Appl Intell (2018) 48:1506–1518
DOI 10.1007/s10489-017-1018-9

A new distance between BPAs based on the power-set-distribution pignistic probability function

Jingwei Zhu[1] · Xiaodan Wang[1] · Yafei Song[1]

Published online: 22 August 2017

Abstract In Dempster-Shafer evidence theory, the pignistic probability function is used to transform the basic probability assignment (BPA) into pignistic probabilities. Since the transformation is from the power set of the frame of discernment to the set itself, it may cause some information loss. The distance between betting commitments is constructed on the basis of the pignistic probability function and is used to measure the dissimilarity between two BPAs. However, it is a pseudo-metric and it may bring unreasonable results in some cases. To solve such problem, we propose a power-set-distribution (PSD) pignistic probability function based on the new explanation of the non-singleton focal elements in the BPA. The new function is directly operated on the power set, so it takes more information contained in the BPA than the pignistic probability function does. Based on the new function, the distance between PSD betting commitments which can better measure the dissimilarity between two BPAs is also proposed, and the proof that it is a metric is provided. In order to demonstrate the performance of the new distance, numerical examples are given to compare it with three existing dissimilarity measures. Moreover, its applications in combining the conflicting BPAs are also presented through two examples.

✉ Xiaodan Wang
afeu_wang@163.com

Jingwei Zhu
jingwei_zhukgd@163.com

Yafei Song
yafei_song@163.com

[1] Air Force Engineering University, Xi'an 710051, People's Republic of China

Keywords Dempster-Shafer evidence theory · Pignistic probability function · Distance between betting commitments · Power-set-distribution (PSD) pignistic probability function · Distance between PSD betting commitments

1 Introduction

Dempster-Shafer evidence theory is widely used in uncertainty reasoning and decision making [1, 2]. Two BPAs derived from distinct sources with a high similarity can be combined by using Dempster's combination rule and the result is always reasonable. However, if they have a high dissimilarity, a counterintuitive result which is unwished will appear [3, 4]. This leaves us with the question of how to determine whether there is a high dissimilarity between two BPAs or not.

For a long time, the conflict coefficient denoted by k [2] in Dempster's combination rule has been regarded as the only way to quantify the dissimilarity between two BPAs, which is the mass of the combined BPA assigned to the empty set before normalization. However, it is inappropriate to use k as a quantitative measure of dissimilarity. In [5], a research is particularly made to point out the weakness of k, indicating that not all high values of k represent a high dissimilarity between two BPAs. In recent years, researchers have proposed many dissimilarity measures. In [6, 7], a survey of the existing dissimilarity measures is provided together with a classification of them, including the composite distances [8, 9], the Minkowski family [10–13], the inner product family [14, 15], the fidelity family [16, 17], the information-based distances [18, 19] and the two-dimensional distances [5].

Among all the dissimilarity measures, the distance between betting commitments proposed in [10] is the maximum difference between the pignistic probabilities transformed from two BPAs. In [5], it is used together with the conflict coefficient k to construct a two-dimensional measure to quantify the dissimilarity between two BPAs. Four conditions in which Dempster's combination rule should be used are defined. As the distance between betting commitments and the conflict coefficient k are complementary in character, the performance of the two-dimensional measure in guiding the use of Dempster's combination rule is feasible. However, the shortcomings in the distance between betting commitments should not be ignored. It is judged as a pseudo-metric in [6] because of its violation of the standard metric properties. In [20], some unreasonable results obtained from it are pointed out and it is suggested to be cautiously used. The essential component of the distance between betting commitments is the pignistic probability function proposed in [22], which is progressively used as a probability measure for decision making in recent years [23–25]. The pignistic probability function is used to transform the BPA into pignistic probabilities. Since the transformation is from the power set to the set, some information contained in the BPA is lost. The loss of information may be a cause of the shortcomings in the distance between betting commitments. In order to make use of the information contained in the BPA as much as possible, a distance directly defined based on the power set is proposed by Jousselme et al. in [11]. As it takes more information contained in the BPA than many other measures, its performance is satisfactory in most cases, and it is widely used in quantifying the dissimilarity between two BPAs.

In this paper, the PSD pignistic probability function is proposed, which can be regarded as an improvement of the pignistic probability function. It is used to transform the BPA into pignistic probabilities on the power set. And based on it, the distance between PSD betting commitments is proposed. The new distance is a metric and the relevant proof is provided. Like the distance proposed by Jousselme et al, the new distance works over the power set and takes more information contained in the BPA. Numerical examples confirm that the results obtained from it are more reasonable.

The rest of the paper is organized as follows. In Section 2, the basic definitions in Dempster-Shafer evidence theory and the standard metric properties are reviewed, the property that the distance between betting commitments doesn't satisfy is also indicated. In Section 3, the definitions of the PSD pignistic probability function and the distance between PSD betting commitments are proposed. The proof that the proposed distance is a metric one is provided through mathematical reasoning. In Section 4, the rationality of the new distance is demonstrated by comparing it with three existing dissimilarity measures. In Section 5, its applications in combining the conflicting BPAs are presented through two examples. In Section 6, the main contribution of this paper is concluded.

2 Background

The background knowledge presented in this section involves the following three main points: (1) the basic definitions which are frequently used in Dempster-Shafer evidence theory, (2) the geometrical interpretation of the BPA, on the basis of which the dissimilarity between two BPAs can be quantified by various distances, (3) the definition of the distance between betting commitments, and one of the standard metric properties it doesn't satisfy.

It should be noted that the BPAs in this paper are always assumed to be derived from distinct sources.

2.1 Basics of Dempster-Shafer evidence theory

In Dempster-Shafer evidence theory, the notation Ω is used to denote the frame of discernment, which is a non-empty set with mutually exclusive and exhaustive elements. The power set of Ω, denoted by 2^{Ω}, consists of all the $2^{|\Omega|}$ subsets of Ω.

Definition 2.1 (Basic probability assignment) [1]. Let Ω be a frame of discernment, the subset of which is denoted by A. The function $m : 2^{\Omega} \rightarrow [0, 1]$ is called a basic probability assignment (BPA) if it satisfies the two conditions as follows:

$$m(\emptyset) = 0 \text{ and } \sum_{A \subseteq \Omega} m(A) = 1 \tag{1}$$

For $\forall A \subseteq \Omega$, $m(A)$ is the basic probability mass of A, which reflects the support degree that the evidence gives to A. If $m(A) > 0$, A is called the focal element of Ω.

According to Definition 2.1, three one-to-one corresponding functions were defined by G. Shafer [2], including the belief function denoted by $Bel(A)$, the plausibility function denoted by $Pl(A)$ and the commonality function denoted by $Q(A)$. The calculation formulas of them are as follows:

$$\begin{aligned} Bel(A) &= \sum_{B \subseteq A} m(B) \\ Pl(A) &= \sum_{B \cap A \neq \emptyset} m(B) \\ Q(A) &= \sum_{A \subseteq B} m(B) \end{aligned} \tag{2}$$

Definition 2.2 (Dempster's combination rule) [1]. Let m_1 and m_2 be two BBAs on the frame of discernment Ω. The combined BPA, denoted by $m_{1\oplus 2}$, is defined as:

$$m_{1\oplus 2}(A) = \begin{cases} \frac{1}{1-k} \sum\limits_{B\cap C=A} m_1(B)m_2(C),\ A \neq \emptyset \\ 0,\ A = \emptyset \end{cases} \tag{3}$$

with

$$k = \sum_{B\cap C=\emptyset} m_1(B)m_2(C) \tag{4}$$

where k is the mass of the combined BPA assigned to the empty set before normalization, called the conflict coefficient.

Definition 2.3 (Pignistic probability function) [21, 22]. Let $\Omega = \{\theta_1, \theta_2, ..., \theta_n\}$ be a frame of discernment, $\theta_i (1 \leq i \leq n)$ is the singleton element which belongs to Ω, m is a BPA on Ω. The corresponding pignistic probability function $BetP_m : \Omega \to [0, 1]$ of m is defined as:

$$BetP_m(\theta_i) = \sum_{A\subseteq\Omega} m(A)\frac{|\theta_i \cap A|}{|A|} = \sum_{\theta_i\in A, A\subseteq\Omega} \frac{m(A)}{|A|} \tag{5}$$

where $A \neq \emptyset$ and $|A|$ is the cardinality of A. For $B \subseteq \Omega$,

$$BetP_m(B) = \sum_{A\subseteq\Omega} m(A)\frac{|B\cap A|}{|A|} \ \ or \ \ BetP_m(B) = \sum_{\theta_i\in B} BetP_m(\theta_i) \tag{6}$$

The pignistic probability function transforms the BPA into pignistic probabilities. Since the transformation is from the power set of Ω to the set itself, some information is lost. The computational process of the transformation can be divided into two steps as follows.

Step 1: Distribute $m(A)$ averagely to the singleton elements θ_j which belong to A, add up the values that θ_j acquires to get $BetP_m(\theta_j)$;
Step 2: For $\theta_k \in B$, add up $BetP_m(\theta_k)$ to get $BetP_m(B)$.

2.2 Geometrical interpretation of the BPA

The geometrical interpretation of the BPA makes it possible to measure the dissimilarity between two BPAs via a distance, which is described clearly in [26, 27].

Definition 2.4 (Geometrical interpretation of the BPA). Let Ω be a frame of discernment, m is a BPA on Ω, Ψ is the vector space which is generated from the subsets of Ω. The corresponding vector of m in Ψ can be defined as:

$$\vec{\boldsymbol{m}} = [m(A_1), m(A_2), ..., m(A_{2^{|\Omega|}})]^T \tag{7}$$

where $A_i \subseteq \Omega, i = 1, 2, ..., 2^{|\Omega|}, \sum_{i=1}^{2^{|\Omega|}} m(A_i) = 1$.

After transforming two BPAs into vectors, researchers can use different kinds of distances to measure the dissimilarity between them.

Based on the satisfaction degree of the standard metric properties, Jousselme and Maupin [6] judged the existing distances into four groups: metric, pseudo-metric, semi-metric and non-metric. The standard metric properties are as follows:

Definition 2.5 (Standard metric properties). Let Ψ be a vector space. If the function $d : \Psi \times \Psi \to R$ satisfies the following properties for $\forall A, B, C \in \Psi$, it is called a metric.

(p1) Non-negativity: $d(A, B) \geq 0$;
(p2) Symmetry: $d(A, B) = d(B, A)$;
(p3) Definiteness: $d(A, B) = 0 \Leftrightarrow A = B$;
(p4) Triangle inequality: $d(A, B) \leq d(A, C) + d(B, C)$.

In [6], (p3) is divided into (p3)′ and (p3)″, which are as follows:

(p3)′ Reflexivity: $d(A, A) = 0$;
(p3)″ Separability: $d(A, B) = 0 \Rightarrow A = B$.

If a function d satisfies all the properties except for (p3)″, it is called a pseudo-metric. A semi-metric satisfies all the conditions except for (p4). If d doesn't satisfy (p1) and (p3)′, it is called a non-metric.

2.3 Distance between betting commitments

Definition 2.6 (Distance between betting commitments). Let m_1 and m_2 be two BBAs on the frame of discernment Ω, $BetP_{m_1}$ and $BetP_{m_2}$ are the corresponding pignistic probability functions of them, respectively. Then the distance between betting commitments is defined as:

$$difBetP_{m_1}^{m_2} = \max_{A\subseteq\Omega}(|BetP_{m_1}(A) - BetP_{m_2}(A)|) \tag{8}$$

Influenced by the pignistic probability function, the distance between betting commitments works over Ω. This leads to the loss of information contained in the BPA. We use $d_{Bet}(m_1, m_2)$ to denote $difBetP_{m_1}^{m_2}$ in further detail below.

d_{Bet} is judged as a pseudo-metric in [6], which means that it doesn't satisfies (p3)″. Here we cite an example to demonstrate it.

Example 2.1 Let m_1 and m_2 be two BPAs on the frame of discernment $\Omega = \{\theta_1, \theta_2, \theta_3\}$, which are defined as follows:

$$m_1(\{\theta_1\}) = \frac{1}{3},\ m_1(\{\theta_2\}) = \frac{1}{3},\ m_1(\{\theta_3\}) = \frac{1}{3};$$

$$m_2(\{\theta_1, \theta_2, \theta_3\}) = 1.$$

Then we have $d_{Bet}(m_1, m_2) = 0$. However, it is obvious that $m_1 \neq m_2$. This example indicates that d_{Bet} doesn't satisfy (p3)$''$.

As the proof that d_{Bet} satisfies the other properties is similar with the proposed distance in Section 3, it will not be listed here in addition.

3 The proposed function and distance

3.1 Power-set-distribution pignistic probability function

Example 3.1 Let $m(\{\theta_1, \theta_2, \theta_3\}) = 1$ be a BPA on the frame of discernment $\Omega = \{\theta_1, \theta_2, \theta_3\}$, $\theta_i (1 \leq i \leq 3)$ is the singleton element in Ω. What's the information contained in the BPA?

The general explanation is as follows: Any of the singleton elements in Ω may have a support degree of 1. Here, the singleton elements refer to $\{\theta_1\}$, $\{\theta_2\}$, $\{\theta_3\}$. In [22], $BetP$ is proposed by Smets on the basis of this explanation. When $BetP$ is used to transform the BPA $m(\{\theta_1, \theta_2, \theta_3\}) = 1$ into pignistic probabilities, the value of $m(\{\theta_1, \theta_2, \theta_3\})$ is distributed equally among the singleton elements of $\{\theta_1, \theta_2, \theta_3\}$. Since the transformation is from the power set of Ω to the set itself, it leads to much information loss.

In order to reduce the information loss, the transformation should be modified to become a bijective mapping. One practical way is to distribute the m value of the non-singleton focal element among its non-empty subsets. In this way, an improved pignistic probability function comes up. It works over the power set of Ω, the information loss of which is less than $BetP$. In this case, how to explain the information contained in the BPA $m(\{\theta_1, \theta_2, \theta_3\}) = 1$? The explanation proposed by us is as follows: Any of the singleton and non-singleton elements may have a support degree of 1. Here, the singleton and non-singleton elements refer to $\{\theta_1\}$, $\{\theta_2\}$, $\{\theta_3\}$, $\{\theta_1, \theta_2\}$, $\{\theta_1, \theta_3\}$, $\{\theta_2, \theta_3\}$, $\{\theta_1, \theta_2, \theta_3\}$.

As the improved function works over the power set of Ω, we call it the power-set-distribution (PSD) pignistic probability function. The definition of it is as follows:

Definition 3.1 (PSD pignistic probability function). Let Ω be a frame of discernment, m is a BPA on Ω. The corresponding power-set-distribution (PSD) pignistic probability function $PBetP_m : 2^{\Omega} \to [0, 1]$ of m is defined as:

$$PBetP_m(B) = \sum_{A \subseteq \Omega} m(A) \frac{2^{|B \cap A|} - 1}{2^{|A|} - 1} \quad (9)$$

where B denotes a subset of Ω, $A \neq \emptyset$ and $|A|$ is the cardinality of set A.

3.2 Distance between PSD betting commitments

On the basis of Definition 3.1, the definition of the distance between PSD betting commitments is proposed as follows:

Definition 3.2 (Distance between PSD betting commitments). Let m_1 and m_2 be two BBAs on the frame of discernment Ω, $PBetP_{m_1}$ and $PBetP_{m_2}$ are the corresponding PSD pignistic probability functions of them, respectively. Then the distance between PSD betting commitments is defined as:

$$difPBetP_{m_1}^{m_2} = \max_{A \subseteq \Omega}(|PBetP_{m_1}(A) - PBetP_{m_2}(A)|) \quad (10)$$

Influenced by the PSD pignistic probability function, the new distance works over the power set of Ω. It takes more information contained in the BPA than d_{Bet}. We use $d_{PBet}(m_1, m_2)$ to denote $difPBetP_{m_1}^{m_2}$ in further detail below. d_{PBet} is most applicable to the BPAs which contain non-singleton focal elements. When d_{PBet} is applied to the BPAs which just contain singleton focal elements, the result obtained from it is consistent with that of d_{Bet}.

3.3 Proof of the properties

d_{PBet} is a metric as it satisfies all the standard metric properties. We will prove these properties in this section.

Property 1 (Non-negativity) $d_{PBet} \geq 0$;

Proof This property is easy to prove via (10).

$$d_{PBet}(m_1, m_2) = \max_{A \subseteq \Omega}(|PBetP_{m_1}(A) - PBetP_{m_2}(A)|) \geq 0$$

□

Property 2 (Symmetry) $d_{PBet}(m_1, m_2) = d_{PBet}(m_2, m_1)$;

Proof This is straightforward via (10).

$$\begin{aligned} d_{PBet}(m_1, m_2) &= \max_{A \subseteq \Omega}(|PBetP_{m_1}(A) - PBetP_{m_2}(A)|) \\ &= \max_{A \subseteq \Omega}(|PBetP_{m_2}(A) - PBetP_{m_1}(A)|) \\ &= d_{PBet}(m_2, m_1) \end{aligned}$$

□

Property 3$'$ (Reflexivity) $d_{PBet}(m_1, m_1) = 0$;

Proof This property can be proved via (10).

$$d_{PBet}(m_1, m_1) = \max_{A\subseteq\Omega}(|PBetP_{m_1}(A) - PBetP_{m_1}(A)|) = 0$$

□

Property 3″ (Separability) $d_{PBet}(m_1, m_2) = 0 \Rightarrow m_1 = m_2$;

Proof Suppose that m_1 and m_2 are two BPAs on the frame of discernment $\Omega = \{\theta_1, \theta_2, ..., \theta_n\}$, which are detailed in Table 1. $A_i \subseteq \Omega$ and $A_i \neq \emptyset$. The total number of A_i is $2^n - 1$. All of them are in sequential order.

If $d_{PBet}(m_1, m_2) = \max_{A\subseteq\Omega}(|PBetP_{m_1}(A) - PBetP_{m_2}(A)|) = 0$, we can get the following equations:

$$\begin{cases} PBetP_{m_1}(A_1) - PBetP_{m_2}(A_1) = 0 \\ PBetP_{m_1}(A_2) - PBetP_{m_2}(A_2) = 0 \\ \vdots \\ PBetP_{m_1}(A_{2^n-1}) - PBetP_{m_2}(A_{2^n-1}) = 0 \end{cases}$$

By (9), we have:

$$\begin{cases} \left(\frac{2^{|A_1\cap A_1|}-1}{2^{|A_1|}-1}x_1 + \frac{2^{|A_1\cap A_2|}-1}{2^{|A_2|}-1}x_2 + \cdots + \frac{2^{|A_1\cap A_{2^n-1}|}-1}{2^{|A_{2^n-1}|}-1}x_{2^n-1}\right) - \left(\frac{2^{|A_1\cap A_1|}-1}{2^{|A_1|}-1}y_1 + \frac{2^{|A_1\cap A_2|}-1}{2^{|A_2|}-1}y_2 + \cdots + \frac{2^{|A_1\cap A_{2^n-1}|}-1}{2^{|A_{2^n-1}|}-1}y_{2^n-1}\right) = 0 \\ \left(\frac{2^{|A_2\cap A_1|}-1}{2^{|A_1|}-1}x_1 + \frac{2^{|A_2\cap A_2|}-1}{2^{|A_2|}-1}x_2 + \cdots + \frac{2^{|A_2\cap A_{2^n-1}|}-1}{2^{|A_{2^n-1}|}-1}x_{2^n-1}\right) - \left(\frac{2^{|A_2\cap A_1|}-1}{2^{|A_1|}-1}y_1 + \frac{2^{|A_2\cap A_2|}-1}{2^{|A_2|}-1}y_2 + \cdots + \frac{2^{|A_2\cap A_{2^n-1}|}-1}{2^{|A_{2^n-1}|}-1}y_{2^n-1}\right) = 0 \\ \vdots \\ \left(\frac{2^{|A_{2^n-1}\cap A_1|}-1}{2^{|A_1|}-1}x_1 + \frac{2^{|A_{2^n-1}\cap A_2|}-1}{2^{|A_2|}-1}x_2 + \cdots + \frac{2^{|A_{2^n-1}\cap A_{2^n-1}|}-1}{2^{|A_{2^n-1}|}-1}x_{2^n-1}\right) - \left(\frac{2^{|A_{2^n-1}\cap A_1|}-1}{2^{|A_1|}-1}y_1 + \frac{2^{|A_{2^n-1}\cap A_2|}-1}{2^{|A_2|}-1}y_2 + \cdots + \frac{2^{|A_{2^n-1}\cap A_{2^n-1}|}-1}{2^{|A_{2^n-1}|}-1}y_{2^n-1}\right) = 0 \end{cases}$$

$$\begin{cases} \frac{2^{|A_1\cap A_1|}-1}{2^{|A_1|}-1}(x_1 - y_1) + \frac{2^{|A_1\cap A_2|}-1}{2^{|A_2|}-1}(x_2 - y_2) + \cdots + \frac{2^{|A_1\cap A_{2^n-1}|}-1}{2^{|A_{2^n-1}|}-1}(x_{2^n-1} - y_{2^n-1}) = 0 \\ \frac{2^{|A_2\cap A_1|}-1}{2^{|A_1|}-1}(x_1 - y_1) + \frac{2^{|A_2\cap A_2|}-1}{2^{|A_2|}-1}(x_2 - y_2) + \cdots + \frac{2^{|A_2\cap A_{2^n-1}|}-1}{2^{|A_{2^n-1}|}-1}(x_{2^n-1} - y_{2^n-1}) = 0 \\ \vdots \\ \frac{2^{|A_{2^n-1}\cap A_1|}-1}{2^{|A_1|}-1}(x_1 - y_1) + \frac{2^{|A_{2^n-1}\cap A_2|}-1}{2^{|A_2|}-1}(x_2 - y_2) + \cdots + \frac{2^{|A_{2^n-1}\cap A_{2^n-1}|}-1}{2^{|A_{2^n-1}|}-1}(x_{2^n-1} - y_{2^n-1}) = 0 \end{cases}$$

It is equivalent to the equation as follows:

$$\begin{bmatrix} \frac{2^{|A_1\cap A_1|}-1}{2^{|A_1|}-1} & \frac{2^{|A_1\cap A_2|}-1}{2^{|A_2|}-1} & \cdots & \frac{2^{|A_1\cap A_{2^n-1}|}-1}{2^{|A_{2^n-1}|}-1} \\ \frac{2^{|A_2\cap A_1|}-1}{2^{|A_1|}-1} & \frac{2^{|A_2\cap A_2|}-1}{2^{|A_2|}-1} & \cdots & \frac{2^{|A_2\cap A_{2^n-1}|}-1}{2^{|A_{2^n-1}|}-1} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{2^{|A_{2^n-1}\cap A_1|}-1}{2^{|A_1|}-1} & \frac{2^{|A_{2^n-1}\cap A_2|}-1}{2^{|A_2|}-1} & \cdots & \frac{2^{|A_{2^n-1}\cap A_{2^n-1}|}-1}{2^{|A_{2^n-1}|}-1} \end{bmatrix} \begin{bmatrix} x_1 - y_1 \\ x_2 - y_2 \\ \vdots \\ x_{2^n-1} - y_{2^n-1} \end{bmatrix} = 0$$

$$\begin{bmatrix} 2^{|A_1\cap A_1|} - 1 & 2^{|A_1\cap A_2|} - 1 & \cdots & 2^{|A_1\cap A_{2^n-1}|} - 1 \\ 2^{|A_2\cap A_1|} - 1 & 2^{|A_2\cap A_2|} - 1 & \cdots & 2^{|A_2\cap A_{2^n-1}|} - 1 \\ \vdots & \vdots & \ddots & \vdots \\ 2^{|A_{2^n-1}\cap A_1|} - 1 & 2^{|A_{2^n-1}\cap A_2|} - 1 & \cdots & 2^{|A_{2^n-1}\cap A_{2^n-1}|} - 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2^{|A_1|}-1} & 0 & \cdots & 0 \\ 0 & \frac{1}{2^{|A_2|}-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{2^{|A_{2^n-1}|}-1} \end{bmatrix} \begin{bmatrix} x_1 - y_1 \\ x_2 - y_2 \\ \vdots \\ x_{2^n-1} - y_{2^n-1} \end{bmatrix} = 0$$

Table 1 Two BPAs

	$A_1 = \{\theta_1\}$	$A_2 = \{\theta_2\}$	$\dots$	$A_{2^n-1} = \{\theta_1, \theta_2, ..., \theta_n\}$
m_1	x_1	x_2	$\dots$	x_{2^n-1}
m_2	y_1	y_2	$\dots$	y_{2^n-1}

We set

$$R = \begin{bmatrix} \frac{2^{|A_1 \cap A_1|}-1}{2^{|A_1|}-1} & \frac{2^{|A_1 \cap A_2|}-1}{2^{|A_2|}-1} & \cdots & \frac{2^{|A_1 \cap A_{2^n-1}|}-1}{2^{|A_{2^n-1}|}-1} \\ \frac{2^{|A_2 \cap A_1|}-1}{2^{|A_1|}-1} & \frac{2^{|A_2 \cap A_2|}-1}{2^{|A_2|}-1} & \cdots & \frac{2^{|A_2 \cap A_{2^n-1}|}-1}{2^{|A_{2^n-1}|}-1} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{2^{|A_{2^n-1} \cap A_1|}-1}{2^{|A_1|}-1} & \frac{2^{|A_{2^n-1} \cap A_2|}-1}{2^{|A_2|}-1} & \cdots & \frac{2^{|A_{2^n-1} \cap A_{2^n-1}|}-1}{2^{|A_{2^n-1}|}-1} \end{bmatrix}$$

$$S = \begin{bmatrix} 2^{|A_1 \cap A_1|} - 1 & 2^{|A_1 \cap A_2|} - 1 & \cdots & 2^{|A_1 \cap A_{2^n-1}|} - 1 \\ 2^{|A_2 \cap A_1|} - 1 & 2^{|A_2 \cap A_2|} - 1 & \cdots & 2^{|A_2 \cap A_{2^n-1}|} - 1 \\ \vdots & \vdots & \ddots & \vdots \\ 2^{|A_{2^n-1} \cap A_1|} - 1 & 2^{|A_{2^n-1} \cap A_2|} - 1 & \cdots & 2^{|A_{2^n-1} \cap A_{2^n-1}|} - 1 \end{bmatrix}$$

$$T = \begin{bmatrix} \frac{1}{2^{|A_1|}-1} & 0 & \cdots & 0 \\ 0 & \frac{1}{2^{|A_2|}-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{2^{|A_{2^n-1}|}-1} \end{bmatrix}$$

Then we can get $R = ST$ and $R \begin{bmatrix} x_1 - y_1 \\ x_2 - y_2 \\ \vdots \\ x_{2^n-1} - y_{2^n-1} \end{bmatrix} = 0.$

The positive definiteness of the Jaccard index matrix has been presented in [28] by M. Bouchard. S is similar to the Jaccard index matrix. Moreover, S is more complex, and it is really difficult to demonstrate its positive definiteness through mathematical proof. In the following, the results of S are listed when $n = 1, 2, 3, 4$. After being diagonalized, each of them becomes an identity matrix, which demonstrates that S is a non-singular matrix when $n = 1, 2, 3, 4$. As the dimension of S is 31×31 when $n = 5$ and it is still growing, there is no need to list the results of S when $n \geq 5$. However, it can be determined that the matrix is still an identity matrix after being diagonalized. The mathematical proof of the positive definiteness is needed in the future work.

For $n = 1$, $S = [1]$; For $n = 2$, $S = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 3 \end{bmatrix} \xrightarrow{diagonalized} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$;

$$\text{For } n = 3,\ S = \begin{bmatrix} 1&0&0&1&1&0&1 \\ 0&1&0&1&0&1&1 \\ 0&0&1&0&1&1&1 \\ 1&1&0&3&1&1&3 \\ 1&0&1&1&3&1&3 \\ 0&1&1&1&1&3&3 \\ 1&1&1&3&3&3&7 \end{bmatrix} \xrightarrow{diagonalized} \begin{bmatrix} 1&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0 \\ 0&0&1&0&0&0&0 \\ 0&0&0&1&0&0&0 \\ 0&0&0&0&1&0&0 \\ 0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&1 \end{bmatrix};$$

$$\text{For } n = 4,\ S = \begin{bmatrix} 1&0&0&0&1&1&1&0&0&0&1&1&1&0&1 \\ 0&1&0&0&1&0&0&1&1&0&1&1&0&1&1 \\ 0&0&1&0&0&1&0&1&0&1&1&0&1&1&1 \\ 0&0&0&1&0&0&1&0&1&1&0&1&1&1&1 \\ 1&1&0&0&3&1&1&1&1&0&3&3&1&1&3 \\ 1&0&1&0&1&3&1&1&0&1&3&1&3&1&3 \\ 1&0&0&1&1&1&3&0&1&1&1&3&3&1&3 \\ 0&1&1&0&1&1&0&3&1&1&3&1&1&3&3 \\ 0&1&0&1&1&0&1&1&3&1&1&3&1&3&3 \\ 0&0&1&1&0&1&1&1&1&3&1&1&3&3&3 \\ 1&1&1&0&3&3&1&3&1&1&7&3&3&3&7 \\ 1&1&0&1&3&1&3&1&3&1&3&7&3&3&7 \\ 1&0&1&1&1&3&3&1&1&3&3&3&7&3&7 \\ 0&1&1&1&1&1&1&3&3&3&3&3&3&7&7 \\ 1&1&1&1&3&3&3&3&3&3&7&7&7&7&15 \end{bmatrix} \xrightarrow{diagonalized} \begin{bmatrix} 1&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&1&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&1&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&1 \end{bmatrix}.$$

Since S is a non-singular matrix. The rank of S is $2^n - 1$, i.e., $rank(S) = 2^n - 1$. Obviously, $rank(T) = 2^n - 1$. As $R = ST$, we can get that $rank(R) = 2^n - 1$.

Because that $rank(R) = 2^n - 1$ and $R \begin{bmatrix} x_1 - y_1 \\ x_2 - y_2 \\ \vdots \\ x_{2^n-1} - y_{2^n-1} \end{bmatrix} = 0$, it is available that $\begin{bmatrix} x_1 - y_1 \\ x_2 - y_2 \\ \vdots \\ x_{2^n-1} - y_{2^n-1} \end{bmatrix} = 0$. That is to say: $x_1 = y_1, x_2 = y_2, \ldots, x_{2^n-1} = y_{2^n-1}$. It is equivalent to that $m_1 = m_2$. □

Property 4 (Triangle inequality)

$d_{PBet}(m_1, m_2) \le d_{PBet}(m_1, m_3) + d_{PBet}(m_3, m_2)$;

Proof Suppose that there are three BPAs on the frame of discernment Ω, which are denoted by m_1, m_2 and m_3. The corresponding $PBetP$ of the three BPAs are detailed in Table 2, where $A_i \subseteq \Omega (1 \le i \le n)$ and $A_i \ne \emptyset$.

Assume that,

$$\begin{aligned} d_{PBet}(m_1, m_2) &= \max_{A \subseteq \Theta}\left(\left|PBetP_{m_1}(A) - PBetP_{m_2}(A)\right|\right) \\ &= \left|PBetP_{m_1}(A_i) - PBetP_{m_2}(A_i)\right| = |x_i - y_i| \end{aligned}.$$

It is certain that the following two inequations are appropriate,

$$\begin{aligned} d_{PBet}(m_1, m_3) &= \max_{A \subseteq \Theta}\left(\left|PBetP_{m_1}(A) - PBetP_{m_3}(A)\right|\right) \\ &\ge \left|PBetP_{m_1}(A_i) - PBetP_{m_3}(A_i)\right| \\ &= |x_i - z_i| \end{aligned}$$

$$\begin{aligned} d_{PBet}(m_3, m_2) &= \max_{A \subseteq \Theta}\left(\left|PBetP_{m_3}(A) - PBetP_{m_2}(A)\right|\right) \\ &\ge \left|PBetP_{m_3}(A_i) - PBetP_{m_2}(A_i)\right| = |z_i - y_i| \end{aligned}$$

Then, we can get the inequation as follows,

$$\begin{aligned} d_{PBet}(m_1, m_3) + d_{PBet}(m_3, m_2) &\ge |x_i - z_i| + |z_i - y_i| \\ &\ge |x_i - z_i + z_i - y_i| \\ &= |x_i - y_i| = d_{PBet}(m_1, m_2) \end{aligned}$$

Table 2 The corresponding $PBetP$ of m_1, m_2 and m_3

	A_1	A_2	...	A_i	...	A_n
$PBetP_{m_1}(A_i)$	x_1	x_2	...	x_i	...	x_n
$PBetP_{m_2}(A_i)$	y_1	y_2	...	y_i	...	y_n
$PBetP_{m_3}(A_i)$	z_1	z_2		z_i		z_n

That is,

$d_{PBet}(m_1, m_2) \le d_{PBet}(m_1, m_3) + d_{PBet}(m_3, m_2)$. □

Through the mathematical proof above, it is certain that d_{PBet} is a metric.

4 Numerical comparisons

In [11], a distance between two BPAs is proposed, which is defined as:

$$d_J(m_1, m_2) = \sqrt{\frac{1}{2}\left(\vec{m}_1 - \vec{m}_2\right)^T \underline{\underline{D}} \left(\vec{m}_1 - \vec{m}_2\right)} \tag{11}$$

where $\vec{m}$ is a $2^{|\Omega|}$-dimensional column vector generated from a BPA, $\underline{\underline{D}}$ is a $2^{|\Omega|} \times 2^{|\Omega|}$-dimensional matrix, the elements of $\underline{\underline{D}}$ are $J(A, B) = |A \cap B| / |A \cup B|$, A and B are the subsets of Ω(we define $|\emptyset \cap \emptyset| / |\emptyset \cup \emptyset| = 0$). d_J is proved to be a metric in [28]. It is widely used to measure the dissimilarity between two BPAs.

In [15], a cosine similarity measure between two BPAs is proposed, which is defined as:

$$Sim(m_1, m_2) = \cos\theta = \frac{\left\langle \vec{m}_1, \vec{m}_2 \right\rangle}{\left\|\vec{m}_1\right\| \left\|\vec{m}_2\right\|} \tag{12}$$

where $\vec{m}$ is a $2^{|\Omega|}$-dimensional column vector generated from a BPA, θ is the angle between $\vec{m}_1$ and $\vec{m}_2$, $\left\langle \vec{m}_1, \vec{m}_2 \right\rangle$ is the inner product of $\vec{m}_1$ and $\vec{m}_2$, $\left\|\vec{m}\right\|$ is the norm of $\vec{m}$. $\cos\theta$ is proved to be a semi-pseudo-metric in [28]. As $\cos\theta$ is a similarity measure, then the corresponding dissimilarity measure can be defined as $1 - \cos\theta$.

In this section, three examples are given to demonstrate the performance of d_{PBet}, including Example 4.1 (one of the cases in Example 1 in [29]), Example 4.2 (Example 5, Fig. 2 in [20]) and Example 4.3 (Example 1, Fig. 6 in [11]). d_{PBet} is compared with d_{Bet}, d_J and $1 - \cos\theta$ in these examples, and each of them can be used to measure the dissimilarity between two BPAs.

Example 4.1 Let m_1, m_2, m_3 and m_4 be four BPAs on the frame of discernment $\Omega = \{\theta_1, \theta_2, \theta_3\}$, which are defined as follows:

$m_1(\{\theta_1\}) = \frac{1}{3}$, $m_1(\{\theta_2\}) = \frac{1}{3}$, $m_1(\{\theta_3\}) = \frac{1}{3}$;

$m_2(\{\theta_1, \theta_2, \theta_3\}) = 1$;

$m_3(\{\theta_1\}) = \frac{1}{3}$, $m_3(\{\theta_2\}) = \frac{1}{3}$, $m_3(\{\theta_1, \theta_2\}) = \frac{1}{3}$;

$m_4(\{\theta_1\}) = 1$.

For m_1 and m_2, the results of d_{Bet}, d_J, $1-\cos\theta$ and d_{PBet} are as follows:

$d_{Bet}(m_1, m_2) = 0$; $d_J(m_1, m_2) = 0.5774$; $1-\cos\theta = 0$; $d_{PBet}(m_1, m_2) = 0.2381$.

Since m_1 and m_2 are not the same, the value of the dissimilarity between them should not be 0. According to this, both $d_{Bet}(m_1, m_2) = 0$ and $1-\cos\theta = 0$ are counterintuitive, while $d_J(m_1, m_2) = 0.5774$ and $d_{PBet}(m_1, m_2) = 0.2381$ are acceptable.

For m_1 and m_3, the results of d_J and d_{PBet} are as follows:

$d_J(m_1, m_3) = 0.3333$; $d_{PBet}(m_1, m_3) = 0.3333$.

It is obvious that m_1 and m_3 are not the same. Both $d_J(m_1, m_3) = 0.3333$ and $d_{PBet}(m_1, m_3) = 0.3333$ prove this.

For m_2 and m_3, the results of d_J and d_{PBet} are as follows:

$d_J(m_2, m_3) = 0.5774$; $d_{PBet}(m_2, m_3) = 0.5714$.

Since m_1 and m_3 are not the same, the dissimilarity between m_1 and m_2 should not be the same as that between m_2 and m_3. But from the results, we can see that $d_J(m_1, m_2) = d_J(m_2, m_3) = 0.5774$, which is counterintuitive. $d_{PBet}(m_1, m_3) \neq d_{PBet}(m_2, m_3)$ can reflect the difference between the two dissimilarities, which demonstrates that d_{PBet} is more reasonable than d_J.

For m_1 and m_4, the results of d_J and d_{PBet} are as follows:

$d_J(m_1, m_4) = 0.5774$; $d_{PBet}(m_1, m_4) = 0.6667$.

As we can see, m_4 is absolutely confident in θ_1, while m_1 and m_2 can be considered both as very uncertain sources. m_1 has a full randomness, and m_2 corresponds to the full ignorant source. Although m_1 and m_2 are different in the intrinsic nature of uncertainty, from a decision-making point of view, the decision-maker is faced with the full uncertainty for taking a decision. Intuitively, since both m_1 and m_2 carry uncertainty and they yield to the complete indeterminacy in the decision-making problem, it is expected that m_1 is closer to m_2 than to m_4. As $d_J(m_1, m_2) = d_J(m_1, m_4) = 0.5774$ is counterintuitive, d_J doesn't characterize well the difference between these two very different cases. As $d_{PBet}(m_1, m_2) = 0.2381 < d_{PBet}(m_1, m_4) = 0.6667$ is consistent with the analysis, d_{PBet} characterize well the difference.

To sum up, in this example, d_{PBet} is more reasonable than the other three dissimilarity measures.

Example 4.2 Let m_1 and m_2 be two BBAs on the frame of discernment $\Omega = \{\theta_1, \theta_2, \theta_3\}$, it is known that the number of the non-empty subsets of Ω is $2^3 - 1 = 7$. Here we set $m_1\{\theta_3\} = 1$ and keep it unchanged. m_2 is constantly changing from steps 1 to 20, which is as follows: in step 1, m_2 is beginning with an identical distribution (the value is $\frac{1}{7}$) of the mass over the 7 non-empty subsets; from steps 2 to 20, each step has an increase of Δ for $m_2(\{\theta_1, \theta_2, \theta_3\})$ together with a decrease of $\frac{\Delta}{6}$ for 6 other non-empty subsets(ensure that the sum is 1). A value of $\frac{6}{133}$ is assigned to Δ in order to make sure that $m_2(\{\theta_1, \theta_2, \theta_3\}) = 1$ in step 20. The comparisons of d_{Bet} , d_J, $1-\cos\theta$ and d_{PBet} in all the 20 steps are detailed in Table 3 and illustrated in Fig. 1.

From steps 1 to 20, $m_1\{\theta_3\} = 1$ is kept unchanged, while $m_2\{\theta_3\}$ dips from $\frac{1}{7}$ to 0. Intuitively, the distance between m_1 and m_2 should increase gradually. In Fig. 1, d_{Bet} has an invariable value of 0.6667, which mistakenly indicates that m_2 stay unchanged as m_1 does. It is obviously unreasonable, and it also demonstrates that d_{Bet} is a pseudo-metric which doesn't satisfy the (p3)$''$in the standard metric properties. The values of $1-\cos\theta$ are acceptable from steps 1 to 19. However, m_1 and m_2 are not completely conflicting in step 20, so the value of the dissimilarity should not be 1 in step 20. According to this, $1-\cos\theta$ is faulty. Since both d_J and d_{PBet} are metrics, the values of them in all the 20 steps increase gradually and are reasonable.

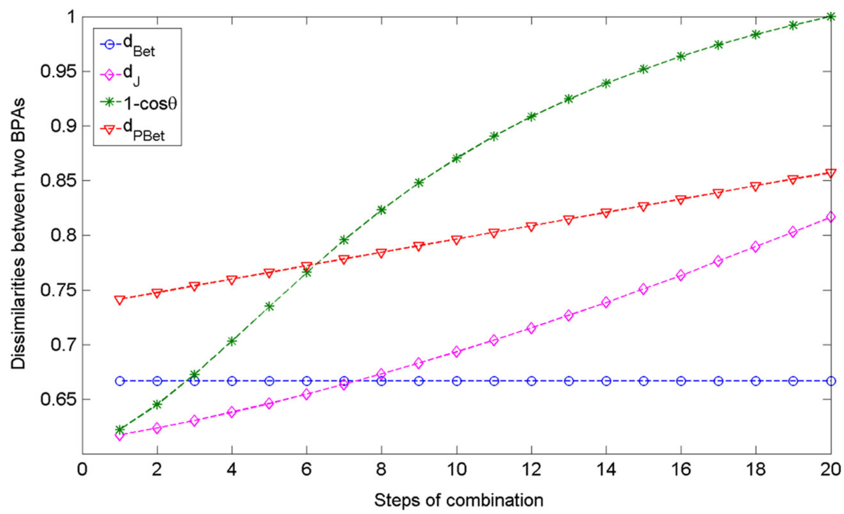
Example 4.3 Let $\Omega = \{\theta_1, \theta_2, ..., \theta_{20}\}$ be a frame of discernment. m_1 and m_2 are two BBAs on Ω defined as follows:

$m_1(\{\theta_2, \theta_3, \theta_4\}) = 0.05$, $m_1(\{\theta_7\}) = 0.05$,

Table 3 Comparisons of the four dissimilarity measures

Step	d_{Bet}	d_J	$1-\cos\theta$	d_{PBet}
1	0.6667	0.6172	0.6220	0.7415
2	0.6667	0.6236	0.6449	0.7476
3	0.6667	0.6305	0.6725	0.7537
4	0.6667	0.6380	0.7031	0.7598
5	0.6667	0.6460	0.7348	0.7658
6	0.6667	0.6546	0.7659	0.7719
7	0.6667	0.6636	0.7954	0.7780
8	0.6667	0.6730	0.8228	0.7841
9	0.6667	0.6830	0.8477	0.7902
10	0.6667	0.6933	0.8701	0.7963
11	0.6667	0.7041	0.8903	0.8024
12	0.6667	0.7152	0.9083	0.8084
13	0.6667	0.7267	0.9244	0.8145
14	0.6667	0.7386	0.9388	0.8206
15	0.6667	0.7509	0.9518	0.8267
16	0.6667	0.7634	0.9635	0.8328
17	0.6667	0.7762	0.9740	0.8389
18	0.6667	0.7894	0.9835	0.8450
19	0.6667	0.8028	0.9921	0.8511
20	0.6667	0.8165	1	0.8571

Fig. 1 Comparisons of the four dissimilarity measures



$m_1(\Omega) = 0.1,\ \ m_1(A) = 0.8;$

$m_2(A^*) = 1\ \ with\ \ A^* = \{\theta_1, \theta_2, \theta_3, \theta_4, \theta_5\}.$

Here we set 20 steps. From steps 1 to 20, m_2 is unchanged, m_1 is constantly changing as follows: in step 1, A is $\{\theta_1\}$; from steps 2 to 20, one more element $\theta_i (i = 2, 3, ...20)$ is added to A at each step. The comparisons of d_{Bet}, d_J, $1 - \cos\theta$ and d_{PBet} in all the 20 steps are detailed in Table 4 and illustrated in Fig. 2.

Table 4 Comparisons of the four dissimilarity measures

Steps	d_{Bet}	d_J	$1 - \cos\theta$	d_{PBet}
$A = \{\theta_1\}$	0.6060	0.7858	1	0.8177
$A = \{\theta_1, \theta_2\}$	0.5517	0.6866	1	0.7604
$A = \{\theta_1, \theta_2, \theta_3\}$	0.3733	0.5633	1	0.6456
$A = \{\theta_1,\ldots,\theta_4\}$	0.1950	0.4286	1	0.4161
$A = \{\theta_1,\ldots,\theta_5\}$	0.1250	0.1322	0.0115	0.1500
$A = \{\theta_1,\ldots,\theta_6\}$	0.2583	0.3883	1	0.4437
$A = \{\theta_1,\ldots,\theta_7\}$	0.3536	0.5029	1	0.7547
$A = \{\theta_1,\ldots,\theta_8\}$	0.4250	0.5705	1	0.8527
$A = \{\theta_1,\ldots,\theta_9\}$	0.4806	0.6187	1	0.9015
$A = \{\theta_1,\ldots,\theta_{10}\}$	0.5250	0.6553	1	0.9258
$A = \{\theta_1,\ldots,\theta_{11}\}$	0.5614	0.6844	1	0.9379
$A = \{\theta_1,\ldots,\theta_{12}\}$	0.5917	0.7081	1	0.9439
$A = \{\theta_1,\ldots,\theta_{13}\}$	0.6173	0.7274	1	0.9470
$A = \{\theta_1,\ldots,\theta_{14}\}$	0.6393	0.7444	1	0.9485
$A = \{\theta_1,\ldots,\theta_{15}\}$	0.6583	0.7592	1	0.9492
$A = \{\theta_1,\ldots,\theta_{16}\}$	0.6750	0.7658	1	0.9496
$A = \{\theta_1,\ldots,\theta_{17}\}$	0.6897	0.7839	1	0.9498
$A = \{\theta_1,\ldots,\theta_{18}\}$	0.7028	0.7944	1	0.9499
$A = \{\theta_1,\ldots,\theta_{19}\}$	0.7145	0.8042	1	0.94995
$A = \{\theta_1,\ldots,\theta_{20}\}$	0.7250	0.8123	1	0.94997

As can be seen from Fig. 2, the value of $1 - \cos\theta$ stays unchanged at 1 from steps 1 to 4 and steps 6 to 20. Since m_1 and m_2 are not completely conflicting in these steps, it is obviously unreasonable. d_{Bet}, d_J and d_{PBet} have a similar variation trend in general. All three decrease when A tends to A^* from steps 1 to 4, and they reach their respective minimum values when A attains A^* at step 5, then they rise as A departs from A^* in the remaining steps.

By analyzing, more information is acquired. From steps 3 to 8, the rangeability of d_{PBet} is greater than that of d_{Bet} and d_J, indicating that the sensibility of d_{PBet} is better. From steps 10 to 20, the rangeability of d_{PBet} is smaller than that of d_{Bet} and d_J, indicating that the sensibility of d_{PBet} is worse. The reason why this happens is as follows. The variation range of the dissimilarity is [0,1]. Steps 3 to 8 are adjacent to step 5 (the minimum value), while steps 10 to 20 are away from step 5. From steps 3 to 8, the dissimilarity has a large variation range which is from the minimum value to 1. From steps 10 to 20, as the dissimilarity has approached its maximum value and is still rising, the variation range is small. On one hand, the better sensibility from steps 3 to 8 enables d_{PBet} to better measure the difference among these steps. On the other hand, it makes d_{PBet} approach its maximum value too quickly, which leads to the worse sensibility from steps 10 to 20.

To summarize, d_{Bet} and $1 - \cos\theta$ are not metrics, sometimes the results obtained from them are obviously unreasonable. d_J and d_{PBet} are metrics, the results obtained from them are reasonable. When d_{PBet} is compared with d_J, it has both an advantage and disadvantage. The advantage is that d_{PBet} is more accurate than d_J in some cases (Example 4.1), and the sensibility of d_{PBet} is better when the variation range is large. The disadvantage is that the sensibility of it is worse when the variation range is small. d_{PBet} is a proper measure for quantifying the dissimilarity between BPAs. As it takes more information (works over the power set) than

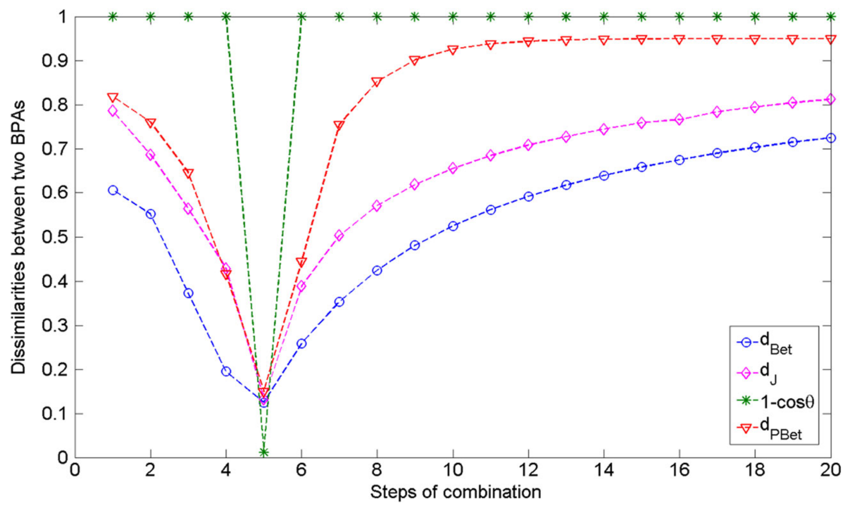


Fig. 2 Comparisons of the four dissimilarity measures

many other dissimilarity measures, the performance of it is better.

To demonstrate the superiority of d_{PBet} further, d_{PBet} and d_J are compared through the applications in combining the conflicting BPAs in Section 5.

5 Applications of the proposed distance

In this section, d_{PBet} is used to replace $1 - \cos\theta$ and d_J in two methods presented in [15] and [30], respectively. Both of the methods are used for combining the conflicting BPAs.

After importing d_{PBet} into the method in [15], Method 1 is generated, which is as follows:

Suppose that there are $n (n \geq 3)$ BPAs on the frame of discernment Ω, $m_i (1 \leq i \leq n)$ and $m_j (1 \leq j \leq n)$ are used to denote any two of them.

Step 1: Calculate the value of $d_{PBet}(m_i, m_j)$, then the corresponding similarity measure between m_i and m_j can be obtained, which is defined as:

$$Sim(m_i, m_j) = 1 - d_{PBet}(m_i, m_j) \tag{13}$$

when $i = j$, it is obvious that $Sim(m_i, m_j) = 1$. For ease of description, we use S_{ij} to denote $Sim(m_i, m_j)$ in further detail below.

Step 2: Use S_{ij} to build the similarity measure matrix (SMM), which is as follows:

$$SMM = \begin{bmatrix} 1 & S_{12} & \cdots & S_{1j} & \cdots & S_{1n} \\ S_{21} & 1 & \cdots & S_{2j} & \cdots & S_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ S_{i1} & S_{i2} & \cdots & S_{ij} & \cdots & S_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ S_{n1} & S_{n2} & \cdots & S_{nj} & \cdots & 1 \end{bmatrix} \tag{14}$$

Step 3: Calculate the support degree of m_i, which is defined as:

$$Sup(m_i) = \sum_{j=1, j \neq i}^{n} S_{ij} \tag{15}$$

Step 4: Calculate the confidence degree of m_i, which is defined as:

$$\omega_i = \frac{Sup(m_i)}{\max_{1 \leq i \leq n} [Sup(m_i)]} \tag{16}$$

Step 5: Modify the original BPAs m_i via the following equation to obtain the new BPAs m_i':

$$m_i'(A) = \begin{cases} \omega_i m_i(A) & A \subset \Omega \\ 1 - \sum\limits_{B \subset \Omega} \omega_i m_i(B) & A = \Omega \end{cases} \tag{17}$$

Step 6: Use Dempster's combination rule to combine the new BPAs m_i', then the combined result is obtained.

The example of fault diagnosis in [15] is quoted to show the performance of Method 1. The details are as follows.

Example 5.1 Let $\Omega = \{\theta_1, \theta_2, \theta_3, \theta_4\}$ be a frame of discernment, $\theta_i (1 \leq i \leq 4)$ represent the running conditions of the equipment, including $\theta_1 = \{normal\}$, $\theta_2 = \{unbalance\}$, $\theta_3 = \{eccentricity\}$ and $\theta_4 = \{baseloosening\}$. m_1, m_2 and m_3 are BPAs obtained from three sensors experimentally, which are detailed in Table 5. It is known that some

Table 5 m_1, m_2 and m_3 obtained from three sensors

	θ_1	θ_2	θ_3	θ_4	Ω
m_1	0.0120	0.6954	0.0470	0.0435	0.2021
m_2	0.0245	0.1027	0.6803	0.0503	0.1422
m_3	0.0161	0.5251	0.2029	0.0404	0.2155

Table 6 Results obtained from different combination methods

	$m_{\oplus}(\theta_1)$	$m_{\oplus}(\theta_2)$	$m_{\oplus}(\theta_3)$	$m_{\oplus}(\theta_4)$	$m_{\oplus}(\Omega)$
Dempster's combination rule [1]	0.0083	0.6256	0.3178	0.0236	0.0247
Deng's method (d_J) [30]	0.0091	0.6611	0.2487	0.0264	0.0547
Wen's method ($\cos\theta$) [15]	0.0096	0.6680	0.2280	0.0276	0.0668
Method 1 (d_{PBet})	0.0092	0.6928	0.2149	0.0267	0.0564

interference is added to the second sensor, meaning m_2 can't reflect the running conditions correctly.

Among the three BPAs, m_2 is incorrect, m_1 and m_3 are correct and they both assign most of their belief to θ_2. Intuitively, the reasonable combined result of the three BPAs should indicate that θ_2 is most likely to happen. Four different methods are used to combine the three BPAs, including Dempster's combination rule, Deng's method (d_J is used), Wen's method ($\cos\theta$ is used) and Method 1 (d_{PBet} is used). The results obtained from different combination methods are detailed in Table 6.

As can be seen from Table 6, since all the combined results obtained from four different methods give their largest mass of belief to θ_2, they are reasonable. Moreover, Method 1 (d_{PBet} is used) gives θ_2 a larger mass of belief than other three methods, it has a better convergence rate, and the result of it is more convenient for decision making.

After importing d_{PBet} into the method in [30], Method 2 is generated. The first three steps of it are the same as Method 1. Step 4, Step 5 and Step 6 of Method 2 are as follows:

Step 4 of Method 2: Calculate the confidence degree of m_i, which is defined as:

$$\omega_i = \frac{Sup(m_i)}{\sum_{i=1}^{n} Sup(m_i)} \tag{18}$$

Step 5 of Method 2: The modified average of the evidence m_i' is given as:

$$m_i'(A) = \sum_{i=1}^{n} (\omega_i \times m_i) \tag{19}$$

Step 6 of Method 2: Use Dempster's combination rule to combine m_i' $n-1$ times.

The example in [30] is quoted to show the performance of Method 2. The details are as follows.

Example 5.2 In a ballistic target identification system, there are five sensors for judging the target. It is known that the types of the target could be warhead, bait and fragment. That is to say, the frame of discernment is $\Omega = \{A(warhead), B(bait), C(fragment)\}$. Suppose the real target is A, the BPAs obtained from the sensors are as follows:

$$m_1(A) = 0.5,\ m_1(B) = 0.2,\ m_1(C) = 0.3;$$

$$m_2(A) = 0,\ m_2(B) = 0.9,\ m_2(C) = 0.1;$$

$$m_3(A) = 0.55,\ m_3(B) = 0.1,\ m_3(A, C) = 0.35;$$

$$m_4(A) = 0.55,\ m_4(B) = 0.1,\ m_4(A, C) = 0.35;$$

$$m_5(A) = 0.6,\ m_5(B) = 0.1,\ m_5(A, C) = 0.3.$$

Table 7 Results obtained from different combination methods

	m_1, m_2	m_1, m_2, m_3	m_1, m_2, m_3, m_4	m_1, m_2, m_3, m_4, m_5
Dempster's combination rule [1]	$m(A) = 0$	$m(A) = 0$	$m(A) = 0$	$m(A) = 0$
	$m(B) = 0.8571$	$m(B) = 0.6316$	$m(B) = 0.3288$	$m(B) = 0.1404$
	$m(C) = 0.1429$	$m(C) = 0.3684$	$m(C) = 0.6712$	$m(C) = 0.8596$
Wen's method ($\cos\theta$) [15]	$m(A) = 0$	$m(A) = 0.6373$	$m(A) = 0.9026$	$m(A) = 0.9631$
	$m(B) = 0.8571$	$m(B) = 0.1436$	$m(B) = 0.0073$	$m(B) = 0.0011$
	$m(C) = 0.1429$	$m(C) = 0.2191$	$m(C) = 0.0832$	$m(C) = 0.0260$
			$m(AC) = 0.0070$	$m(AC) = 0.0098$
Deng's method (d_J) [30]	$m(A) = 0.1543$	$m(A) = 0.7369$	$m(A) = 0.9484$	$m(A) = 0.9869$
	$m(B) = 0.7469$	$m(B) = 0.1618$	$m(B) = 0.0120$	$m(B) = 0.0010$
	$m(C) = 0.0988$	$m(C) = 0.0915$	$m(C) = 0.0310$	$m(C) = 0.0088$
		$m(AC) = 0.0098$	$m(AC) = 0.0086$	$m(AC) = 0.0032$
Method 2 (d_{PBet})	$m(A) = 0.1543$	$m(A) = 0.7640$	$m(A) = 0.9520$	$m(A) = 0.9874$
	$m(B) = 0.7469$	$m(B) = 0.1351$	$m(B) = 0.0095$	$m(B) = 0.0008$
	$m(C) = 0.0988$	$m(C) = 0.0901$	$m(C) = 0.0296$	$m(C) = 0.0085$
		$m(AC) = 0.0108$	$m(AC) = 0.0090$	$m(AC) = 0.0033$

The BPAs are combined in turn. First, m_1 and m_2 are combined. Then one more BPA is added each time until all of them are combined. The results obtained from different combination methods are detailed in Table 7.

As can be seen from Table 7, although more BPAs which support A take part in the combination constantly, Dempster's combination rule concludes that the target cannot be A all the time. When the number of BPAs is more than 2, such results are unreasonable. Because the majority BPAs assign most of their belief to A, but only one BPA distributes its largest mass of belief to B. Such unexpected behavior shows that Dempster's combination rule is risky to combine conflicting BPAs. In the same circumstances, the results obtained from other three methods indicate that the target is most likely to be A, and they are reasonable. The difference of the three methods is the convergence rate. As Method 2 (d_{PBet} is used) gives A a larger mass of belief than the other two methods, it has a better convergence rate, and the result of it is more convenient for decision making.

To sum up, when d_{PBet} is used in the methods for combining the conflicting BPAs, the results of the new methods (Method 1 and Method 2) are reasonable. Moreover, the new methods have better convergence rates, which are more convenient for decision making.

6 Conclusion

In this paper, the PSD pignistic probability function was proposed, which transforms the BPA into pignistic probabilities on the power set of the frame of discernment. Compared with the pignistic probability function which works over the frame of discernment, it takes more information contained in the BPA. Based on the proposed function, a new dissimilarity measure called the distance between PSD betting commitments was defined. The new distance is a metric, and the relevant proof was provided. In order to demonstrate the performance of the new distance, we compared it with three existing dissimilarity measures. Its applications in combining the conflicting BPAs were also presented. The results indicate that it is a better measure for quantifying the dissimilarity between BPAs.

For the new distance, the disadvantage lies in the calculation burden, which will increase exponentially with the rise of the singleton elements in Ω. But as its calculation burden is at the same level as Jousselme's distance, it is acceptable.

Acknowledgments This work is supported by the National Natural Science Foundation of China under grants 61273275, 60975026 and 61573375. The authors also want to thank the anonymous reviewers for their detailed comments that have helped greatly in improving the quality of this paper.

References

1. Dempster AP (1967) Upper and lower probabilities induced by a multivalued mapping. Ann Math Stat 38(1):325–339
2. Shafer G (1976) A mathematical theory of evidence. Princeton University Press, Princeton
3. Zadeh L (1984) Review of a mathematical theory of evidence. AI Mag 5(2):235–247
4. Zadeh L (1986) A simple view of the Dempster–Shafer theory of evidence and its implication for the rule of combination. AI Mag 7(1):85–90
5. Liu W (2006) Analyzing the degree of conflict among belief functions. Artif Intell 170(9):909–924
6. Jousselme AL, Maupin P (2012) Distances in evidence theory: Comprehensive survey and generalizations. Int J Approx Reason 53(1):118–145
7. Jousselme AL, Maupin P On some properties of distances in evidence theory Workshop on the theory of belief functions, Brest, France, p 2010
8. Perry WL, Stephanou HE (1991) Belief function divergence as a classifier Proceedings of the 1991 IEEE international symposium on intelligent control, Arlington, VA, USA
9. Blackman S, Popoli R (1999) Design and analysis of modern tracking systems. Artech House, Norwood
10. Tessem B (1993) Approximations for efficient computation in the theory of evidence. Artif Intell 61(1):315–329
11. Jousselme AL, Grenier D, Bossé É. (2001) A new distance between two bodies of evidence. Information Fusion 2(1):91–101
12. Cuzzolin F Consistent approximations of belief functions 6Th international symposium on imprecise probability: Theories and applications, Durham, UK, p 2009
13. Zouhal L, Denoeux T (1998) An evidence-theoretic k–NN rule with parameter optimization. IEEE Trans Syst Man Cybern Part C Appl Rev 28(1):263–271
14. Ristic B, Smets P (2006) The TBM global distance measure for the association of uncertain combat ID declarations. Information Fusion 7:276–284
15. Wen C, Wang Y, Xu X (2008) Fuzzy information fusion algorithm of fault diagnosis based on similarity measure of evidence Advances in neural networks, lecture notes in computer science, vol 5264. Springer, Berlin, pp 506-515
16. Florea MC, Bossé E. Crisis management using Dempster Shafer theory: using dissimilarity measures to characterize sources' reliability C3i in crisis, emergency and consequence management, RTO–MP–IST–086, Bucharest, Romania, p 2009
17. Hellinger E (1909) Neaue begründung der theorie der quadratischen formen von unendlichen vielen veränderlichen. Journal fü,r die Reine und Aangewandte Mathematik 136:210–271
18. Denoeux T (2001) Inner and outer approximation of belief structures using a hierarchical clustering approach. Int J Uncertainty Fuzziness Knowledge Based Syst 9(3):437–460
19. Denoeux T (1997) A neural network classifier based on Dempster–Shafer theory. IEEE Trans Syst Man Cybern Syst Hum 30(1):131–150
20. Han DQ, Deng Y, Han CZ et al. (2012) Some notes on betting commitment distance in evidence theory. Science China–Information Sciences 55(2):558–565
21. Smets Ph, Kennes R (1994) The transferable belief model. Artif Intell 66(1):191–234
22. Smets Ph (1990) The combination of evidence in the transferable belief model. IEEE Trans Pattern Anal Mach Intell 12(4):447–458
23. Chen S, Deng Y, Wu J (2013) Fuzzy sensor fusion based on evidence theory and its application. Appl Artif Intell 27(2):235–248

24. Du HL, Lv F, Du N (2012) A method of multi–classifier combination based on Dempster–Shafer evidence theory and the application in the fault diagnosis. Adv Mater Res 490–495:1402–1406
25. Huang SY, Su XY, Hu Y et al (2014) A new decision–making method by incomplete preferences based on evidence distance. Knowl-Based Syst 56(2):264–272
26. Cuzzolin F (2004) Geometry of Dempster's rule of combination. IEEE Trans Syst Man Cybern B Cybern 34(1):961–977
27. Cuzzolin F (2008) A geometric approach to the theory of evidence. IEEE Trans Syst Man Cybern Part C Appl Rev 38(3):522–534
28. Bouchard M, Jousselme AL, Dore PE (2013) A proof for the positive definiteness of the Jaccard index matrix. Int J Approx Reason 54(4):615–626
29. Liu ZG, Dezert J, Pan Q, Mercier G (2011) Combination of sources of evidence with different discounting factors based on a new dissimilarity measure. Decis Support Syst 52(1):133–141
30. Deng Y, Shi WK, Zhu ZF, Liu Q (2004) Combining belief functions based on distance of evidence. Decis Support Syst 38(2):489–493

Jingwei Zhu is a Ph.D. candidate at the College of Air and Missile Defense, Air Force Engineering University (AFEU). His research interest covers pattern recognition, intelligent information processing, and evidential reasoning.

Xiaodan Wang received her Ph.D. of computer science from North Western Polytechnical University of China. She is now a professor and Ph.D. advisor in Air Force Engineering University of China. Her research interest includes: pattern recognition, machine learning, and artificial intelligence. Corresponding author of this paper.

Yafei Song is a lecturer at the College of Air and Missile Defense, Air Force Engineering University (AFEU). He received his Ph.D. from AFEU in 2015. He is working as a postdoctoral researcher in AFEU since April 2017. His research interest covers pattern recognition, intelligent information processing, and evidential reasoning.