quad (14)$$

Similar to the proof of Theorems 3 and 4 can be obtained.

Now, let us discuss a special situation of the PHFFWG operator with respect to r.

Corollary 3 *If $r \to 1$, then the PHFFWG operator reduces to the probabilistic hesitant fuzzy weighted geometric (PHFWG) operator, which is defined as follows.*

$$PHFWG(H_1, H_2, ..., H_n) = \bigcup_{\gamma_i \in H_i, p_i \in H_i} \left\{ (\prod_{i=1}^{n} (\gamma_i)^{w_i}, \prod_{i=1}^{n} p_i) \right\}. \quad (15)$$

The proof of Corollary 3 is similar to that of Corollary 1.

Corollary 4 *Let $H_i(i = 1, 2, .., n)$ be a collection of PHFEs. Then, the membership dimension of the PHFFWG operator is monotonically increasing with respect to r.*

Proof We prove the monotonicity of $g(t)$ and $g^{-1}(t)$ with respect to r, respectively.

Since $g(t) = \log(\frac{r-1}{r^t-1}), r > 1, 0 < t < 1$, we have

$$\frac{\partial g(t)}{\partial r} = \frac{r^t-1}{r-1} \cdot \frac{(r^t-1) - tr^{t-1}(r-1)}{(r^t-1)^2} = \frac{(1-t)r^t + tr^{t-1} - 1}{(r-1)(r^t-1)}.$$

Let $f_1(r) = (1-t)r^t + tr^{t-1} - 1, r > 1, 0 < t < 1$, then $f_1'(r) = t(1-t)(r-1)r^{t-2} > 0$, which means that $f_1(r)$ is a monotonically increasing function, and we have $f_1(r) = (1-t)r^t + tr^{t-1} - 1 > f_1(1)$ =0. Since $r - 1 > 0, r^t - 1 > 0$, then $\frac{\partial g(t)}{\partial r} = \frac{(1-t)r^t+tr^{t-1}-1}{(r-1)(r^t-1)} > 0$. Therefore, $g(t)$ is monotonically increasing with respect to r.

As is demonstrated in corollary 2, $g^{-1}(t)$ is monotonically increasing with respect to r. Because $g(t)$ and $g^{-1}(t)$ are both monotonically increasing with respect to r, we can deduce $g^{-1}(\sum_{i=1}^{n} w_i g(\gamma_i)) = \log_r \left(1 + (r-1)\left(\prod_{i=1}^{n}(\frac{r-1}{r^{\gamma_i}-1})^{w_i}\right)^{-1}\right)$ is monotonically increasing with respect to r. This completes the proof of Corollary 4. □

4.3 Relationship between PHFFWA operator and PHFFWG operator

The aggregated value computed using the PHFFWA operator is always larger than or at least equal to that by the PHFFWG operator, which is demonstrated in the following.

Lemma 1 *Suppose that $H = \{(h_i, p_i)\}(i = 1, 2, ..., n)$ is a PHFE such that $0 \le h_i \le 1$, $\sum_{i=1}^{n} p_i = 1$, then the variance arithmetic-mean score function denoted by $S(H) = \bar{h}(1 - \delta^2)$ where $\bar{h} = \sum_{i=1}^{n} h_i p_i$ and $\sigma(h)^2 = \sum_{i=1}^{n}(h_i - \bar{h})^2 p_i$ is monotonically increasing with respect to h_i.*

Proof Let $F(h_1, h_2, ..., h_n) = S(H)$; it is clearly a multivariate function. For any variable h_i such that $h_i \geq \bar{h}$, we have

$$\begin{aligned}\frac{\partial F}{\partial h_i} &= p_i(1-\sum\nolimits_{i=1}^{n}(h_i-\bar{h})^2 p_i) - 2\bar{h}(h_i-\bar{h})(1-p_i)p_i \\ &= p_i(1-\sum\nolimits_{i=1}^{n}h_i^2 p_i - \sum\nolimits_{i=1}^{n}\bar{h}^2 p_i + 2\sum\nolimits_{i=1}^{n}h_i p_i \bar{h} \\ &\quad -2\bar{h}(h_i-\bar{h})(1-p_i)) \\ &\geq p_i(1-\sum\nolimits_{i=1}^{n}h_i^2 p_i + \bar{h}^2 - 2\bar{h}(h_i-\bar{h})) \\ &= p_i(1-\sum\nolimits_{i=1}^{n}h_i^2 p_i + \bar{h}^2 - 2\bar{h}(h_i-\bar{h})(1-p_i)) \\ &= p_i(1+3\bar{h}^2 - \sum\nolimits_{i=1}^{n}h_i^2 p_i - 2\bar{h}h_i) \\ &\geq p_i(1+3\bar{h}^2-3\bar{h}).\end{aligned}$$

For the function $W(x) = 3x^2 - 3x + 1$ such that $0 \leq x \leq 1$, it is known that $\partial W / \partial x < 0$ if $0 \leq x < 0.5$, $\partial W / \partial x = 0$ if $x = 0.5$, and $\partial W / \partial x > 0$ if $0.5 < x \leq 1$. Thus, we can infer that $W(x) \geq W(0.5) = 0.25$, thus deducing that $\partial F / \partial h_i \geq 0.25p > 0$.

On the other hand, for any variable h_i such that $h_i < \bar{h} \partial F / \partial h_i > 0$ clearly holds. As a whole, $F(h_1, ...h_n)$ is a monotonically increasing function with respect to h_i ($\forall h_i \in H$), which completes the proof. □

Theorem 5 *Given a probabilistic hesitant fuzzy decision matrix $Q = (H_{ij})_{m\times n}$ and attribute weight vector $w = (w_1, w_2, ..., w_n)^T$ for a MAGDM problem, the score of the aggregated value for PHFFWA $(H_{i1}, H_{i2}, ..., H_{in})$ is larger than that of PHFFWG $(H_{i1}, H_{i2}, ..., H_{in})$, i.e.,*

$$\begin{aligned}&S(PHFFWA(H_{i1}, H_{i2}, ..., H_{in})) \\ &\geq S(PHFFWG(H_{i1}, H_{i2}, ..., H_{in})),\end{aligned} \tag{16}$$

which is independent of r for $r \in (1, +\infty)$.

Proof Corollaries 2 and 4 indicate that S(PHFFWA(H_{i1}, H_{i2}, ..., H_{in})) and S(PHFFWG(H_{i1}, H_{i2}, ..., H_{in})) are monotonically decreasing and increasing with respect to r, respectively. Thus, (16) can be transformed into

$$\begin{aligned}&\lim_{r\to\infty} S(\mathrm{PHFFWA}(H_{i1}, H_{i2}, ..., H_{in})) \\ &\geq \lim_{r\to\infty} S(\mathrm{PHFFWG}(H_{i1}, H_{i2}, ..., H_{in})).\end{aligned}$$

This can be further converted into

$$\begin{aligned}&\lim_{r\to\infty}(1-\log_r(1+\frac{r-1}{\prod_{i=1}^{n}(\frac{r-1}{r^{1-\gamma_i}-1})^{w_i}})) \\ &\geq \lim_{r\to\infty}(\log_r(1+\frac{r-1}{\prod_{i=1}^{n}(\frac{r-1}{r^{\gamma_i}-1})^{w_i}})).\end{aligned}$$

We can reason from L'Hopital's rule that

$$\begin{aligned}I_1 &= \lim_{r\to+\infty}(1-\log_r(1+\frac{r-1}{\prod_{i=1}^{n}(\frac{r-1}{r^{1-\gamma_i}-1})^{w_i}})) \\ &\geq \lim_{r\to+\infty}(1-\log_r(1+\frac{r}{\prod_{i=1}^{n}(\frac{r}{r^{1-\gamma_i}})^{w_i}})) \\ &= \lim_{r\to\infty}(1-\log_r(1+r^{1-\sum_{i=1}^{n}w_i\gamma_i})) \\ &= \lim_{r\to+\infty}\frac{\log\frac{r}{(1+r^{1-\sum_{i=1}^{n}w_i\gamma_i})}}{\log r} \\ &= \lim_{r\to+\infty}\frac{1+r^{1-\sum_{i=1}^{n}\gamma_i w_i}}{r} \\ &\quad \cdot\frac{1+r^{\sum_{i=1}^{n}w_i\gamma_i} - r(1-\sum_{i=1}^{n}w_i\gamma_i)r^{\sum_{i=1}^{n}w_i\gamma_i}}{(1+r^{\sum_{i=1}^{n}w_i\gamma_i})^2} \\ &= \lim_{\theta\to\infty}\frac{\sum_{i=1}^{n}w_i\gamma_i(1-\sum_{i=1}^{n}\gamma_i w_i)r^{-\sum_{i=1}^{n}w_i\gamma_i}}{(1-\sum_{i=1}^{n}\gamma_i w_i)r^{-\sum_{i=1}^{n}w_i\gamma_i}} \\ &= \sum\nolimits_{i=1}^{n}\gamma_i w_i, \\ I_2 &= \lim_{r\to+\infty}(\log_r(1+\frac{r-1}{\prod_{i=1}^{n}(\frac{r-1}{r^{\gamma_i}-1})^{w_i}})) \\ &\leq \lim_{r\to+\infty}(\log_r(1+\frac{r}{\prod_{i=1}^{n}(\frac{r}{r^{\gamma_i}})^{w_i}})) \\ &= \lim_{r\to+\infty}(\log_r(1+r^{\sum_{i=1}^{n}w_i\gamma_i})) \\ &= \lim_{r\to+\infty}\frac{\log(1+r^{\sum_{i=1}^{n}w_i\gamma_i})}{\log r} \\ &= \lim_{r\to+\infty}\frac{\frac{\sum_{i=1}^{n}w_i\gamma_i r^{\sum_{i=1}^{n}w_i\gamma_i-1}}{1+r^{\sum_{i=1}^{n}w_i\gamma_i}}}{\frac{1}{r}} \\ &= \lim_{r\to\infty}\frac{\sum_{i=1}^{n}w_i\gamma_i r^{\sum_{i=1}^{n}w_i\gamma_i}}{1+r^{\sum_{i=1}^{n}w_i\gamma_i}} \\ &= \sum\nolimits_{i=1}^{n}\gamma_i w_i.\end{aligned}$$

As a result, $I_1 \geq I_2$. The conclusion in (16) is verified for any $r \in (1, +\infty)$. □

5 A MAGDM approach with probabilistic hesitant fuzzy information

In this section, the probabilistic hesitant fuzzy Frank aggregation operators and the variance arithmetic-mean score function are used to address the MAGDM problem in a probabilistic hesitant fuzzy information environment.

The MAGDM problem in probabilistic hesitant fuzzy information can be described as follows:

Let $X = \{x_1, x_2, ..., x_m\}$ be a discrete set of alternatives, $C = \{c_1, c_2, ..., c_n\}$ be a collection of n attributes with the weight vector $w = (w_1, w_2, ..., w_n)^T$ such that $w_j \in [0, 1], (j = 1, 2, ..., n)$ and $\sum_{i=1}^{n} w_i = 1$. Let $A = (H_{ij})_{m\times n}$ be a probabilistic hesitant fuzzy decision matrix, where $H_{ij} = \bigcup_{\gamma_{ij}\in h_{ij}}\{(\gamma_{ij}, p_{ij})\}$ is a PHFE, which is a set of all the possible values that the alternative $x_i \in X$ can take with respect to attribute $c_j \in C$ according to the decision maker.

Normally, there are both benefit attributes and cost attributes in MAGDM problems. We utilize the probabilistic hesitant fuzzy operational laws to transform the cost attributes into benefit attributes. For a probabilistic hesitant fuzzy decision matrix $Q = (H_{ij})_{m\times n}$, we have $H_{ij} = \begin{cases} H_{ij}, \text{for benefit attribute } \mathrm{c_j} \\ (H_{ij})^c, \text{for cost attribute } \mathrm{c_j} \end{cases}, i = 1, 2, ..., m, j = 1, 2, ..., n$.

Then, we utilize the PHFFWA (or PHFFWG) operator and the variance arithmetic- mean score function to propose an approach for MAGDM, where the following steps are involved.

Step 1 The decision makers give their evaluations of alternative $x_i \in X$ with respect to attributes $c_j \in C$, which are expressed by PHFEs $H_{ij} (i = 1, 2, ..., m, j = 1, 2, ..., n)$.

Step 2 Transform the decision matrix $Q = (H_{ij})_{m\times n}$ into a normalized matrix $\widehat{Q} = [\widehat{H}_{ij}]_{m\times n}$ by using the complementary operation of the PHFEs (see Definition 10).

Step 3 Aggregate all probabilistic hesitant fuzzy values $\widehat{H}_{ij} (j = 1, 2, ..., n)$ using the PHFFWA operator (or PHFFWG operator).

Step 4 Utilize the variance arithmetic-mean score function in Definition 12 to calculate the score $S(H_i)$ $(i = 1, 2, ..., m)$.

Step 5 Rank all alternatives $x_i (i = 1, 2, ..., m)$ in terms of $S(H_i)$ $(i = 1, 2, ..., m)$ in Step 4 and select the best one.

6 Application of the proposed approach in efficiency evaluation problem

In this section, to show the applicability of the proposed approach, we test our approach on the efficiency evaluation problem, and then we compare our method with the existing method of Zhang et al. [43].

There are six public companies to be investigated and evaluated with regards to the transformation efficiency of science and technological achievements: the Sun Create Electronic Company (A_1), the Changan Automobile Company (A_2), Westone Information Industry Inc. (A_3), the East China Computer Company (A_4), the GCI Science and Technology Company (A_5) and the Lida Optical and Electronic Company (A_6). With the help of ten experts from the commission, government and a collaborative university, the decision maker evaluated these companies based on seven attributes. Two of them were input attributes, the input of technical staff (c_1) and research spending (c_2). The others were output attributes, the number of invention patents (c_3), sales revenue of new products (c_4), the labor productively level (c_5), promotion of employment (c_6), and waste disposal rates (c_7). After studying documents concerning the seven attributes during the evaluation the process, the decision maker specified that $w = (0.1, 0.12, 0.15, 0.18, 0.16, 0.15, 0.14)^T$.

Step 1: The decision maker evaluates the six companies $A_i (i = 1, 2, 3, 4, 5, 6)$ based on the attributes $c_j (j = 1, 2, 3, 4, 5, 6, 7)$ and constructs a probabilistic hesitant fuzzy decision matrix $Q = [H_{ij}]_{m\times n} = [\bigcup (h_{ij}, p_{ij})]_{m\times n}$. The decision matrix $Q = [H_{ij}]_{6\times 7}$ is transformed into a normalized matrix $\widehat{Q} = [\widehat{H}_{ij}]_{6\times 7}$, which is shown in Table 1.

Step 2: The decision maker specifies $r = 2$ based on his preference. By using the BHFFWA operator in (9), the aggregated assessments of the six companies $A_i (i = 1, 2, 3, 4, 5, 6)$ are produced but are not presented to save space.

Step 3: The following can be derived from the aggregated assessments of the six companies and Definition 12:

Step 4: A rank-order of the six companies can be deduced from $S(\widehat{H}_i)(i = 1, 2, \cdots, 6)$ and Definition 11, $A_2 \succ A_3 \succ A_5 \succ A_1 \succ A_6 \succ A_4$ to determine that Changan Automobile Company (A_2) is the best one.

Similarly, under the condition of $r = 2$, and using the PHFFWG operator in (11), the application of the PHFFWG operator leads to $S(\widehat{H}_i)(i = 1, 2, 3, 4, 5, 6) = (0.5514, 0.5077, 0.5617, 0.5284, 0.5664, 0.4721)$. The results of the evaluation are converted into $A_2 \succ A_3 \succ A_5 \succ A_1 \succ A_6 \succ A_4$ to determine the Changan Automobile Company (A_2) is the best one, which is consistent with the results generated

Table 1 The normalized matrix

$S(\widehat{H}_1) = 0.6025$, $S(\widehat{H}_2) = 0.6964$, $S(\widehat{H}_3) = 0.6437$,
$S(\widehat{H}_4) = 0.5977$, $S(\widehat{H}_5) = 0.6216$, $S(\widehat{H}_6) = 0.6024$.

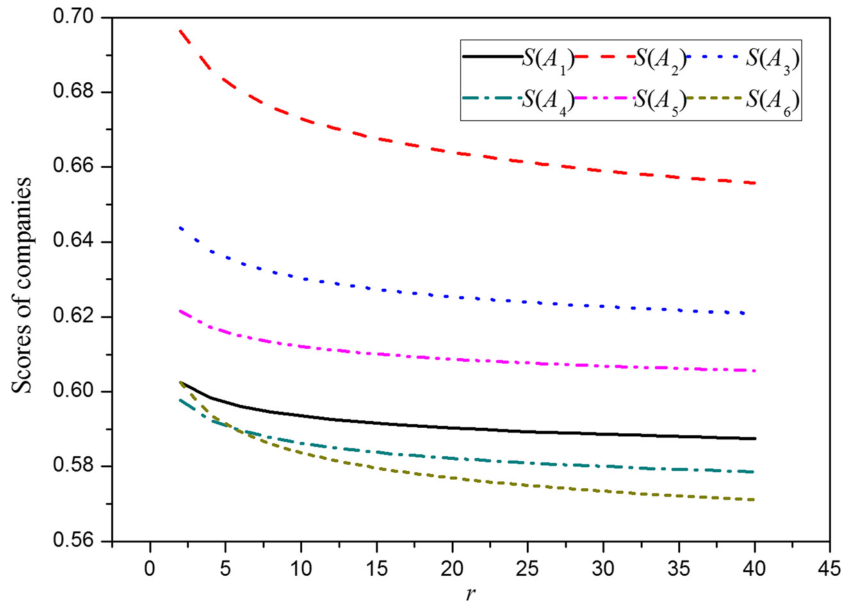

Fig. 1 The arithmetic scores of the six companies with a function of r

by the PHFFWA operator. Moreover, the scores of the six companies are smaller. The causes of the difference will be analyzed in the next subsection.

In the process of generating the above results, the parameter r in (9) and (12) was determined by the preference of the decision maker. When different assessments are provided by the decision maker, different values for r will be used. This may further result in a change to the solution to the problem. The arithmetic and geometric score curves of the six companies with respect to r are plotted in Figs. 1 and 2, respectively.

Figure 1 reveals that the arithmetic scores of the companies decrease as the value of r increases from 2 to 40 with a step size of 0.5, which demonstrates Corollary 2. Figure 2 reveals that the geometric scores of the companies increase as the value of r increases from 2 to 80 with a step size of 0.5, which validates Corollary 4. Furthermore, the scores of the arithmetic aggregated assessment are larger than those determined by the geometric operator no matter what the value r is, which validates Theorem 5.

With the increase of r, which represents the preference of the decision maker, the rank-ordering of the six companies may be different. However, when r is large enough, such as 100, the rank-ordering becomes fixed with respect to both the arithmetic scores and the geometric scores of the six companies change only slightly. Furthermore, since

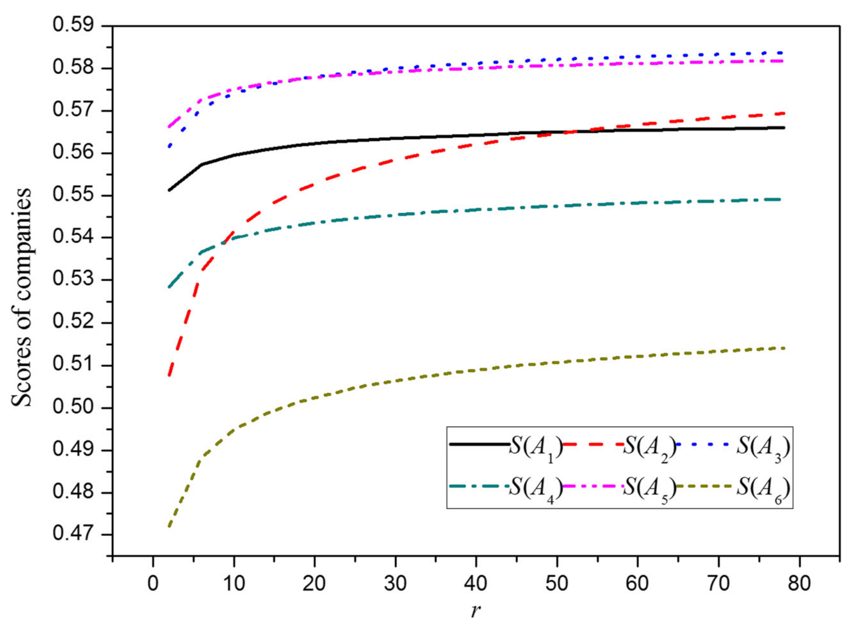

Fig. 2 The geometric scores of the six companies as a function of r

there are fewer intersections in Fig. 1 compared with Fig. 2, the solutions computed using the BHFFWA operator can be considered to be more stable than those computed using the BHFFWG operator with increasing r.

In what follows, we apply the PHFWA operator proposed by Zhang et al. [43] to address the aforementioned problem. The ranking result and effectiveness will be compared with that determined by our proposed MAGDM approach. The following steps are involved:

Step 1': See Step 1.

Step 2': Utilize the PHFWA operator proposed by Zhang et al. [43] to derive the overall BHFEs $\widehat{H}_i (i = 1, 2, \cdots, 6)$ of the six public companies A_i, which are also not presented to save space.

Step 3': Based on the above overall BHFEs, $\widehat{H}_i (i = 1, 2, \cdots, 6)$ and Definition 12, we can calculate the score functions $S(\widehat{H}_i)(i = 1, 2, \cdots, 6)$ as follows:

$$S(\widehat{H}_1) = 0.5916, S(\widehat{H}_2) = 0.6801, S(\widehat{H}_3) = 0.6677,$$
$$S(\widehat{H}_4) = 0.5745, S(\widehat{H}_5) = 0.6039, S(\widehat{H}_6) = 0.5930.$$

Step 4': Since $S(\widehat{H}_2) > S(\widehat{H}_3) > S(\widehat{H}_5) > S(\widehat{H}_6) > S(\widehat{H}_1) > S(\widehat{H}_4)$, then the ranking of all the public companies $A_i (i = 1, 2, \cdots, 6)$ is $A_2 \succ A_3 \succ A_5 \succ A_6 \succ A_1 \succ A_4$, and the most desirable public company is the Changan Automobile Company A_2.

Through the above example, we find that compared with the method developed by Zhang et al. [43] to rank the companies and determine the most desirable company, our proposed approach has some advantages.

(1) Comparing the six public company rankings derived by our approach with that of Zhang et al. [43], we find that the method of Zhang et al. [43] determined a slightly different ranking of these companies. However, reviewing the original probabilistic hesitant fuzzy decision matrix $\widehat{Q} = [\widehat{H}_{ij}]_{6\times7}$, we find that most of the attribute values of company A_1 are greater than A_6, which means that the A_1 is preferable to A_6. Therefore, our MAGDM approach is more rational than that of Zhang et al. [43] in this case.

(2) In addition, we can consider a wide range of specific operators, including the PHFFWA operator and the PHFFWG operator. The parameter r can be viewed as a measure of the decision maker's attitude; therefore, when people make a decision, they can choose a different parameter r depending on their attitude towards risk.

7 Conclusion

In this paper, we introduced the concept of PHFS. In the context of PHFSs based on the Frank t-conorm and t-norm, some operational laws were presented. The score functions of the PHFEs were designed to compare PHFEs. We developed some probabilistic hesitant fuzzy Frank aggregation operators, including a PHFFWA operator and a PHFFWG operator. In particular the monotonicity of the PHFFWA operator and the PHFFWG operator with respect to the parameter in Frank t-conorm and t-norm and their relationship were discussed A new method for approaching the MAGDM problem under probabilistic hesitant fuzzy information environment was developed. This method confirms the DMs' original decision based on information to the greatest extent possible. A numerical example was provided to illustrate that the method is both valid and practical in dealing with efficiency evaluation problems.

In the future, we plan to generalize the proposed probabilistic hesitant fuzzy Frank aggregation operators to an interval-valued PHFS environment, and investigate novel operators. It is worth further research to construct the GDM with granular computing techniques. Further research may provide practical applications in many fields including digital libraries, mobile internet, recommendation systems, supplier selection, pattern recognition and medical diagnosis.

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