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Total utility of Z-number

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Abstract Z-numbers, combined with "constraint" and "reliability", has more power to express human knowledge. How to determine the ordering of Z-numbers and how to make a decision with Z-numbers are both meaningful and open issues. In this paper, a new notion of the total utility of Z-number is proposed to measure the total effects of a Z-number. The proposed total utility of Z-number can be used to determine the ordering of Z-numbers, and can also be simply applied in the application of multi-criteria decision making under uncertain environments. Two particular cases of Z-number (Gaussian and triangular), and some mathematical properties of the total utility of Z-number are discussed in this paper. Several applications and comparative analyses are shown to demonstrate the effectiveness of the proposed total utility of Z-number in the application of ordering Z-numbers and multi-criteria decision making.

Keywords Z-number · Utility · Fuzzy number · Preference of Z-number · Decision making

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1 Introduction

Relevant information for real-world decision making often has an element of uncertainty, is imprecise and only partially reliable. In 2011, Zadeh proposed a Z-number framework, which is able to account for the restriction and reliability of natural human language. The concept of a Z-number has the potential capability of representing human knowledge. In the past several years, Z-number has received plenty of attention from multiple mathematic and scientific disciplines. We briefly review the work relevant to Z-numbers in both theory and application.

1.1 Theory of Z-number

Yager [1] used Z-numbers to provide information about an uncertain variable V in the form of a Z-valuation. The Z-valuation expresses the probability that V is A is equal to B. Yager [1] showed that Z-valuations essentially induce a possibility distribution over the probability distributions associated with V. Aliev et al. [2] discussed the arithmetic of discrete Z-numbers, including addition, subtraction, multiplication, division, square roots and other operations of Z-numbers. Aliev et al. [3] also established a general theory of decisions based on the concept of Z-numbers, discussing the method of determining the preference of Z-numbers. Banerjee and Pal [4] presented an extended Z-number with $Z^* =< T, C, A, B, AG >$, including factors: time, context, restriction, reliability and affect group. Z^* is inspired by the study of human psychology.

1.2 Z-number in application

Soroudi and Amraee [5] proposed an uncertain decision making method with the framework of Z-numbers to model the uncertainty of an energy system. Pal et al. [6] discussed the application of Z-numbers in computing with words (CWW). Z-numbers extended the basic philosophy of CWW to include the inherent uncertainty of the information conveyed by human language. Yaakob and Gegov [7] introduced a novel modification of the TOPSIS method to facilitate multi criteria decision making problems based on the concept of Z-numbers called Z-TOPSIS. Aliev et al. [8] introduced the linear programming in the context of Z-numbers to extend the ability of the framework to account for uncertain information associated with a classical fuzzy linear programming method. Aliev and Memmedova [9] also applied Z-numbers in the modeling of psychological research. Aliev and Memmedova [9] used Z-numbers to increase precision and reliability of data processing results in the presence of uncertainty of input data obtained from completed questionnaires. Aliev et al. [10] proposed expected utility based decision making under Z-Information to establish a model of multi-criteria decision making. Kang et al. [11] proposed a methodology of multi-criteria decision making in suppler selection based on Z-numbers with a genetic algorithm and FAHP. Jiang et al. [12] utilized Z-numbers in fault diagnosis based on sensor data fusion.

From the work reviewed, it can be concluded that ranking of Z-numbers is a necessary operation in the arithmetic of Z-numbers and is a challenging practical issue, just as Zadeh [13] presented the interesting question: “Is (approximately 100, likely) greater than (approximately 90, very likely)?” To address this problem, the authors believe it is necessary to briefly review the recent literature related to the ranking of fuzzy numbers. Ureña et al. [14] reviewed the incomplete preference relation in decision making and divided the issue into two categories: numerical preference and linguistic preference, Ureña et al. [14] also analyzed the advantages and disadvantages of preference relations. Wan et al. [15] utilized the closeness degree to characterize the amount of information according to the geometrical representation of an intuitionistic fuzzy sets (IFSs) inspired by the similarity to the ideal solution (TOPSIS). Das and De [16] defined a distance measure for interval numbers based on L-p metric and further generalized the idea to intuitionistic fuzzy numbers. The authors proposed forming the interval with their respective value and ambiguity indices, then ranked the IFSs by the new distance measure. Zhang et al. [17] proposed a framework for comparing two interval sets through inclusion measures, the authors presented similarity measures and distances of interval sets and investigated their relationship with inclusion measures and proposed a fuzziness measure and ambiguity measure to show the uncertainty embedded in an interval set. Destercke and Couso [18] investigated ranking rules based on different statistical features (mean, median) and orderings, and related the obtained (partial) orders to some classical proposals. The authors then proposed a new method of ranking of fuzzy intervals in the context of imprecise probabilities. Rezvani [19] calculated ranking of exponential trapezoidal fuzzy numbers based on variance. The authors calculated the values by finding expected values using the probability density function corresponding to the membership functions of the given fuzzy number and provided the correct ordering of exponential trapezoidal fuzzy numbers. Ban and Coroianu [20] proved that a ranking index used to order a subset of fuzzy numbers can be reduced to a simpler ranking index to generate an equivalent order. Wang [21] proposed a fuzzy preference relation using a membership function representing the preference degree between two fuzzy numbers, Wang [21] then constructed a relative preference relation based on the fuzzy preference relation to rank a set of fuzzy numbers. Shi and Yuan [22] presented a possibility-based method for ranking fuzzy numbers and applied this method to decision making. Duzce [23] presented a new method for ranking trapezoidal fuzzy numbers, by generalizing trapezoidal fuzzy numbers with different left and right heights. Xu [24] investigated the ranking methods of alternatives on the basis of intuitionistic preference relation in fuzzy decision-making environments. Due to the efficiency of handling uncertain information, evidence theory is also widely used in decision making [25]. Recently, a new ranking method based on evidence theory is presented [26]. Other work related to ranking fuzzy number includes [27–39] etc.

Based on the reviewed literature, the authors of this paper conclude that little attention has been paid to the important issue of measuring the utility of Z-number and ranking Z- numbers. The author in [40] proposed a methodology of multi-layer decision methodology for ranking Z-numbers by converting Z-number to standardized generalized fuzzy numbers. The authors in [7] introduced a novel modification of the TOPSIS method to facilitate multi criteria decision making problems based on the concept of Z-numbers called Z-TOPSIS. However, both methods require a procedure for converting Z-numbers to classical fuzzy numbers, which is not a direct index for ranking Z-numbers. Another solution is proposed by Aliev et al. [10]. The authors proposed expected utility-based decision making under Z-Information to establish a model of multi-criteria decision making. The shortcoming of the method in [10] is that the ranking of Z-numbers is based on a subjective membership function (Fig. 4 in paper [10]). Another open issue regarding Z-numbers is the effective application of Z-numbers in decision making. Most of the examples from the reviewed literature established the decision models with other fuzzy technologies, such as TOPSIS, fuzzy logic rule, etc. The inherent meaning of the question cannot be described clearly for each example since most authors have not accounted for the inherent utility of Z-numbers.

In this paper, a new notion of the total utility of Z-numbers is proposed to measure the total effects of a Z-number, which is dependent on the inherent mathematical characteristics of the Z-number. Then the proposed notion of total utility of Z-numbers is used to determine the ordering of Z-numbers. The proposed method can easily be used in the application of multi-criteria decision making. Some examples and applications are used to illustrate the effectiveness of the proposed method.

This paper is organized as follows: Section 2 briefly presents the definition about fuzzy number, and Z-number; Section 3 develops the mathematical notion of the Total Utility of Z-number and two special cases (Gaussian fuzzy number and triangular fuzzy number); Section 4 discusses some of the properties of the total utility of Z-number; in Section 5, the effectiveness analysis of the proposed total utility of Z-number is presented; Section 6 introduces the application of the total utility of Z-number in ranking Z-number and in multi-criteria decision making in uncertain environments. An application of the total utility of Z-number and FEMA (failure modes and effect analysis) in the failure modes risk assessment with a case study of the geothermal power plant (GPP) is also discussed; and finally, conclusion are made in Section 7.

2 Preliminaries

2.1 Fuzzy sets

In 1965, the notion of fuzzy sets was firstly introduced by Zadeh [41], providing a natural way of dealing with problems in which the source of information is imprecise and there is a lack of a sharply defined criteria for class membership. The fuzzy set theory can be used in a wide range of domains, such as clustering [42], fault diagnosis [43], risk and reliability analysis [44, 45], supplier selection [46], job-shop scheduling problems [47], evaluation of network vulnerability [48, 49], medical diagnosis [50], and other decision making [51–62] etc. A brief introduction of fuzzy sets is given as follows.

Definition 1 A fuzzy set A, defined for universe X may be given as:

$$A = \{\langle x, \mu_A(x)\rangle \, | x \in X\}$$

where $\mu_A : X \rightarrow [0, 1]$ is the membership function A. The membership value $\mu_A(x)$ describes the degree of belongingness of $x \in X$ to A.

In real-world applications, the domain experts may provide their opinions in the form of fuzzy numbers. For example, when pricing a new product, one expert may give

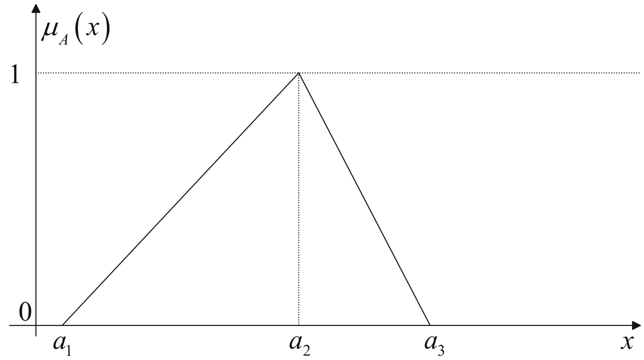


Fig. 1 A triangular fuzzy number

his opinion as: the lowest possible price is \$2.00, the most probable price of the product may be \$3.00, the highest possible price of this product will not be greater than \$4.00. Hence, we can use a triangular fuzzy number (2, 3, 4) to represent the expert's opinion. The triangular fuzzy numbers can be defined as follows.

Definition 2 A triangular fuzzy number $\widetilde{A}$ can be defined by a triplet (a_1, a_2, a_3), where the membership can be determined by (1)

A triangular fuzzy number $\widetilde{A} = (a_1, a_2, a_3)$ can be shown in Fig. 1.

$$\mu_{\widetilde{A}}(x) = \begin{cases} 0, & x \in (-\infty, a_1) \\ \frac{x-a_1}{a_2-a_1}, & x \in [a_1, a_2] \\ \frac{c-x}{a_3-a_2}, & x \in [a_2, a_3] \\ 0, & x \in (a_3, +\infty) \end{cases} \quad (1)$$

Definition 3 A trapezoidal fuzzy number $\widetilde{A}$ can be defined by a quadruplet (a_1, a_2, a_3, a_4), where the membership can be determined by (2)

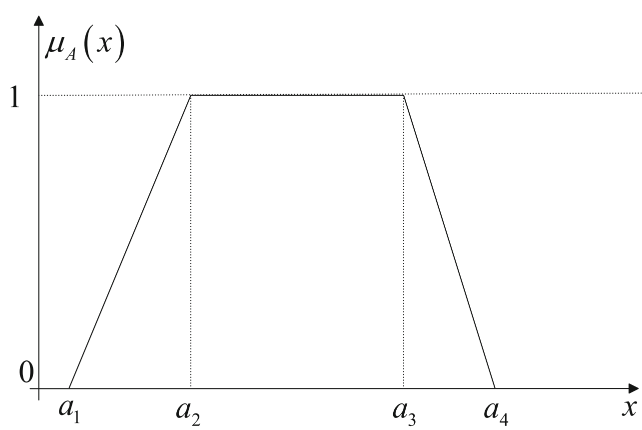


Fig. 2 A trapezoidal fuzzy number

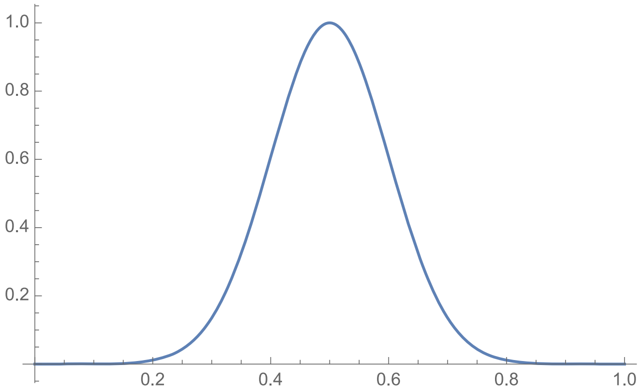


Fig. 3 Gaussian fuzzy number [c=0.5,σ=0.1]

A trapezoidal fuzzy number $\widetilde{A} = (a_1, a_2, a_3, a_4)$ can be shown in Fig. 2.

$$\mu_{\tilde{A}}(x) = \begin{cases} 0, & x \in (-\infty, a_1) \\ \frac{1}{(a_2-a_1)}x - \frac{a_1}{a_2-a_1}, & x \in [a_1, a_2] \\ 1\ , & x \in [a_2, a_3] \\ \frac{-1}{(a_4-a_3)}x + \frac{a_4}{a_4-a_3}, & x \in [a_3, a_4] \\ 0, & x \in (a_4, +\infty) \end{cases} \tag{2}$$

Definition 4 Gaussian fuzzy number $\widetilde{A}$ can be defined by a binary (c, σ), where c determines the center of the function, σ determines the width of the function. The Gaussian membership function can be determined by (3)

$$\mu_{\tilde{A}}(x) = e^{-\frac{(x-c)^2}{2\sigma^2}} \tag{3}$$

A Gaussian fuzzy number $\widetilde{A} = \text{Gauss}(0.5, 0.1)$ is shown in Fig. 3. For the simplicity of theoretical analysis, we will use Gaussian fuzzy numbers in this paper.

Definition 5 Let $\mu_{\tilde{A}}(x) \to [0, 1]$, $\alpha \in [0, 1]$, and $\tilde{A}^{\alpha}$ or $\left[\mu_{\tilde{A}}(x)\right]^{\alpha}$ called the α-cut set of μ, is denoted by (4).

$$\tilde{A}^{\alpha} = \left[\mu_{\tilde{A}}(x)\right]^{\alpha} = \left\{x \in X \,\middle|\, \mu_{\tilde{A}}(x) \geq \alpha\right\} \tag{4}$$

where $\mu_{\tilde{A}}(x)$ is the membership function of fuzzy number $\tilde{A}$.

In the real world, uncertainty is a pervasive phenomenon. Much of the information on which decisions are based is uncertain. Humans have a remarkable capability to make rational decisions based on information which is uncertain, imprecise and/or incomplete. Formalization of this process, at least to some degree, is a challenging task. Zadeh [13] proposed a method using an ordered pair of fuzzy numbers, namely Z-number, (A, B). The first component, A, plays the role of a fuzzy restriction, and the second component, B, represents the reliability of the first component [13]. The definition of Z-number is shown in the Section 2.2.

2.2 Z-numbers

A new concept, Z-numbers, is proposed by Zadeh [13] to model uncertain information. A Z-number can be defined as an ordered pair of fuzzy numbers as follows:

Definition 6 A Z-number is an ordered pair of fuzzy numbers denoted as $Z = \left(\widetilde{A}, \widetilde{R}\right)$. The first component $\widetilde{A}$, a restriction on the values, is a real-valued uncertain variable X. The second component $\widetilde{R}$ is a measure of reliability of the first component.

Zadeh [13] points out that *R is a restriction on the possibility measure of A rather than on the probability of A. Conversely, if R is a restriction on the probability of A rather than on the possibility measure of A, then (A, R) is not a Z-number.* This means that R measures the sureness, confidence, and reliability of measurement of restriction of A.

Z-numbers can be used to model uncertain information in real-world situations. For example, in risk analysis, when the loss of severity of the fifth component is very low, and the confidence is very likely, the Z-number is written as $Z = (\textit{very low}, \textit{very likely})$. Figure 4 shows a Z-number

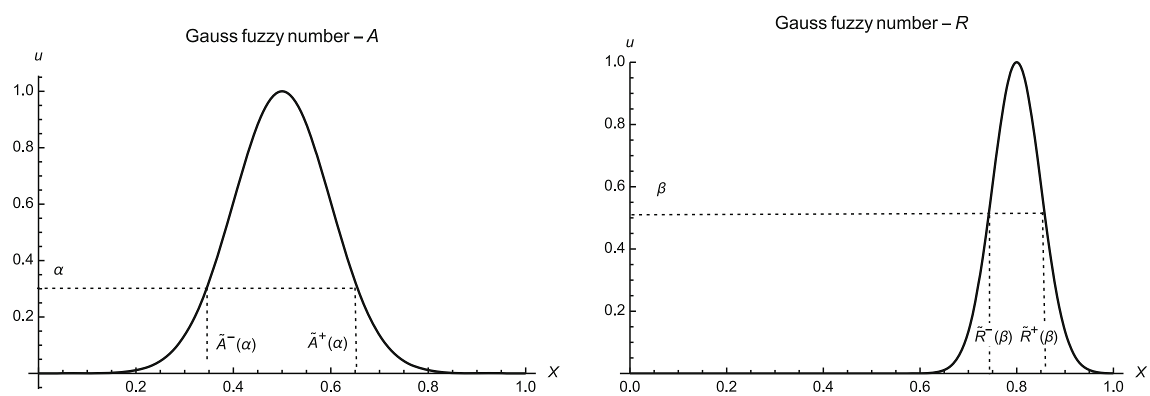


Fig. 4 $Z = (\tilde{A}, \tilde{R})$ with $\tilde{A} = \text{Gauss}\ [0.5,0.1]$, $\tilde{R} = \text{Gauss}\ [0.8,0.05]$

with Z = $(\tilde{A}, \tilde{R})$ with $\tilde{A}$ = Gauss [0.5,0.1], $\tilde{R}$ = Gauss [0.8,0.05].

Recently, a new uncertain framework, namely D-number, has also received plenty of attention. D-numbers are relevant to the situations of dependence of the propositions, and has been applied in failure modes and effect analysis [63], linguistic decision making [64], and human resources selection [65] etc.

In the Section 3, the notion of the total utility of Z-number is proposed in detail.

3 Total utility of Z-number

Total Utility (TU) is proposed to estimate the total utility of a Z-number, which is based on the α-cut set of restraint $(\tilde{A})$ and reliability $(\tilde{R})$ with respect to the interaction of both restraint $(\tilde{A})$ and reliability $(\tilde{R})$.

Definition 7 Assume a Z-number is denoted as $Z = (\tilde{A}, \tilde{R})$, $-1 \leq \tilde{A} \leq 1$, $0 \leq \tilde{R} \leq 1$, the total utility of Z-number is denoted as $TU(Z)$,

$$TU(Z) = TU\left(\tilde{A}, \tilde{R}\right) = \int_0^1\int_0^1\int_{-1/2}^{1/2}\int_{-1/2}^{1/2}\left\{\begin{array}{l}\left[\frac{\tilde{A}^-(\alpha)+\tilde{A}^+(\alpha)}{2} + x\left(\tilde{A}^+(\alpha) - \tilde{A}^-(\alpha)\right)\right]e^{-\left[\tilde{A}^+(\alpha)-\tilde{A}^-(\alpha)\right]^2} \\ \times\left[\frac{\tilde{R}^-(\beta)+\tilde{R}^+(\beta)}{2} + y\left(\tilde{R}^+(\beta) - \tilde{R}^-(\beta)\right)\right]e^{-\left[\tilde{R}^+(\beta)-\tilde{R}^-(\beta)\right]^2}\end{array}\right\}dxdyd\alpha d\beta \tag{5}$$

where $\tilde{A}$, $\tilde{R}$ are two regular fuzzy numbers, representing the "constraint" and "reliability" of a Z-number, $-1 \leq \tilde{A} \leq 1$, $0 \leq \tilde{R} \leq 1$. $[\tilde{A}^-(\alpha), \tilde{A}^+(\alpha)]$ is the α-cut set of fuzzy number $\tilde{A}$ ($\alpha \in [0, 1]$), $[\tilde{R}^-(\beta), \tilde{R}^+(\beta)]$ is the β-cut set of fuzzy number $\tilde{R}$ ($\beta \in [0, 1]$), which are shown in Fig. 4.

Especially, if a Z-number is denoted by two interval numbers with [A,R], where $A = [a^-, a^+]$, and $R = [r^-, r^+]$, then (5) is degenerated as

$$TU(Z) = TU(A, R) = \int_{-1/2}^{1/2}\int_{-1/2}^{1/2}\left\{\begin{array}{l}\left[\frac{a^+ + a^-}{2} + x\left(a^+ - a^-\right)\right]e^{-(a^+-a^-)^2} \\ \times\left[\frac{r^+ + r^-}{2} + y\left(r^+ - r^-\right)\right]e^{-(r^+-r^-)^2}\end{array}\right\}dxdy \tag{6}$$

where $-1 \leq A \leq 1, 0 \leq R \leq 1$.

Let

$$\tilde{A}_1 = \tilde{A}^-(\alpha) + \tilde{A}^+(\alpha) \tag{7}$$
$$\tilde{A}_2 = \tilde{A}^+(\alpha) - \tilde{A}^-(\alpha) \tag{8}$$
$$\tilde{R}_1 = \tilde{R}^-(\beta) + \tilde{R}^+(\beta) \tag{9}$$
$$\tilde{R}_2 = \tilde{R}^+(\beta) - \tilde{R}^-(\beta) \tag{10}$$

Then

$$TU(Z) = TU\left(\tilde{A}, \tilde{R}\right) \tag{11}$$

$$= \int_0^1\int_0^1\int_{-1/2}^{1/2}\int_{-1/2}^{1/2}\left\{\begin{array}{l}\left(\frac{\tilde{A}_1}{2} + x\tilde{A}_2\right)e^{-\tilde{A}_2^2} \\ \times\left(\frac{\tilde{R}_1}{2} + y\tilde{R}_2\right)e^{-\tilde{R}_2^2}\end{array}\right\} \times dxdyd\alpha d\beta \tag{12}$$

$$= \int_0^1\int_0^1 e^{-\tilde{A}_2^2}e^{-\tilde{R}_2^2}\int_{-1/2}^{1/2}\int_{-1/2}^{1/2} \times\left[\begin{array}{l}\frac{\tilde{A}_1}{2}\frac{\tilde{R}_1}{2} + \frac{\tilde{A}_1\tilde{R}_2}{2}y \\ +\frac{\tilde{A}_2\tilde{R}_1}{2}x + \tilde{A}_2\tilde{R}_2xy\end{array}\right]dxdyd\alpha d\beta \tag{13}$$

$$= \int_0^1\int_0^1 e^{-\tilde{A}_2^2}e^{-\tilde{R}_2^2}\frac{\tilde{A}_1}{2}\frac{\tilde{R}_1}{2}d\alpha d\beta \tag{14}$$

$$= \frac{1}{4}\int_0^1\int_0^1 \tilde{A}_1\tilde{R}_1e^{-\tilde{A}_2^2}e^{-\tilde{R}_2^2}d\alpha d\beta \tag{15}$$

Hence,

$$TU(Z) = TU(A, R) \tag{16}$$

$$= \frac{1}{4}\int_0^1\int_0^1 \tilde{A}_1\tilde{R}_1e^{-\tilde{A}_2^2}e^{-\tilde{R}_2^2}d\alpha d\beta \tag{17}$$

$$= \frac{1}{4}\int_0^1\int_0^1\left\{\begin{array}{l}\left[\tilde{A}^-(\alpha) + \tilde{A}^+(\alpha)\right]\left[\tilde{R}^-(\beta) + \tilde{R}^+(\beta)\right] \\ \times e^{-\left[\tilde{A}^+(\alpha)-\tilde{A}^-(\alpha)\right]^2}e^{-\left[\tilde{R}^+(\beta)-\tilde{R}^-(\beta)\right]}\end{array}\right\}d\alpha d\beta \tag{18}$$

Case 1 Assume $Z = (\tilde{A}, \tilde{R})$, and $\tilde{A}$, $\tilde{R}$ are two Gaussian fuzzy number, whose membership functions are respectively denoted as

$$\mu_{\tilde{A}}(x) = e^{-\frac{(x-c_1)^2}{2\sigma_1^2}} \tag{19}$$

where $-1 =< c_1 <= 1$, and $\sigma_1 > 0$.

$$\mu_{\tilde{R}}(x) = e^{-\frac{(x-c_2)^2}{2\sigma_2^2}} \tag{20}$$

where $0 =< c_2 <= 1$, and $\sigma_2 > 0$.

Let $\alpha = \mu_{\tilde{A}}(x)$, the solution of x is

$$x = c_1 \pm \sqrt{-2\sigma_1^2 \ln \alpha} \tag{21}$$

Table 1 Total utility of special Z-number with Gaussian fuzzy number

Item	$\widetilde{A}_1$	$\widetilde{A}_2$	$\widetilde{R}_1$	$\widetilde{R}_2$	Total utility
$Z = (1, 0)$	2	0	0	0	0
$Z = (0, 1)$	0	0	2	0	0
$Z = (0.5, 0.5)$	1	0	1	0	0.25
$Z = (0.5, 0.6)$	1	0	1.2	0	0.30
$Z = (1, 1)$	2	0	2	0	1
$Z = (Gauss(0.5, 0.1), Gauss(0.5, 0.3))$	–	–	–	–	0.135
$Z = (Gauss(0.5, 0.1), Gauss(0.5, 0.2))$	–	–	–	–	0.175
$Z = (Gauss(0.5, 0.1), Gauss(0.5, 0.1))$	–	–	–	–	0.214
$Z = (Gauss(0.5, 0.1), Gauss(0.6, 0.1))$	–	–	–	–	0.257
$Z = (Gauss(0.6, 0.1), Gauss(0.5, 0.1))$	–	–	–	–	0.257
$Z = (Gauss(0.6, 0.1), Gauss(0.9, 0.1))$	–	–	–	–	0.463
$Z = (Gauss(0.9, 0.1), Gauss(0.9, 0.1))$	–	–	–	–	0.694
$Z = (Gauss(0.999, 0.1), Gauss(0.999, 0.1))$	–	–	–	–	0.856
$Z = (Gauss(0.999, 0.01), Gauss(0.999, 0.01))$	–	–	–	–	0.996

1. Gauss(c, σ) is a Gaussian fuzzy number, where c is mean of Gaussian fuzzy number and σ is the variance of Gaussian fuzzy number
2. Z = $(\tilde{A}, \tilde{R}) = (0.5, 0.5)$, $\tilde{A}$ and $\tilde{R}$ is degenerated into two distinct numbers

Hence

$$\tilde{A}_1 = \tilde{A}^-(\alpha) + \tilde{A}^+(\alpha) = 2c_1 \tag{22}$$

$$\tilde{A}_2 = \tilde{A}^+(\alpha) - \tilde{A}^-(\alpha) = 2\sqrt{-2\sigma_1{}^2 \ln\alpha} \tag{23}$$

Similarly

$$\tilde{R}_1 = \tilde{R}^-(\beta) + \tilde{R}^+(\beta) = 2c_2 \tag{24}$$

$$\tilde{R}_2 = \tilde{R}^+(\beta) - \tilde{R}^-(\beta) = 2\sqrt{-2\sigma_2{}^2 \ln\alpha} \tag{25}$$

$$TU(Z) = TU(A, R) \tag{26}$$

$$= \frac{1}{4}\int_0^1\int_0^1 \tilde{A}_1\tilde{R}_1 e^{-\tilde{A}_2^2} e^{-\tilde{R}_2^2} d\alpha d\beta \tag{27}$$

$$= \frac{1}{4}\int_0^1\int_0^1 (2c_1)(2c_2)\, e^{-\left(2\sqrt{-2\sigma_1{}^2\ln\alpha}\right)^2} \times e^{-\left(2\sqrt{-2\sigma_2{}^2\ln\beta}\right)^2} d\alpha d\beta \tag{28}$$

$$= c_1c_2\int_0^1\int_0^1 e^{8\sigma_1{}^2\ln\alpha} e^{8\sigma_2{}^2\ln\beta} d\alpha d\beta \tag{29}$$

$$= c_1c_2\int_0^1\int_0^1 \alpha^{8\sigma_1{}^2}\beta^{8\sigma_2{}^2} d\alpha d\beta \tag{30}$$

$$= \frac{c_1c_2}{\left(1+8\sigma_1{}^2\right)\left(1+8\sigma_2^2\right)} \tag{31}$$

Samples of the total utility of special Z-number with Gaussian fuzzy number are shown in Table 1.

Table 2 Total utility of special Z-number with triangle fuzzy number

Item	$\widetilde{A}_1$	$\widetilde{A}_2$	$\widetilde{R}_1$	$\widetilde{R}_2$	Total utility
$Z = (1, 0)$	2	0	0	0	0
$Z = (0, 1)$	0	0	2	0	0
$Z = (0.5, 0.5)$	1	0	1	0	0.250
$Z = (0.5, 0.6)$	1	0	1.2	0	0.300
$Z = (1, 1)$	2	0	2	0	1
$Z = (Triangle(0.4, 0.6), Triangle(0.2, 0.8))$	–	–	–	–	0.220
$Z = (Triangle(0.4, 0.6), Triangle(0.3, 0.7))$	–	–	–	–	0.234
$Z = (Triangle(0.4, 0.6), Triangle(0.4, 0.6))$	–	–	–	–	0.244
$Z = (Triangle(0.4, 0.6), Triangle(0.5, 0.7))$	–	–	–	–	0.292
$Z = (Triangle(0.5, 0.7), Triangle(0.4, 0.6))$	–	–	–	–	0.292
$Z = (Triangle(0.5, 0.7), Triangle(0.8, 1))$	–	–	–	–	0.526
$Z = (Triangle(0.8, 1), Triangle(0.9, 1))$	–	–	–	–	0.841
$Z = (Triangle(0.999, 1), Triangle(0.999, 1))$	–	–	–	–	0.999

1. Triangle(a_1, a_3) is a symmetrical triangle fuzzy number with $(a_1 + a_3) = 2a_2$
2. Z = $(\tilde{A}, \tilde{R}) = (0.5, 0.5)$, $\tilde{A}$ and $\tilde{R}$ is degenerated into two distinct numbers

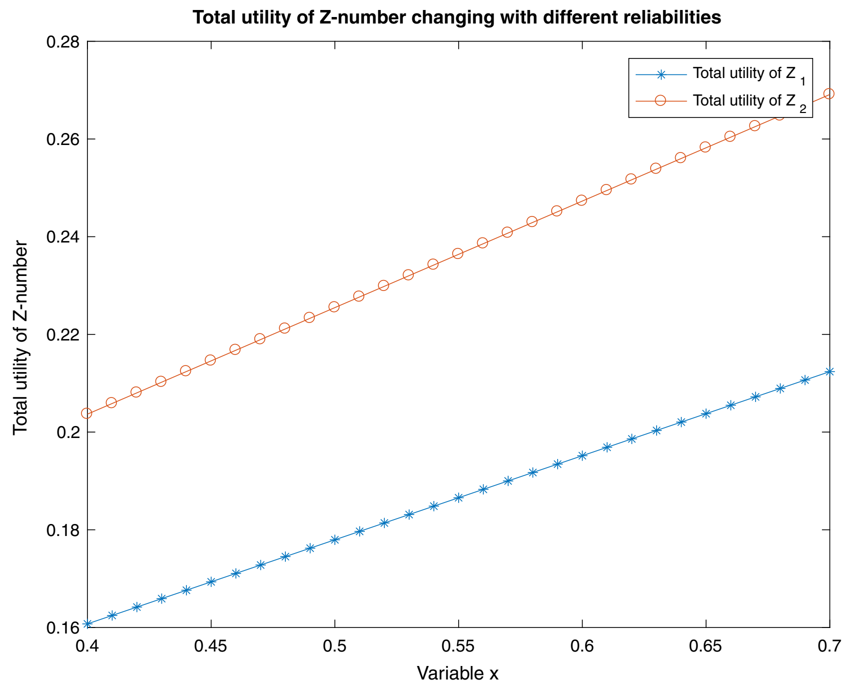


Fig. 5 Total utility of Z_1 and Z_2 changing with different reliabilities

Case 2 Assume $Z = (\widetilde{A}, \widetilde{R})$, and $\widetilde{A}$, $\widetilde{R}$ are two triangle fuzzy numbers, whose membership functions are respectively denoted as

$$\mu_{\widetilde{A}}(x) = \begin{cases} 0, & x \in (-\infty, a_1) \\ \frac{x-a_1}{a_2-a_1}, & x \in [a_1, a_2] \\ \frac{a_3-x}{a_3-a_2}, & x \in [a_2, a_3] \\ 0, & x \in (a_3, +\infty) \end{cases} \tag{32}$$

$$\mu_{\widetilde{R}}(x) = \begin{cases} 0, & x \in (-\infty, r_1) \\ \frac{x-r_1}{r_2-r_1}, & x \in [r_1, r_2] \\ \frac{r_3-x}{r_3-r_2}, & x \in [r_2, r_3] \\ 0, & x \in (r_3, +\infty) \end{cases} \tag{33}$$

Assume $\widetilde{A}$ and $\widetilde{R}$ are two symmetrical fuzzy number, $a_2 - a_1$ is equal to $a_3 - a_2$, and $r_2 - r_1$ is equal to $r_3 - r_2$, then the α-cut of $\widetilde{A}$ and $\widetilde{R}$ can be denoted as

$$[\mu_{\tilde{A}}(x)]^{\alpha} = \begin{cases} [a_1+\alpha(a_2-a_1), a_3-\alpha(a_3-a_2)], & if\ 0<\alpha\le 1 \\ X, & if\ \alpha=0 \end{cases} \tag{34}$$

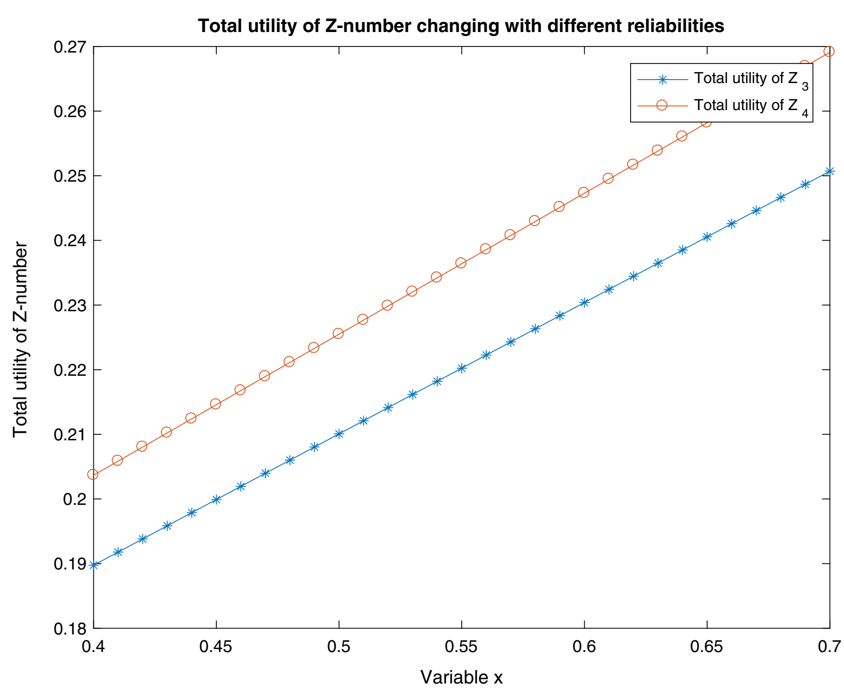


Fig. 6 Total utility of Z_3 and Z_4 changing with different reliabilities

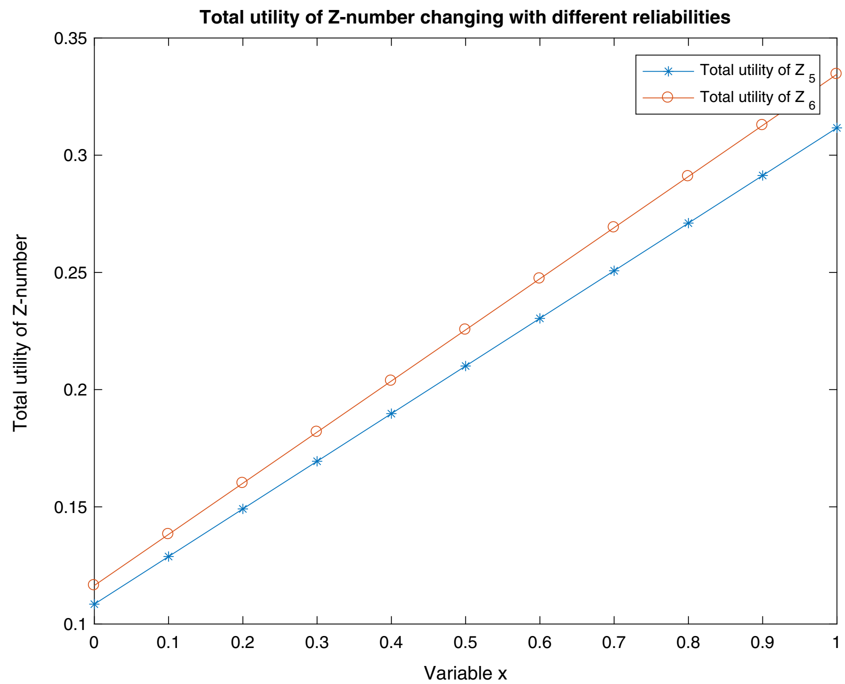


Fig. 7 Total utility of Z_5 and Z_6 changing with different reliabilities

Then

$$\tilde{A}_1 = \tilde{A}^-(\alpha) + \tilde{A}^+(\alpha) \tag{35}$$
$$= a_1 + \alpha(a_2 - a_1) + a_3 - \alpha(a_3 - a_2) \tag{36}$$
$$= a_1 + a_3 \tag{37}$$

$$\tilde{A}_2 = \tilde{A}^+(\alpha) - \tilde{A}^-(\alpha) \tag{38}$$
$$= a_3 - \alpha(a_3 - a_2) - [a_1 + \alpha(a_2 - a_1)] \tag{39}$$
$$= a_3 - a_1 - 2\alpha(a_3 - a_2) \tag{40}$$
$$= a_3 - a_1 - \alpha(a_3 - a_1) \tag{41}$$
$$= (1 - \alpha)(a_3 - a_1) \tag{42}$$

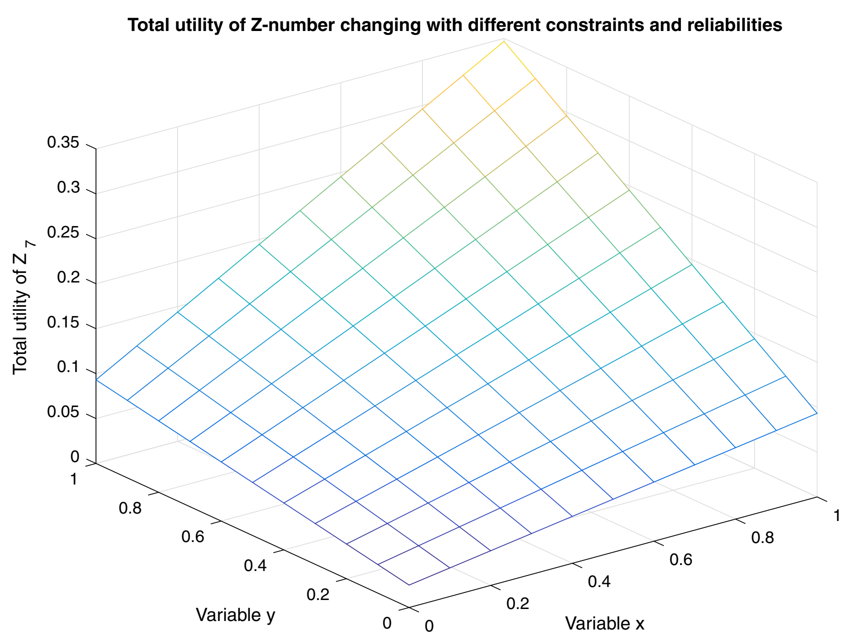


Fig. 8 Total utility of Z_7 changing with different constraints and reliabilities

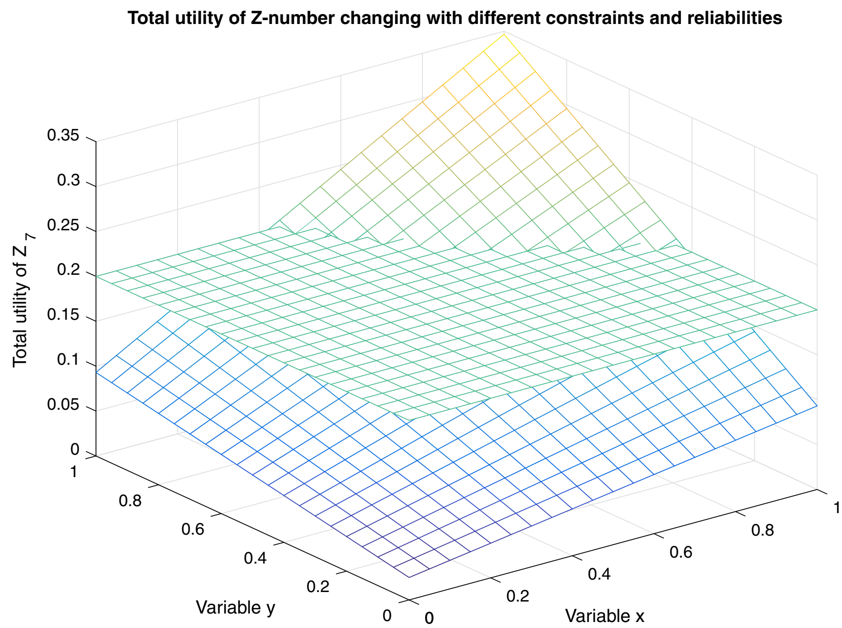


Fig. 9 Total utility of Z_7 is equal where the pitch arc intersected between the curved surface and the horizontal plane

Similarly, we can get

$$\tilde{R}_1 = \tilde{R}^-(\beta) + \tilde{R}^+(\beta) = r_1 + r_3 \quad (43)$$

$$\tilde{R}_2 = \tilde{R}^+(\beta) - \tilde{R}^-(\beta) = (1-\beta)(r_3 - r_1) \quad (44)$$

Hence

$$TU(Z) = TU(A, R) \quad (45)$$

$$= \frac{1}{4}\int_0^1\int_0^1 \tilde{A}_1 \tilde{R}_1 e^{-\tilde{A}_2^2} e^{-\tilde{R}_2^2} d\alpha d\beta \quad (46)$$

$$= \frac{1}{4}\int_0^1\int_0^1 (a_1+a_3)(r_1+r_3)\, e^{-[(1-\alpha)(a_3-a_1)]^2} \times e^{-[(1-\beta)(r_3-r_1)]^2} d\alpha d\beta \quad (47)$$

$$= \frac{(a_1+a_3)(r_1+r_3)}{4} e^{-(a_3-a_1)^2} e^{-(r_3-r_1)^2} \int_0^1\int_0^1 \times e^{-(1-\alpha)^2} e^{-(1-\beta)^2} d\alpha d\beta \quad (48)$$

$$= \frac{(a_1+a_3)(r_1+r_3)}{4} e^{-(a_3-a_1)^2} e^{-(r_3-r_1)^2} \frac{\pi\left(1-e^{-1}\right)}{2} \quad (49)$$

$$= \frac{\pi\left(1-e^{-1}\right)}{8} \frac{(a_1+a_3)(r_1+r_3)}{e^{(a_3-a_1)^2} e^{(r_3-r_1)^2}} \quad (50)$$

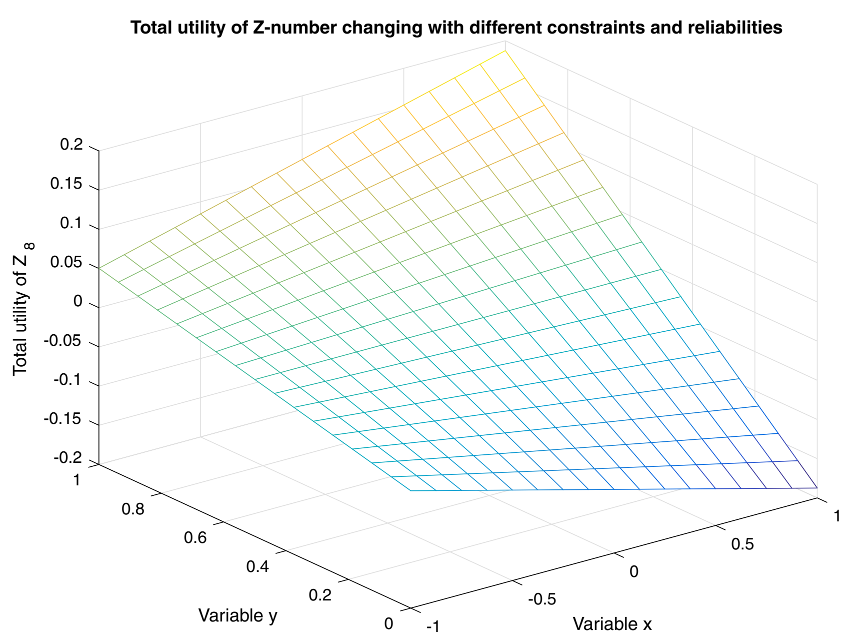


Fig. 10 Total utility of Z_8 changing with different constraint and reliability

where

$$\int_0^1 \int_0^1 e^{-(1-\alpha)^2} e^{-(1-\beta)^2} d\alpha d\beta \tag{51}$$

$$\underline{\underline{x = 1-\alpha,\ y = 1-\beta}} \int_0^1 \int_0^1 e^{-x^2} e^{-y^2} dx dy \tag{52}$$

$$= \int_0^1 \int_0^1 e^{-\left(x^2+y^2\right)} dx dy \tag{53}$$

$$\underline{\underline{x = \rho \sin\theta,\ y = \rho\cos\theta}} \int_0^{\frac{\pi}{2}} d\theta \int_0^1 e^{-\rho^2} \rho d\rho \tag{54}$$

$$= -\frac{1}{2} \int_0^{\frac{\pi}{2}} e^{-\rho^2} \Big|_0^1 d\theta \tag{55}$$

$$= \frac{\pi}{2}\left(1 - e^{-1}\right) \tag{56}$$

Samples of the total utility of special Z-number with triangle fuzzy number are shown in Table 2

In Section 4, we discuss some mathematical properties of the total utility of Z-number.

4 Properties of the total utility of Z-number

Proposition 1 *Given a $Z = (\tilde{A}, \tilde{R})$ with ($-1 \le \tilde{A} \le 1$, $0 \le \tilde{R} \le 1$), $TU(Z)$ is monotonically increasing with $\tilde{A}_1$ when $-1 \le \tilde{A} \le 1$, $0 \le \tilde{R} \le 1$, $TU(Z)$ is monotonically increasing with $\tilde{A}_2$ when $-1 \le \tilde{A}_1 < 0$, $0 \le \tilde{R} \le 1$, $TU(Z)$ is monotonically decreasing with $\tilde{A}_2$ when $0 \le \tilde{A}_1 < 1$, $0 \le \tilde{R} \le 1$, where $\widetilde{A}_1 = \widetilde{A}^-(\alpha) + \widetilde{A}^+(\alpha)$, $\widetilde{A}_2 = \widetilde{A}^+(\alpha) - \widetilde{A}^-(\alpha)$.*

Proof Assume $\tilde{R}$ is a constant fuzzy number. For $0 \le \tilde{A} \le 1$ and $0 \le \tilde{R} \le 1$, $\frac{1}{4}\int_0^1 \tilde{R}_1 e^{-\tilde{R}_2^2} d\beta >= 0$,

$$TU(Z) = TU\left(\tilde{A}, \tilde{R}\right) \tag{57}$$

$$= \frac{1}{4}\int_0^1 \int_0^1 \tilde{A}_1 \tilde{R}_1 e^{-\tilde{A}_2^2} e^{-\tilde{R}_2^2} d\alpha d\beta \tag{58}$$

$$= \frac{1}{4}\int_0^1 \tilde{R}_1 e^{-\tilde{R}_2^2} d\beta \int_0^1 \tilde{A}_1 e^{-\tilde{A}_2^2} d\alpha \tag{59}$$

$$\underline{\underline{\frac{1}{4}\int_0^1 \tilde{R}_1 e^{-\tilde{R}_2^2} d\beta = C \ge 0}} \int_0^1 C \tilde{A}_1 e^{-\tilde{A}_2^2} d\alpha \tag{60}$$

and

$$TU(Z) = \int_0^1 C \tilde{A}_1 e^{-\tilde{A}_2^2} d\alpha \propto \tilde{A}_1 e^{-\tilde{A}_2^2} \tag{61}$$

Let

$$F = \tilde{A}_1 e^{-\tilde{A}_2^2} \tag{62}$$

we can get

$$\frac{\partial F}{\partial \tilde{A}_1} = e^{-\tilde{A}_2^2} > 0 \tag{63}$$

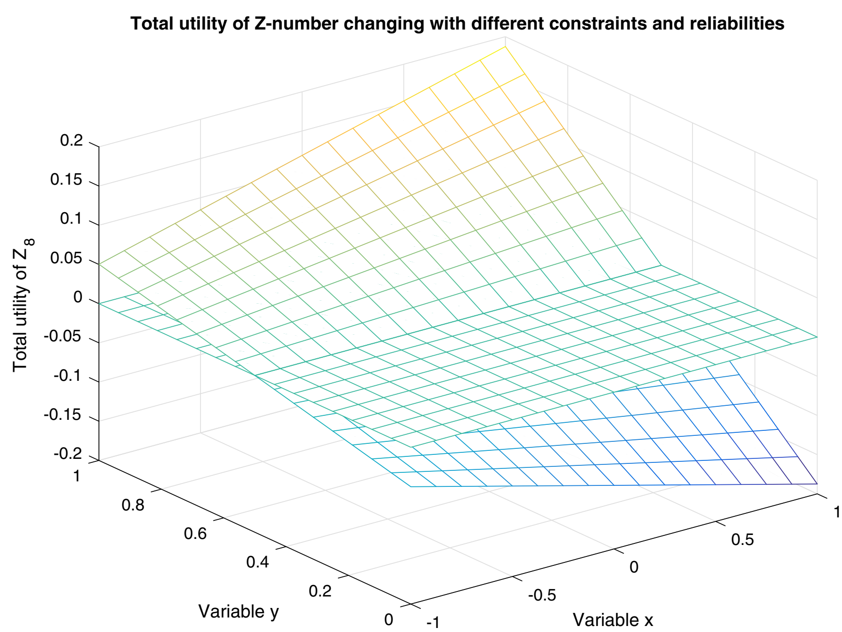


Fig. 11 Total utility of Z_8 is 0 where the pitch arc intersected between the curved surface and the horizontal plane

$$\frac{\partial F}{\partial \tilde{A}_2} = \begin{cases} \tilde{A}_1 e^{-\tilde{A}_2^2}\left(-2\tilde{A}_2\right) < 0, & 0 < \tilde{A}_1 \le 1 \\ \tilde{A}_1 e^{-\tilde{A}_2^2}\left(-2\tilde{A}_2\right) = 0 & \tilde{A}_1 = 0 \\ \tilde{A}_1 e^{-\tilde{A}_2^2}\left(-2\tilde{A}_2\right) > 0 & -1 \le \tilde{A}_1 < 0 \end{cases} \tag{64}$$

End proof. □

Proposition 2 *Given a* $Z = (\tilde{A}, \tilde{R})$ *with* $(-1 \le \tilde{A} \le 1, 0 \le \tilde{R} \le 1)$*, when* $0 < \tilde{A}_1 \le 1$*,* $TU(Z)$ *is monotonically increasing with* $\tilde{R}_1$ *and monotonically decreasing with* $\tilde{R}_2$*, when* $-1 \le \tilde{A}_1 < 0$*,* $TU(Z)$ *is monotonically decreasing with* $\tilde{R}_1$ *and monotonically increasing with* $\tilde{R}_2$*, where* $\widetilde{R}_1 = \widetilde{R}^-(\beta) + \widetilde{R}^+(\beta)$, $\widetilde{R}_2 = \widetilde{R}^+(\beta) - \widetilde{R}^-(\beta)$.

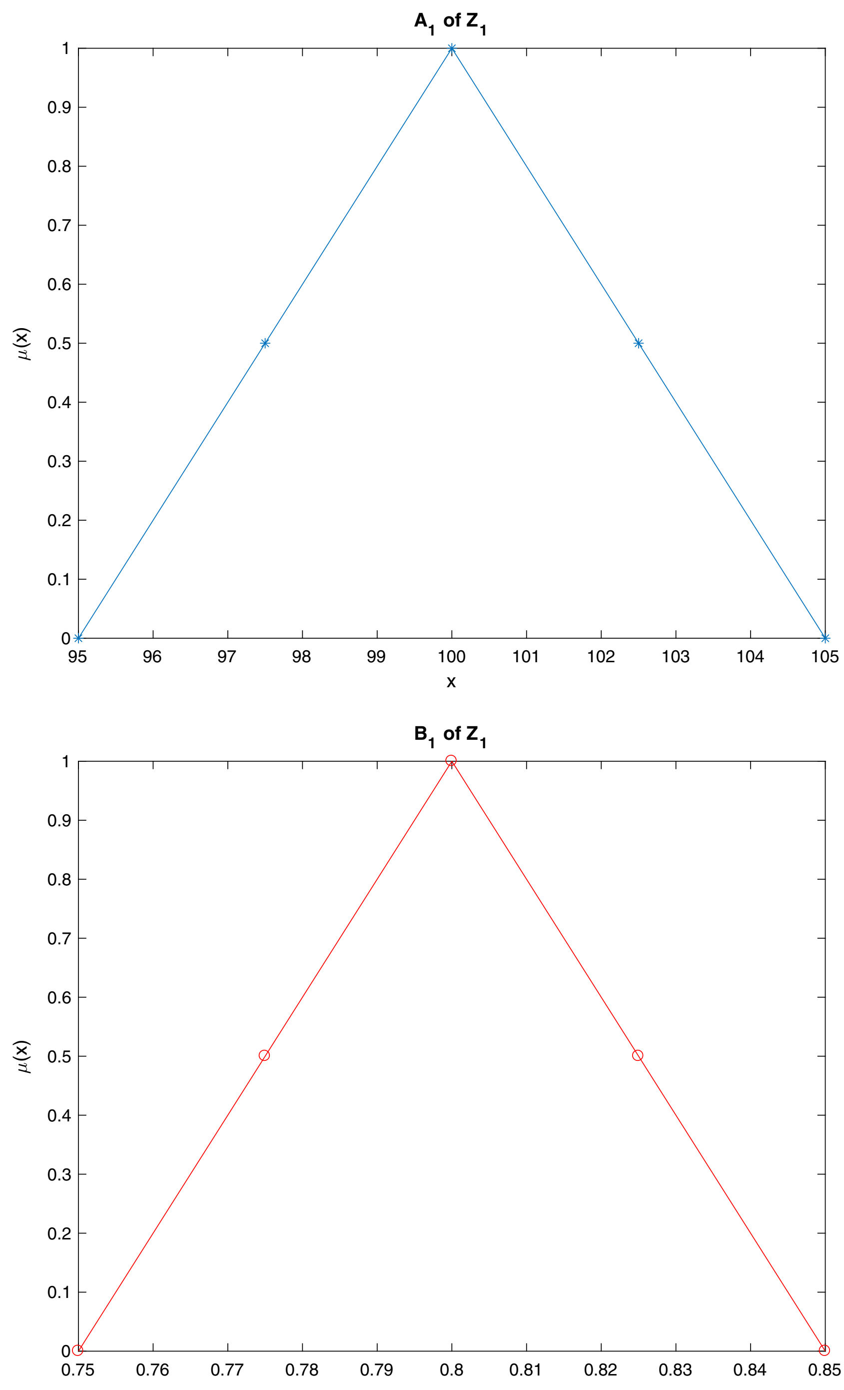


Fig. 12 A sample of Z_1 from paper [2]

Proof

$$TU(Z) = TU\left(\tilde{A}, \tilde{R}\right) \tag{65}$$

$$= \frac{1}{4}\int_0^1\int_0^1 \tilde{A}_1\tilde{R}_1 e^{-\tilde{A}_2^2} e^{-\tilde{R}_2^2} d\alpha d\beta \tag{66}$$

$$= \frac{1}{4}\int_0^1 \tilde{R}_1 e^{-\tilde{R}_2^2} d\beta \int_0^1 \tilde{A}_1 e^{-\tilde{A}_2^2} d\alpha \tag{67}$$

$$= \begin{cases} \underline{\underline{\frac{1}{4}\int_0^1 \tilde{A}_1 e^{-\tilde{A}_2^2} d\alpha = C_1 > 0}} \int_0^1 C_1\tilde{R}_1 e^{-\tilde{R}_2^2} d\beta, & 0 < \tilde{A}_1 \le 1 \\ \underline{\underline{\frac{1}{4}\int_0^1 \tilde{A}_1 e^{-\tilde{A}_2^2} d\alpha = C_2 < 0}} \int_0^1 C_2\tilde{R}_1 e^{-\tilde{R}_2^2} d\beta, & -1 \le \tilde{A}_1 < 0 \end{cases} \tag{68}$$

When $0 < \tilde{A}_1 \le 1$, Let

$$F1 = C_1\tilde{R}_1 e^{-\tilde{R}_2^2}, \quad C_1 > 0 \tag{69}$$

$$\frac{\partial F1}{\partial \tilde{R}_1} = C_1 e^{-\tilde{R}_2^2} > 0 \tag{70}$$

$$\frac{\partial F1}{\partial \tilde{R}_2} = C_1\tilde{R}_1 e^{-\tilde{R}_2^2}\left(-2\tilde{R}_2\right) < 0 \tag{71}$$

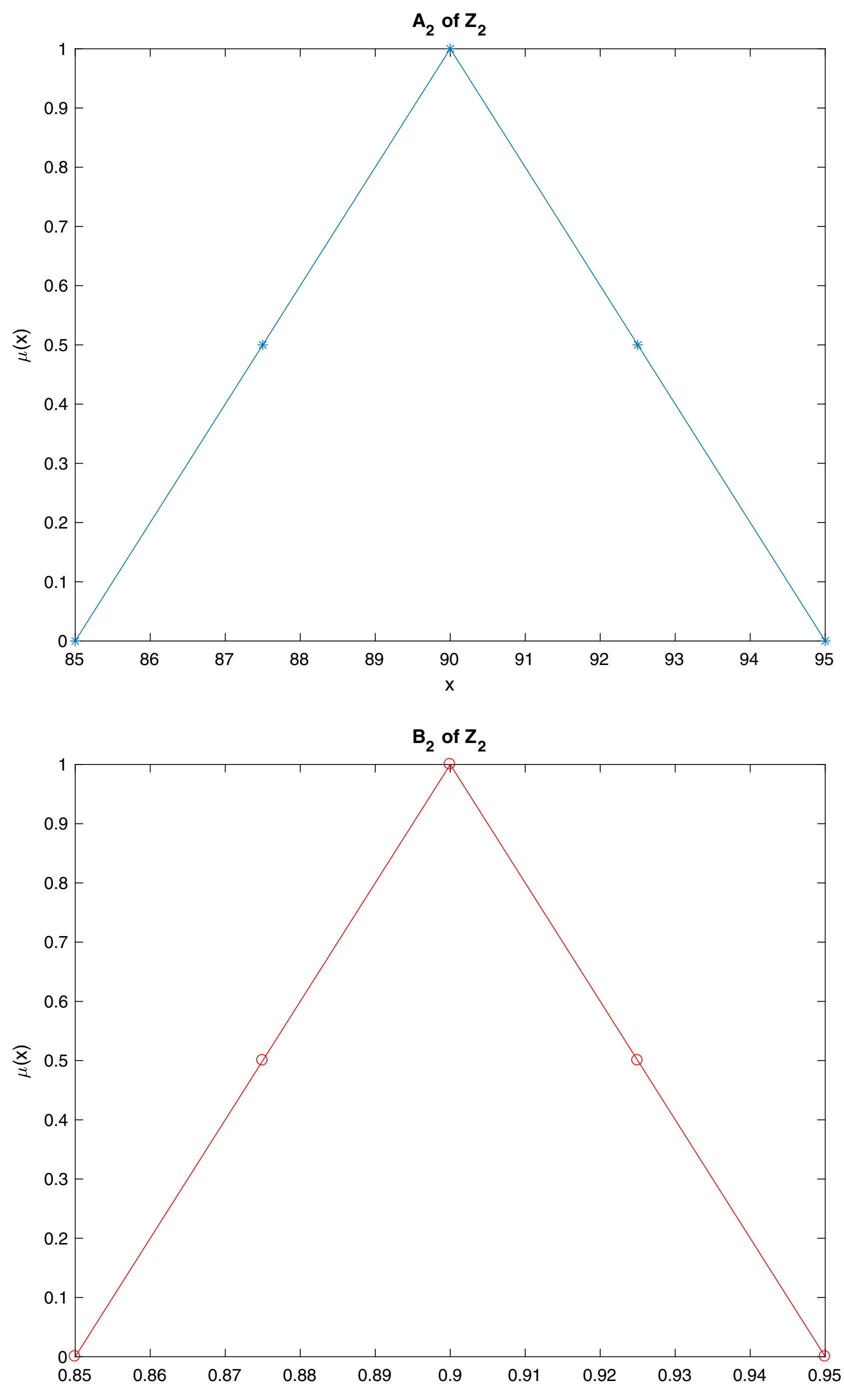


Fig. 13 A sample of Z_2 from paper [2]

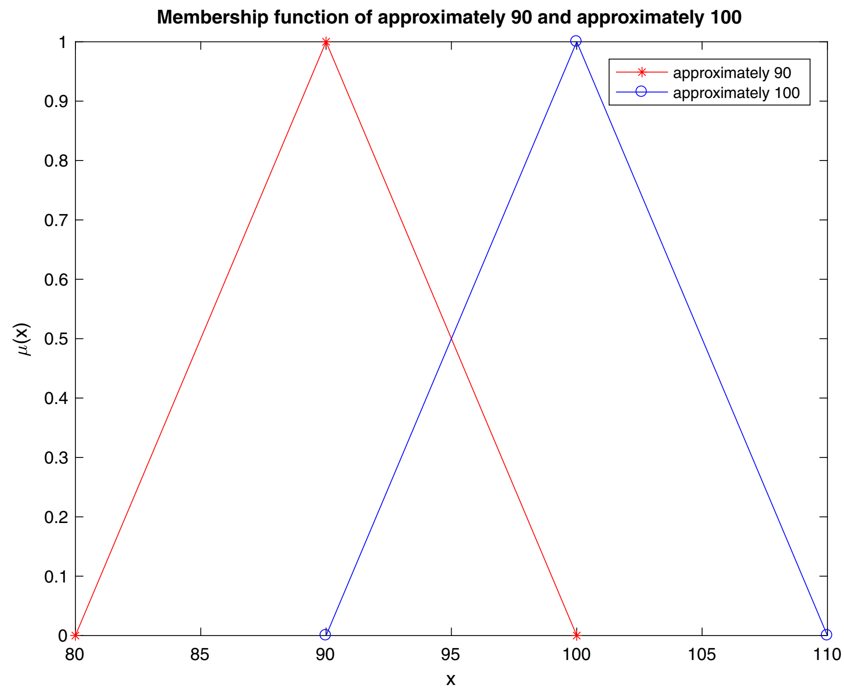


Fig. 14 Membership function of approximately 90 and approximately 100

When $-1 \leq \tilde{A}_1 < 0$, Let

$$F2 = C_2 \tilde{R}_1 e^{-\tilde{R}_2^2}, \quad C_2 < 0 \tag{72}$$

$$\frac{\partial F2}{\partial \tilde{R}_1} = C_2 e^{-\tilde{R}_2^2} < 0 \tag{73}$$

$$\frac{\partial F1}{\partial \tilde{R}_2} = C_1 \tilde{R}_1 e^{-\tilde{R}_2^2} \left(-2\tilde{R}_2\right) > 0 \tag{74}$$

End proof. □

Proposition 3 *Given a* $Z = (\tilde{A}, \tilde{R})$ *with* $(-1 \leq \tilde{A} \leq 1$, $0 \leq \tilde{R} \leq 1)$*, the range of* $TU(Z) \in [-1, 1]$.

Proof (1) For $0 \leq \tilde{A} \leq 1, 0 \leq \tilde{R} \leq 1$,

$$\widetilde{A}_1 = \widetilde{A}^-(\alpha) + \widetilde{A}^+(\alpha) \in [0, 2] \tag{75}$$

$$\widetilde{A}_2 = \widetilde{A}^+(\alpha) - \widetilde{A}^-(\alpha) \in [0, 1] \tag{76}$$

$$\widetilde{R}_1 = \widetilde{R}^-(\beta) + \widetilde{R}^+(\beta) \in [0, 2] \tag{77}$$

$$\widetilde{R}_2 = \widetilde{R}^+(\beta) - \widetilde{R}^-(\beta) \in [0, 1] \tag{78}$$

According to Proposition 1, and 2, when $0 \leq \tilde{A} \leq 1$, $TU(Z)$ is monotonically increasing with $\tilde{A}_1$ and monotonically decreasing with $\tilde{A}_2$, $TU(Z)$ is monotonically increasing with $\tilde{R}_1$ and monotonically decreasing with $\tilde{R}_2$.

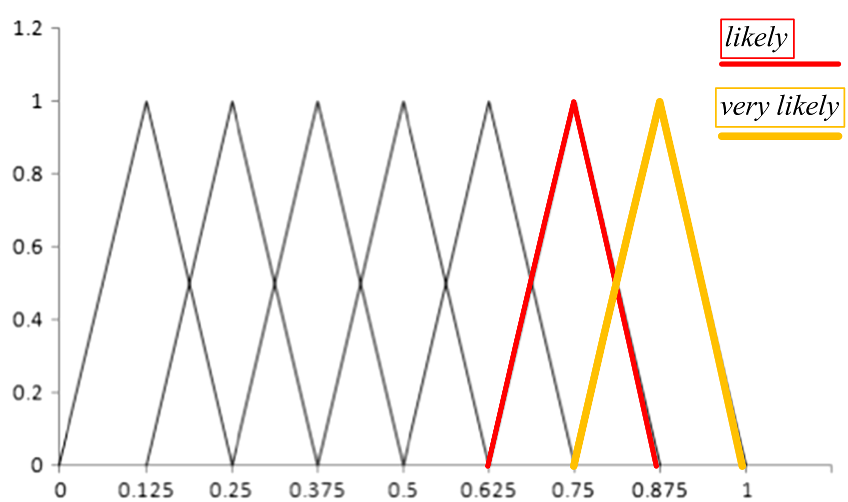
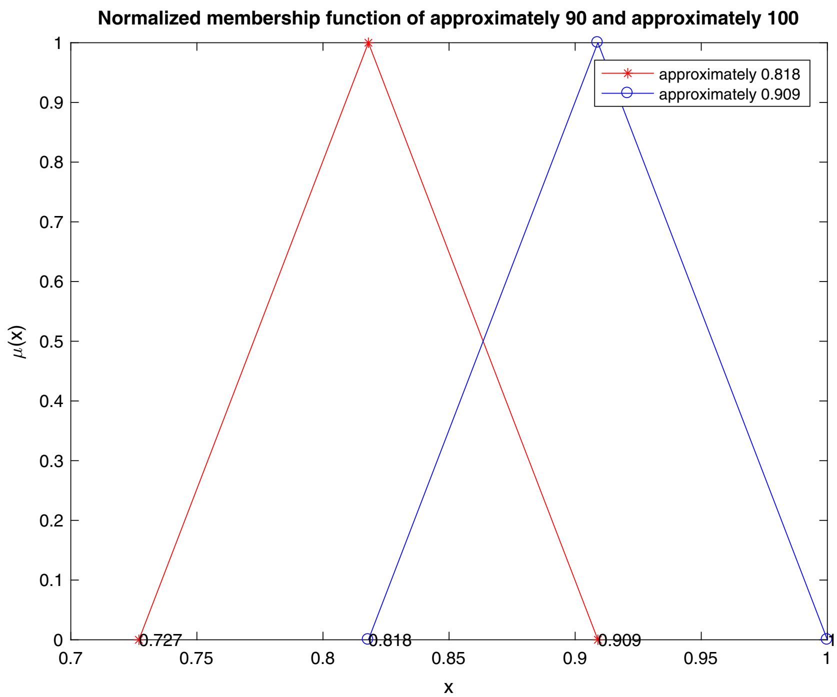


Fig. 15 Membership function of likely and very likely

Fig. 16 Normalized membership function of approximately 90 and approximately 100



When

$$\tilde{A}_1 = \tilde{R}_1 = 2 \tag{79}$$
$$\tilde{A}_2 = \tilde{R}_2 = 0 \tag{80}$$

TU(Z) gets the maximum value as

$$\max TU(Z) = \max TU\left(\tilde{A}, \tilde{R}\right) \tag{81}$$
$$= \frac{1}{4}\int_0^1\int_0^1 \tilde{A}_1\tilde{R}_1 e^{-\tilde{A}_2^2} e^{-\tilde{R}_2^2} d\alpha d\beta \tag{82}$$
$$= \frac{1}{4}\int_0^1\int_0^1 2\times 2e^0e^0 0 d\alpha d\beta \tag{83}$$
$$= 1 \tag{84}$$

when

$$\tilde{A}_1 = \tilde{R}_1 = 0 \tag{85}$$
$$\tilde{A}_2 = \tilde{R}_2 = 1 \tag{86}$$

TU(Z) gets the minimum value as

$$\min TU(Z) = \min TU\left(\tilde{A}, \tilde{R}\right) \tag{87}$$
$$= \frac{1}{4}\int_0^1\int_0^1 \tilde{A}_1\tilde{R}_1 e^{-\tilde{A}_2^2} e^{-\tilde{R}_2^2} d\alpha d\beta \tag{88}$$
$$= 0 \tag{89}$$

Hence, $TU(Z) \in [0, 1]$ if $0 \le \tilde{A} \le 1, 0 \le \tilde{R} \le 1$.
(2) For $-1 \le \tilde{A} \le 0, 0 \le \tilde{R} \le 1$,

$$\widetilde{A}_1 = \widetilde{A}^-(\alpha) + \widetilde{A}^+(\alpha) \in [-2, 0] \tag{90}$$
$$\widetilde{A}_2 = \widetilde{A}^+(\alpha) - \widetilde{A}^-(\alpha) \in [0, 1] \tag{91}$$
$$\widetilde{R}_1 = \widetilde{R}^-(\beta) + \widetilde{R}^+(\beta) \in [0, 2] \tag{92}$$
$$\widetilde{R}_2 = \widetilde{R}^+(\beta) - \widetilde{R}^-(\beta) \in [0, 1] \tag{93}$$

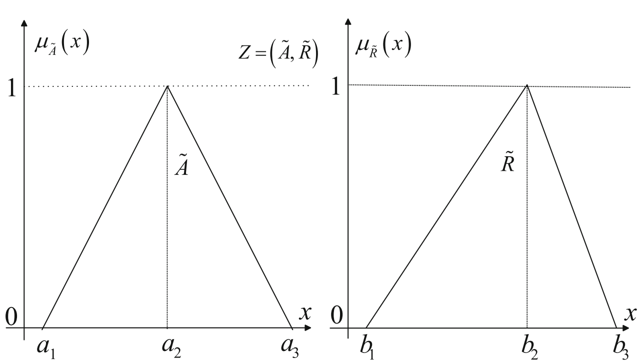


Fig. 17 A simple Z-number with triangular fuzzy number

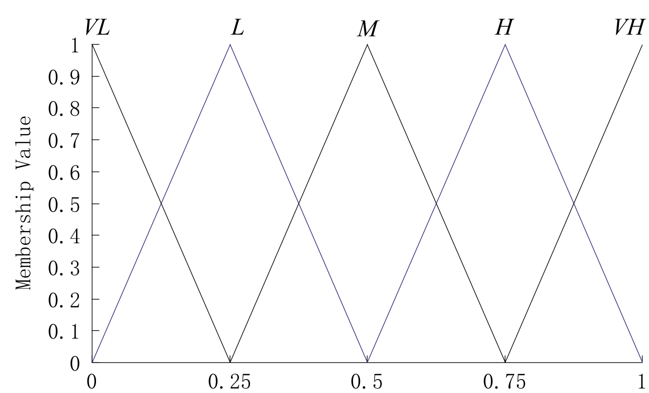


Fig. 18 Membership function of criteria

Table 3 Decision matrix with linguistic values

	Price (pounds) (VH, VH)	Journey time (min) (H, VH)	Comfort (M, VH)
Car	((9,10,12),VH)	((70,100,120),M)	((4,5,6), H)
Taxi	((20,24,25), H)	((60,70,100),VH)	((7,8,10),H)
Train	((15,15,15), H)	((70,80,90), H)	((1,4,7), H)

According to Propositions 1, and 2, when $-1 \le \tilde{A} \le 0$, $TU(Z)$ is monotonically decreasing with $\tilde{A}_1$ and monotonically increasing with $\tilde{A}_2$, $TU(Z)$ is monotonically decreasing with $\tilde{R}_1$ and monotonically increasing with $\tilde{R}_2$.

When

$$\tilde{A}_1 = \tilde{R}_1 = 0 \tag{94}$$

$$\tilde{A}_2 = \tilde{R}_2 = 1 \tag{95}$$

TU(Z) gets the maximum value as

$$\max TU(Z) = \max TU\left(\tilde{A}, \tilde{R}\right) \tag{96}$$

$$= \frac{1}{4}\int_0^1\int_0^1 \tilde{A}_1\tilde{R}_1 e^{-\tilde{A}_2^2} e^{-\tilde{R}_2^2} d\alpha d\beta \tag{97}$$

$$= \frac{1}{4}\int_0^1\int_0^1 0 \times 0e^{-1}e^{-1}0 d\alpha d\beta \tag{98}$$

$$= 0 \tag{99}$$

when

$$\tilde{A}_1 = -2 \tag{100}$$

$$\tilde{R}_1 = 2 \tag{101}$$

$$\tilde{A}_2 = \tilde{R}_2 = 0 \tag{102}$$

TU(Z) gets the minimum value as

$$\min TU(Z) = \min TU\left(\tilde{A}, \tilde{R}\right) \tag{103}$$

$$= \frac{1}{4}\int_0^1\int_0^1 \tilde{A}_1\tilde{R}_1 e^{-\tilde{A}_2^2} e^{-\tilde{R}_2^2} d\alpha d\beta \tag{104}$$

$$= \frac{1}{4}\int_0^1\int_0^1 -2 \times 2e^0 e^0 0 d\alpha d\beta \tag{105}$$

$$= -1 \tag{106}$$

Hence, $TU(Z) \in [-1, 0]$ if $-1 \le \tilde{A} \le 0, 0 \le \tilde{R} \le 1$.

From proof (1) and (2), the conclusion can be made that TU(Z) ranges $[-1, 1]$ if $-1 \le \tilde{A} \le 1, 0 \le \tilde{R} \le 1$.

End proof. □

5 Effectiveness analysis of the proposed total utility of Z-number

In this part, we use several examples and two comparisons with the previous methods to illustrate the effectiveness of the proposed total utility of Z-number.

5.1 Effectiveness analysis using several examples

Example 1 Assume there are two Z-numbers, $Z_1 = ((0.2, 0.3, 0.4, 0.6), (0.4, x, 0.7))$, and $Z_2 = ((0.3, 0.4, 0.5, 0.7), (0.4, x, 0.7))$. The total utility of Z_1 and Z_2 changing with x is shown in Fig. 5.

Example 2 Assume there are two Z-numbers, $Z_3 = ((0.2, 0.4, 0.5, 0.8), (0.4, x, 0.7))$, and $Z_4 = ((0.3, 0.4, 0.5, 0.7), (0.4, x, 0.7))$. The total utility of Z_3 and Z_4 changing with x is shown in Fig. 6.

Example 3 Assume there are two Z-numbers, $Z_5 = ((0.2, 0.4, 0.5, 0.8), (0, x, 1))$, and $Z_6 = ((0.3, 0.4, 0.5, 0.7), (0, x, 1))$, The total utility of Z_5 and Z_6 changing with x is shown in Fig. 7.

Example 4 Assume there is a Z-number $Z_7 = ((0, x, 1), (0, y, 1))$, the total utility of Z_7 changing with x is shown in Fig. 8. The total utility of Z_7 is equal to the line represented by the intersection of the horizontal plane and the curved surface, and is shown in Fig. 9.

Example 5 Assume there is a Z-number $Z_8 = ((-1, x, 1), (0, y, 1))$, the total utility of Z_8 changing with x is shown in Fig. 10. The total utility of Z_8 is equal to the line represented by the intersection of the horizontal plane and the curved surface, and is shown in Fig. 11.

According to these simple examples, we can get that total utility is determined by the mean (or central) value and the range (or variance) of a Z-number. For a fuzzy number, the mean (or central) value represents the expectation of the Z-number, and the range (or variance) refers to the uncertainty of the Z-number. The total utility of a Z-number is based on the following assumptions: for a positive Z-number (positive restriction and positive reliability), the larger the mean (or central) value, the larger the value of

Table 4 Decision matrix with numerical values

	Price (pounds) ((0.75,1,1),(0.75,1,1))	Journey time (min) ((0.5,0.75,1),(0.75,1,1))	Comfort ((0.25,0.5,0.75),(0.75,1,1))
Car	((9,10,12),(0.75,1,1))	((70,100,120),(0.25,0.5,0.75))	((4,5,6),(0.5,0.75,1))
Taxi	((20,24,25),(0.5,0.75,1))	((60,70,100),(0.75,1,1))	((7,8,10),(0.5,0.75,1))
Train	((15,15,15), (0.5,0.75,1))	((70,80,90),(0.5,0.75,1))	((1,4,7),(0.5,0.75,1))

Table 5 Normalized decision matrix

	Price (pounds) ((0.75,1,1),(0.75,1,1))	Journey time (min) ((0.5,0.75,1),(0.75,1,1))	Comfort ((0.25,0.5,0.75),(0.75,1,1))
Car	((0.52,0.6,0.64),(0.75,1,1))	((0,0.17,0.42),(0.25,0.5,0.75))	((0.4,0.5,0.6),(0.5,0.75,1))
Taxi	((0,0.04,0.2),(0.5,0.75,1))	((0.17,0.42,0.5),(0.75,1,1))	((0.7,0.8,1),(0.5,0.75,1))
Train	((0.4,0.4,0.4), (0.5,0.75,1))	((0.25,0.33,0.42),(0.5,0.75,1))	((0.1,0.4,0.7),(0.5,0.75,1))

a Z-number, whereas the larger the range (or variance), the smaller the value of a Z-number.

5.2 Comparison with other methods in [2]

In this section, we use the example of [2] to illustrate the effectiveness of the proposed total utility of Z-number. In paper [2], two Z-numbers, $Z_1 = (A_1, B_1), and Z_2 = (A_2, B_2)$ are constructed as follows:

$$A_1 = 0/95 + 0.5/97.5 + 1/100 + 0.5/102.5 + 0/105,$$
$$B_1 = 0/0.75 + 0.5/0.775 + 1/0.8 + 0.5/0.825 + 0/0.85;$$
$$A_2 = 0/85 + 0.5/87.5 + 1/90 + 0.5/92.5 + 0/95,$$
$$B_2 = 0/0.85 + 0.5/0.875 + 1/0.9 + 0.5/0.925 + 0/0.95.$$

Z_1 and Z_2 can be simulated, as seen Figs. 12 and 13 respectively. Easily we know that A_1, B_1, A_2, and B_2 are all symmetrical triangular fuzzy numbers. At the same time, the span of A_1 and A_2 are both 10 ($|[A_1]_R^{\alpha=0} - [A_1]_L^{\alpha=0}| = |[A_2]_R^{\alpha=0} - [A_2]_L^{\alpha=0}| = 10$), and the span of B_1 and B_2 are both 0.1 ($|[B_1]_R^{\alpha=0} - [B_1]_L^{\alpha=0}| = |[B_2]_R^{\alpha=0} - [B_2]_L^{\alpha=0}| = 0.1$). Therefore, the ordering of Z_1 and Z_2 should be mainly determined by the center of A_1, B_1, A_2, and B_2, where $\mu_{A_1}(x) = 1$, $\mu_{B_1}(x) = 1$, $\mu_{A_2}(x) = 1$, and $\mu_{B_2}(x) = 1$ (i.e. 100 for A_1, 0.8 for B_1, 90 for A_2, and 0.9 for B_2 can be reasonably used to determine the order of Z_1 and Z_2). Then, the commonly used method of weighted average is used to determine the order of Z_1 and Z_2. Hence, $Z_1 < Z_2$ (for $Z_1 \doteq 100 \times 0.8 < Z_2 \doteq 90 \times 0.9$). Aliev et al. [2] obtained the result of $Z_1 > Z_2$ due to a subjective possibility measure to fuzzy terms n_b, n_e and n_w (refer to page 152 in [2]).

Then, we used the proposed the total utility of Z-number to compare the two Z-numbers.

$$A_1 = (95, 100, 105),$$
$$B_1 = (0.75, 0.8, 0.85);$$
$$A_2 = (85, 90, 95),$$
$$B_2 = (0.85, 0.9, 0.95).$$

Firstly, we normalized the two Z-number as follows, then we used the (5) or (50) to obtain the total utility of the two Z-number.

$$A_1 = (95, 100, 105)/105 = (0.9048, 0.9524, 1),$$

$$B_1 = (0.75, 0.8, 0.85);$$

$$A_2 = (85, 90, 95)/105 = (0.8095, 0.8571, 0.9048),$$

$$B_2 = (0.85, 0.9, 0.95).$$

$\mathrm{TU}(Z_1) = 0.7422$, $\mathrm{TU}(Z_1) = 0.7515$, Hence $Z_1 < Z_2$.

From the result, we can see that the proposed notion of the total utility of Z-number can be used to more reasonably order the two Z-numbers.

5.3 Comparison with other methods in [66]

We outline illustrative example in [66] to present the effectiveness of the proposed method. In [66], triangular fuzzy numbers were used to aid in manufacturing-related decision making, accounting for the opinions of three decision makers. The triangular fuzzy numbers are as follows:

$$\tilde{C}_1 = ((0.20, 0.32, 0.44), (0.24, 0.36, 0.48))$$
$$\tilde{C}_2 = ((0.45, 0.56, 0.68), (0.36, 0.48, 0.06))$$
$$\tilde{C}_3 = ((0.48, 0.57, 0.65), (0.00, 0.12, 0.24))$$

A high level outline of the methodology used by the authors of [66] is as follows: firstly, Z-numbers are converted to classical fuzzy numbers, then the Z-numbers are ranked based on comparing the converted classical fuzzy numbers. As an example, using the situation of Case 2 and using (50), the following TUs can be calculated:

$$TU\left(\tilde{C}_1\right) = 0.102$$
$$TU\left(\tilde{C}_2\right) = 0.241$$
$$TU\left(\tilde{C}_3\right) = 0.062$$

Table 6 Decision matrix with crisp number

	Price (pounds) 0.8845	Journey time (min) 0.6539	Comfort 0.4239
Car	0.5396	0.0825	0.3414
Taxi	0.0477	0.3354	0.5539
Train	0.2768	0.2279	0.2469

Table 7 Decision matrix with crisp number

	Price (pounds) 0.8845	Journey time (min) 0.6539	Comfort 0.4239	Priority weight
Car	0.5396	0.0825	0.3414	0.3428
Taxi	0.0477	0.3354	0.5539	0.2565
Train	0.2768	0.2279	0.2469	0.2538

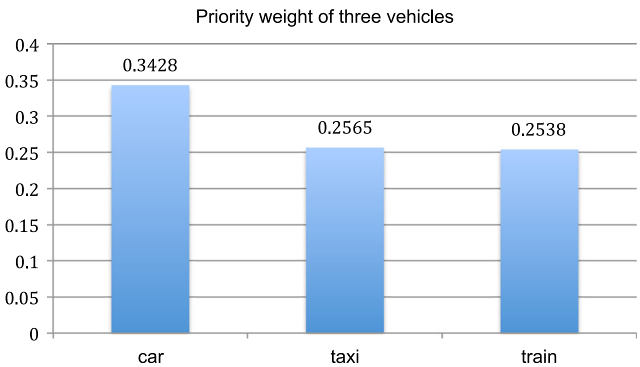


Fig. 19 Priority weight of three vehicles

Hence, the ranking order is $\tilde{C}_2 \succ \tilde{C}_1 \succ \tilde{C}_3$. This order is the same result as [66]. Compared to the methods used in [66], the proposed method in this paper is easier to use.

6 Applications of the proposed total utility of Z-number

In Section 6.1, the authors apply the newly proposed method to answer the question proposed by Zadeh: "Is (approximately 100, likely) greater than (approximately 90, very likely)?" Two steps are necessary to arrive at the final solution. The first step is the normalization of Z-numbers. The second is to rank the Z-numbers based on total utility. In Section 6.2, a simple application of the total utility of Z-numbers is presented to illustrate the procedure of multi-criteria decision making with total utility of Z-numbers. In Section 6.3, a real-world application of total utility of Z-numbers in failure modes risk assessment of the geothermal power plant (a case study) is presented to illustrate the effectiveness of the proposed total utility of Z-numbers. Firstly, we present the application of the total utility of Z-numbers to determine the ordering of Z-numbers.

6.1 Application of total utility of Z-number to determine the ordering of Z-numbers

With the Z-nubmer framework, the natural language of "approximately 100, likely" and "approximately 90, very likely" can be denoted as Z_1 and Z_2 respectively.

$$Z_1 = (approximately\ 100, likely) \tag{107}$$

$$Z_2 = (approximately\ 90, very\ likely) \tag{108}$$

We use the symmetrical triangular fuzzy numbers to model Z_1 and Z_2, as shown in Figs. 14 and 15, and Fig. 14 represents the constraint part of Z_1 and Z_2 with the red line and yellow line respectively, Fig. 15 represents the reliability of Z_1 and Z_2 with the red line and yellow line respectively.

6.1.1 Normalization of fuzzy numbers

Normalization is used to eliminate the influence of different dimensions. All the variables within same category should be converted into numbers ranging between -1 and 1. Assume we have n Z-numbers $Z_i = \left(\tilde{A}_i, \tilde{R}_i\right),\ i = 1, \ldots n$. For the constraint part $\tilde{A}_i$ of the ith Z-number, the $(\alpha = 0)$-cut set of $\tilde{A}_i$ is denoted as

$$\left[\tilde{A}_i\right]^{\alpha=0} = \left[\left[\tilde{A}_i\right]_L^{\alpha=0}, \left[\tilde{A}_i\right]_U^{\alpha=0}\right] \tag{109}$$

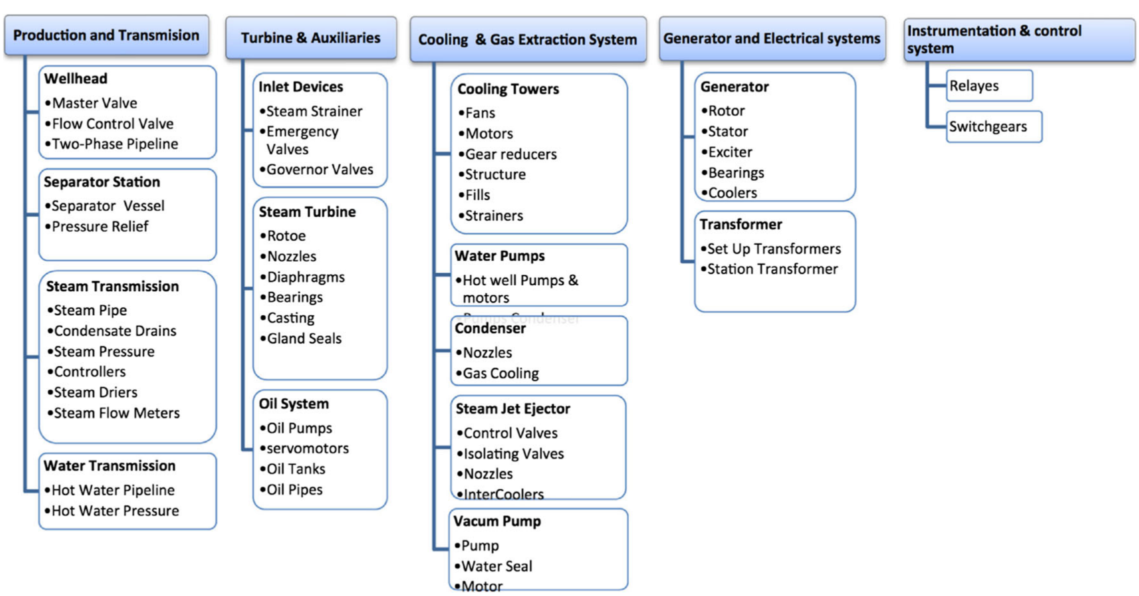


Fig. 20 Equipment block diagram (EBD) of GPP (refers to [67])

Then we can get the max $\left[\tilde{A}_i\right]_U^{\alpha=0}$ for all $\tilde{A}_i$, $i = 1...n$, which can be denoted as k_1,

$$k_{\tilde{A}} = \max\left\{\left|\left[\tilde{A}_i\right]_L^{\alpha=0}\right|, \left|\left[\tilde{A}_i\right]_U^{\alpha=0}\right|\right\},\ i = 1, \ldots n \tag{110}$$

Similarly, for $\tilde{R}_i$,

$$k_{\tilde{R}} = \max\left\{\left|\left[\tilde{R}_i\right]_L^{\alpha=0}\right|, \left|\left[\tilde{R}_i\right]_U^{\alpha=0}\right|\right\},\ i = 1, \ldots n \tag{111}$$

Then we can get the normalized $\tilde{A}_i$, which can be denoted as $\tilde{A}_i'$

$$\tilde{A}_i' = \mu_{\tilde{A}_i'}\left(X'\right) = \mu_{\tilde{A}_i}\left(\frac{X}{k_{\tilde{A}}}\right) \tag{112}$$

Similarly, for $\tilde{R}_i$,

$$\tilde{R}_i' = \mu_{\tilde{R}_i'}\left(X'\right) = \mu_{\tilde{R}_i}\left(\frac{X}{k_{\tilde{R}}}\right) \tag{113}$$

Table 8 Failure modes of the GPP (refers to [67])

	Failure	Failure mode	Cause	Effect
Production and Transmission	PFM1	Sticking valves	Environmental effect	Valves lost disk, scaling
	PFM2	Leaking glands	Separator, wrong quality	Split, crack
	PFM3	Blocked pipes	Deformation, pipeline burst	Deformation
	PFM4	Worn valve disks	Leakage, rupture	Loss of well
	PFM5	Failed traps	Pressure devices	Wrong specification
	PFM6	Dislodged pipes	Wrong operation	Wet steam, downtime
	PFM7	Steam quality degradation	Turbine damage, damage of blades	Reduced turbine efficiency
	PFM8	Scaling problems (calcium, silica, sulfide compounds, etc.)	The plugging and deposit problems in brine handling system, well pipe, injection lines, etc.	Production losses, reduced efficiency
	PFM9	Corrosion problems (carbon dioxide, iron sulfide, oxygen, etc.)	Stress corrosion cracking (SCC) in steam turbines, failure of pipe, production lines, well injections, and equipment	Reduced safety efficiency and power transmission lines. Production losses
Turbine and auxiliaries	PFM10	Scaling on rotor and diaphragms blades	Turbine worn blades, vibration	Reduced efficiency, vibration of rotor, loss of control
	PFM11	Wear and corrosion	Blocked blades	Reduced safety
	PFM12	Sticking of valves	Sticking, leaking	Reduced efficiency
	PFM13	Rotor vibration	Inadequate flow, low pressure	Loss of control
Cooling and NCG extraction system	PFM14	Fouling of condenser tubes	Corrosion on tubes	Poor cooling, loss of efficiency
	PFM15	Blocking of nozzles	Scaling, corrosion	Poor cooling, loss of efficiency
	PFM16	Fouled cooling tower fins	Fan blade failure	Poor cooling, loss of efficiency
	PFM17	Vacuum pump water seal breaking	Water seal break	Loss of vacuum
Generator and electrical systems	PFM18	Rotor vibration	Poor lubrication of bearing	Misalignment
	PFM19	Loose stator coils	Wrong operation	Cost of repair, downtime
	PFM20	Arcing of switch gears	Wrong operation	Poor cooling, corona effect
	PFM21	Failure of motors	Excitation under voltage	Downtime
	PFM22	Failure of transformers	Excitation under voltage	Downtime
Instrumentation and control system	PFM23	H2S damage of copper	Faulty instrument	Safety risk
	PFM24	Wrong control signal	Damage cables	Inefficiency, downtime
	PFM25	Failure of protective relay	Wrong calibration	Inefficiency, downtime

Table 9 Z-numbers for the importance weight of risk factors (refers to [67])

$\tilde{A}$ (restriction component)		$\tilde{R}$ (reliability component)	
Linguistic variable	TFNs and TPFNs	Linguistic variable	TFNs
Equally important (EI)	(0, 0, 0.1, 0.2)	Very low (VL)	(0, 0, 0.1)
Very weakly important (VWI)	(0.1, 0.2, 0.2, 0.3)	Low (L)	(0, 0.1, 0.3)
Weakly important (WI)	(0.2, 0.3, 0.4, 0.5)	Medium low (ML)	(0.1, 0.3, 0.5)
Medium important (MI)	(0.4, 0.5, 0.5, 0.6)	Medium (M)	(0.3, 0.5, 0.7)
Strong important (SI)	(0.5, 0.6, 0.7, 0.8)	Medium high (MH)	(0.5, 0.7, 0.9)
Very strongly important (VSI)	(0.7, 0.8, 0.8, 0.9)	High (H)	(0.7, 0.9, 1)
Absolutely important (AI)	(0.8, 0.9, 1, 1)	Very high (VH)	(0.9, 1, 1)

At last, we can get the normalized Z_i,, which is denoted as $NormalZ_i$,

$$NormalZ_i = \left(\tilde{A}_i', \tilde{R}_i'\right),\ i = 1, \ldots n \tag{114}$$

For $Z_1 = (approximately\ 100, likely), Z_2 = (approximately\ 90, very\ likely)$, $k_{\tilde{A}} = 110$, we can get the normalized Z_1 and normalized Z_2 as

$$NormalZ_1 = (approximately\ 0.909, likely) \tag{115}$$

$$NormalZ_2 = (approximately\ 0.818, very\ likely) \tag{116}$$

and "*approximately* 0.909" and "*approximately* 0.818" are shown in Fig. 16.

6.1.2 Get the total utility of Z-number

$$TU\ (Z_1) \tag{117}$$

$$\leftrightarrow TU\ (NormalZ_1) \tag{118}$$

$$= TU(Triangle\ (0.818, 1)\ , Triangle(0.625, 0.875)) \tag{119}$$

$$= 0.591 \tag{120}$$

$$TU\ (Z_2) \tag{121}$$

$$\leftrightarrow TU\ (NormalZ_2) \tag{122}$$

$$= TU\ (Triangle\ (0.727, 0.909)\ , Triangle(0.75, 1)) \tag{123}$$

$$= 0.646 \tag{124}$$

Then

$$TU\ (Z_1) < TU\ (Z_2) \tag{125}$$

Hence

$$Z_1 < Z_2 \tag{126}$$

The answer to Zadeh' question is "(approximately 100, likely) is less than (approximately 90, very likely)".

In Section 6.2, the application of the total utility of Z-number in multi-criteria decision making is presented. The crisp decision matrix, which is finally converted by the proposed notion of total utility of Z-number, is used to determine the priority weights of each selection.

6.2 Application of total utility of Z-number in decision making

6.2.1 Construct the fuzzy decision-making matrix

Let the matrix M be the multi-criteria decision-making matrix, m is the basic element of the matrix, where $m_{ij} = Z_{ij}(\widetilde{A}, \widetilde{R}), i = 1, ..., m;\ j = 1, ..., n$, and $Z_{ij}(\widetilde{A}, \widetilde{R})$ is the evaluation of the jth criteria for the ith selection. $\widetilde{A}$ and $\widetilde{R}$ represent the constraint and reliability of a Z-number respectively. As an example, the following statement, "The journey time is critical, very surely", contains elements of human opinion, and can be described using Z-number as

Table 10 Z-numbers for the fuzzy rates of potential failure modes (PFMs) (refers to [67])

$\tilde{A}$ (restriction component)		$\tilde{R}$ (reliability component)	
Linguistic variable	TFNs and TPFNs	Linguistic variable	TFNs
Very poor (VP)	(0, 0, 1, 2)	Very low (VL)	(0, 0, 0.1)
Poor (P)	(1, 2, 2, 3)	Low (L)	(0, 0.1, 0.3)
Medium poor (MP)	(2, 3, 4, 5)	Medium low (ML)	(0.1, 0.3, 0.5)
Medium (M)	(4, 5, 5, 6)	Medium (M)	(0.3, 0.5, 0.7)
Medium good (MG)	(5, 6, 7, 8)	Medium high (MH)	(0.5, 0.7, 0.9)
Good (G)	(7, 8, 8, 9)	High (H)	(0.7, 0.9, 1)
Very good (VG)	(8, 9, 10, 10)	Very high (VH)	(0.9, 1, 1)

Table 11 Evaluation of the potential failure modes (PFMs) with regard to the risk factors using Z-numbers (refers to [67])

	O										S										D									
	DM1		DM2		DM3		DM4		DM5		DM1		DM2		DM3		DM4		DM5		DM1		DM2		DM3		DM4		DM5	
	A	R	A	R	A	R	A	R	A	R	A	R	A	R	A	R	A	R	A	R	A	R	A	R	A	R	A	R	A	R
FM1	P	VH	P	H	MP	VH	MP	VH	MP	VH	G	H	VG	VH	G	VH	G	H	G	VH	P	H	MP	VH	P	H	M	VH	MP	VH
FM2	G	H	MG	VH	VG	VH	G	H	G	VH	P	H	P	VH	MP	M	M	M	MP	VH	P	VH	MP	VH	MP	VH	M	VH	P	H
FM3	MG	H	MP	VH	M	VH	M	VH	G	H	P	VH	MP	VH	MG	VH	MP	VH	P	VH	MG	VH	G	H	G	VH	VG	VH	G	VH
FM4	P	VH	MP	VH	MG	VH	MP	VH	P	H	MP	H	MP	H	M	VH	M	VH	G	VH	M	VH	M	M	M	M	MG	VH	M	M
FM5	VG	VH	G	VH	G	H	G	VH	G	H	M	VH	M	VH	MG	H	M	VH	M	VH	VG	VH	VG	VH	VG	VH	VG	H	VG	MH
FM6	VG	H	VG	H	VG	H	VG	VH	VG	VH	P	VH	P	VH	MP	VH	MP	H	MP	VH	P	VH	P	H	MP	H	MP	VH	MP	VH
FM7	VG	VH	VG	VH	G	VH	VG	VH	VG	VH	M	VH	M	VH	MG	H	M	H	M	H	P	VH	P	H	P	H	P	VH	P	VH
FM8	VG	H	VG	VH	VG	VH	VG	VH	VG	H	VG	VH	VG	H	VG	H	VG	VH	VG	H	P	VH	P	VH	MP	VH	P	H	M	MH
FM9	G	VH	G	VH	G	VH	VG	VH	VG	VH	VG	H	VG	VH	VG	VH	G	VH	VG	VH	P	VH	P	VH	P	H	P	H	P	VH
FM10	G	VH	G	VH	VG	H	MG	VH	G	VH	P	H	M	H	MG	VH	M	VH	MP	VH	MP	VH	MP	VH	MG	VH	MG	VH	G	H
FM11	G	VH	VG	VH	VG	VH	G	H	MG	VH	P	H	P	VH	MP	VH	M	VH	M	VH	MG	VH	M	VH	M	VH	MP	H	M	VH
FM12	P	H	P	H	P	VH	MP	VH	MP	H	MG	VH	G	VH	G	VH	G	VH	MG	H	P	VH	MP	VH	MP	H	G	VH	P	VH
FM13	P	H	MP	MH	G	VH	MP	VH	P	VH	MG	VH	VG	VH	VG	VH	VG	H	G	MH	P	MH	M	VH	MP	VH	P	H	MP	VH
FM14	P	VH	MP	VH	M	H	P	H	P	VH	MP	VH	MP	VH	MG	VH	M	VH	MP	H	M	H	MG	H	MG	VH	M	VH	G	VH
FM15	MG	VH	MG	H	G	VH	G	VH	G	VH	P	H	MP	H	M	VH	MG	VH	MP	VH	MG	H	G	VH	MG	H	VG	H	G	VH
FM16	VG	VH	VG	VH	G	VH	G	H	VG	VH	G	H	M	VH	MP	VH	M	VH	MP	VH	MG	VH	MG	VH	G	VH	G	H	VG	VH
FM17	P	H	MP	VH	P	VH	P	VH	P	VH	MP	VH	MP	VH	MP	VH	G	H	M	VH	G	VH	G	VH	MG	H	MG	VH	MG	VH
FM18	P	H	MP	VH	MP	H	MP	VH	M	VH	G	H	G	VH	VG	VH	G	VH	G	VH	MP	VH	M	H	MP	VH	P	VH	P	VH
FM19	MP	VH	P	H	P	H	P	VH	P	VH	G	VH	MG	M	VG	H	G	VH	MG	VH	MP	VH	MP	VH	P	VH	P	VH	MP	VH
FM20	M	VH	M	H	MG	VH	M	MH	M	VH	MG	H	P	VH	G	VH	M	VH	M	VH	P	MH	P	VH	P	VH	P	VH	MP	H
FM21	VG	VH	VG	VH	VG	VH	VG	VH	VG	VH	MP	VH	G	H	MP	VH	MP	VH	P	MH	G	MH	MG	VH	MG	VH	G	H	G	VH
FM22	VG	H	VG	VH	G	VH	VG	MH	VG	VH	MP	H	P	H	P	VH	MP	VH	P	VH	VG	VH	G	MH	G	MH	M	VH	MP	VH
FM23	M	VH	M	VH	MP	VH	M	MH	M	VH	MP	VH	MP	VH	M	H	MP	H	M	VH	P	VH	MP	VH	P	VH	P	VH	P	MH
FM24	VG	VH	VG	VH	VG	VH	G	H	VG	H	MG	VH	MG	H	G	VH	G	VH	MG	MH	G	VH	G	VH	VG	H	VG	H	VG	H
FM25	P	VH	P	H	MP	VH	M	VH	MP	VH	M	VH	MG	MH	MP	VH	G	H	M	VH	MG	VH	MG	VH	MP	VH	MP	H	M	VH

(H, VH). In Section 6.2, if not specially denoted, all Z-numbers $m_{ij} = Z_{ij}(\widetilde{A}, \widetilde{R})$ are combined with triangular fuzzy number, e.g. Fig. 17, unless specifically stated.

Let X be the universe of discourse, which include five linguistic variables describing the degree of security, $X = \{Very\ Low, Low, Medium, High, Very\ High\}$, assuming that only two adjacent linguistic variables have an overlap of their meanings. Let $\widetilde{A}$ be a fuzzy set of the universe of discourse X subjectively defined as follows:

$$f_{Very\ Low}(x) = -4x + 1, \ \ 0 \le x \le 0.25 \tag{127}$$

$$f_{Low}(x) = \begin{cases} 4x, & 0 \le x \le 0.25 \\ -4x + 2, & 0.25 \le x \le 0.5 \end{cases} \tag{128}$$

$$f_{Medium}(x) = \begin{cases} 4x - 1, & 0.25 \le x \le 0.5 \\ -4x + 3, & 0.5 \le x \le 0.75 \end{cases} \tag{129}$$

$$f_{High}(x) = \begin{cases} 4x - 2, & 0.5 \le x \le 0.75 \\ -4x + 4, & 0.75 \le x \le 1 \end{cases} \tag{130}$$

$$f_{Very\ High}(x) = 4x - 3, \quad 0.75 \le x \le 1 \tag{131}$$

where $f_{Very\ Low}$, f_{Low}, f_{Medium}, f_{High} and $f_{Very\ High}$ are the membership functions of the fuzzy sets, which are shown in Fig. 18.

6.2.2 Transform the linguistic values to numerical values

Some knowledge/opinions are presented as linguistic values. In order to deal with these linguistic values, these linguistic variables should be converted into numerical values under the frame of fuzzy set which is described by Fig. 18. For example, if the Z-number is (H, VH) according to linguistic values, then according the linguistic membership function of linguistic, the numerical value is ((0.5, 0.75, 1), (0.75, 1, 1)).

6.2.3 Normalize the fuzzy decision-making matrix

To avoid the complexity of mathematical operations in the decision-making process, the linear-scale transformation is used here to transform the various criteria scales into comparable scales. The set of criteria can be divided into benefit criteria (the larger the rating, the greater the preference) and cost criteria (the smaller the rating, the greater the preference). The normalized fuzzy matrix of the part of constraint $\widetilde{A}$ can be represented as

$$M\left(\widetilde{A}\right) = \left[\widetilde{a_{ij}}\right]_{m \times n} \tag{132}$$

$$\widetilde{a_{ij}} = \left(\frac{a_{ij}^l}{c_j^+}, \frac{a_{ij}^m}{c_j^+}, \frac{a_{ij}^u}{c_j^+}\right) \ \ j \in B \tag{133}$$

$$\widetilde{a_{ij}} = \left(\frac{a_j^-}{a_{ij}^u}, \frac{a_j^-}{a_{ij}^m}, \frac{a_j^-}{a_{ij}^l}\right) \ \ j \in C \tag{134}$$

where B in (133) and C in (134) are the sets of benefit criteria and cost criteria, respectively, and

$$c_j^+ = \max_i \left(a_{ij}^u\right) \quad a_j^- = \min_i \left(a_{ij}^l\right)$$

6.2.4 Convert the Z-numbers to crisp numbers using proposed total utility of Z-number

After normalizing the decision matrix M, the proposed total utility of Z-number is used to determine the utility of each element with Z-number, and then converts the decision matrix of Z-numbers into a crisp decision matrix. In Section 3, the notion of the total utility of Z-number has been discussed in details and some special cases have been introduced, including symmetrical triangular fuzzy numbers, Gaussian fuzzy numbers. In real-world applications, some asymmetrical fuzzy number are always taken into consideration to satisfy the flexibility of the knowledge of human beings. Here, the initial definition of the total utility of Z-number must be denoted again to emphasize the generalization of the total utility of Z-number.

Assume a Z-number is denoted as $Z = \left(\tilde{A}, \tilde{R}\right)$, $-1 \le \tilde{A} \le 1$, $0 \le \tilde{R} \le 1$, and the total utility of Z-number is denoted as $TU(Z)$.

Table 12 Z-numbers for the fuzzy rates of failure modes (PFMs)

$\tilde{A}$ (restriction component)		$\tilde{R}$ (reliability component)	
Linguistic variable	TFNs and TPFNs	Linguistic variable	TFNs
Very poor (VP)	(0, 0, 0.1, 0.2)	Very low (VL)	(0, 0, 0.1)
Poor (P)	(0.1, 0.2, 0.2, 0.3)	Low (L)	(0, 0.1, 0.3)
Medium poor (MP)	(0.2, 0.3, 0.4, 0.5)	Medium low (ML)	(0.1, 0.3, 0.5)
Medium (M)	(0.4, 0.5, 0.5, 0.6)	Medium (M)	(0.3, 0.5, 0.7)
Medium good (MG)	(0.5, 0.6, 0.7, 0.8)	Medium high (MH)	(0.5, 0.7, 0.9)
Good (G)	(0.7, 0.8, 0.8, 0.9)	High (H)	(0.7, 0.9, 1)
Very good (VG)	(0.8, 0.9, 1, 1)	Very high (VH)	(0.9, 1, 1)

Table 13 Evaluation of the PFMs with regard to the risk factors using utility of Z-numbers

	O					S					D				
	DM1	DM2	DM3	DM4	DM5	DM1	DM2	DM3	DM4	DM5	DM1	DM2	DM3	DM4	DM5
FM1	0.1918	0.1677	0.3258	0.3258	0.3258	0.6709	0.8783	0.7672	0.6709	0.7672	0.1677	0.3258	0.1677	0.4795	0.3258
FM2	0.6709	0.605	0.8783	0.6709	0.7672	0.1677	0.1918	0.1591	0.2342	0.3258	0.1918	0.3258	0.3258	0.4795	0.1677
FM3	0.5291	0.3258	0.4795	0.4795	0.6709	0.1918	0.3258	0.605	0.3258	0.1918	0.605	0.6709	0.7672	0.8783	0.7672
FM4	0.1918	0.3258	0.605	0.3258	0.1677	0.2849	0.2849	0.4795	0.4795	0.7672	0.4795	0.2342	0.2342	0.605	0.2342
FM5	0.8783	0.7672	0.6709	0.7672	0.6709	0.4795	0.4795	0.5291	0.4795	0.4795	0.8783	0.8783	0.8783	0.7681	0.6005
FM6	0.7681	0.7681	0.7681	0.8783	0.8783	0.1918	0.1918	0.3258	0.2849	0.3258	0.1918	0.1677	0.2849	0.3258	0.3258
FM7	0.8783	0.8783	0.7672	0.8783	0.8783	0.4795	0.4795	0.5291	0.4193	0.4193	0.1918	0.1677	0.1677	0.1918	0.1918
FM8	0.7681	0.8783	0.8783	0.8783	0.7681	0.8783	0.7681	0.7681	0.8783	0.7681	0.1918	0.1918	0.3258	0.1677	0.3278
FM9	0.7672	0.7672	0.7672	0.8783	0.8783	0.7681	0.8783	0.8783	0.7672	0.8783	0.1918	0.1918	0.1677	0.1677	0.1918
FM10	0.7672	0.7672	0.7681	0.605	0.7672	0.1677	0.4193	0.605	0.4795	0.3258	0.3258	0.3258	0.605	0.605	0.6709
FM11	0.7672	0.8783	0.8783	0.6709	0.605	0.1677	0.1918	0.3258	0.4795	0.4795	0.605	0.4795	0.4795	0.2849	0.4795
FM12	0.1677	0.1677	0.1918	0.3258	0.2849	0.605	0.7672	0.7672	0.7672	0.5291	0.1918	0.3258	0.2849	0.7672	0.1918
FM13	0.1677	0.2227	0.7672	0.3258	0.1918	0.605	0.8783	0.8783	0.7681	0.5245	0.1311	0.4795	0.3258	0.1677	0.3258
FM14	0.1918	0.3258	0.4193	0.1677	0.1918	0.3258	0.3258	0.605	0.4795	0.2849	0.4193	0.5291	0.605	0.4795	0.7672
FM15	0.605	0.5291	0.7672	0.7672	0.7672	0.1677	0.2849	0.4795	0.605	0.3258	0.5291	0.7672	0.5291	0.7681	0.7672
FM16	0.8783	0.8783	0.7672	0.6709	0.8783	0.6709	0.4795	0.3258	0.4795	0.3258	0.605	0.605	0.7672	0.6709	0.8783
FM17	0.1677	0.3258	0.1918	0.1918	0.1918	0.3258	0.3258	0.3258	0.6709	0.4795	0.7672	0.7672	0.5291	0.605	0.605
FM18	0.1677	0.3258	0.2849	0.3258	0.4795	0.6709	0.7672	0.8783	0.7672	0.7672	0.3258	0.4193	0.3258	0.1918	0.1918
FM19	0.3258	0.1677	0.1677	0.1918	0.1918	0.7672	0.2955	0.7681	0.7672	0.605	0.3258	0.3258	0.1918	0.1918	0.3258
FM20	0.4795	0.4193	0.605	0.3278	0.4795	0.5291	0.1918	0.7672	0.4795	0.4795	0.1311	0.1918	0.1918	0.1918	0.2849
FM21	0.8783	0.8783	0.8783	0.8783	0.8783	0.3258	0.6709	0.3258	0.3258	0.1311	0.5245	0.605	0.605	0.6709	0.7672
FM22	0.7681	0.8783	0.7672	0.6005	0.8783	0.2849	0.1677	0.1918	0.3258	0.1918	0.8783	0.5245	0.5245	0.4795	0.3258
FM23	0.4795	0.4795	0.3258	0.3278	0.4795	0.3258	0.3258	0.4193	0.2849	0.4795	0.1918	0.3258	0.1918	0.1918	0.1311
FM24	0.8783	0.8783	0.8783	0.6709	0.7681	0.605	0.5291	0.7672	0.7672	0.4137	0.7672	0.7672	0.7681	0.7681	0.7681
FM25	0.1918	0.1677	0.3258	0.4795	0.3258	0.4795	0.4137	0.3258	0.6709	0.4795	0.605	0.605	0.3258	0.2849	0.4795

Let $m_{ij} = Z_{ij}(\widetilde{A}, \widetilde{R}), i = 1, ..., m; j = 1, ..., n, \widetilde{A} = (a_{ij}^l, a_{ij}^m, a_{ij}^u), \widetilde{R} = (r_{ij}^l, r_{ij}^m, r_{ij}^u)$

$$TU\left(Z_{ij}\right) = TU\left(\tilde{A}, \tilde{R}\right) = \int_0^1 \int_0^1 \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} \left\{ \begin{array}{l} \left[\frac{\tilde{A}^-(\alpha)+\tilde{A}^+(\alpha)}{2} + x\left(\tilde{A}^+(\alpha) - \tilde{A}^-(\alpha)\right)\right] e^{-\left[\tilde{A}^+(\alpha)-\tilde{A}^-(\alpha)\right]^2} \\ \times \left[\frac{\tilde{R}^-(\beta)+\tilde{R}^+(\beta)}{2} + y\left(\tilde{R}^+(\beta) - \tilde{R}^-(\beta)\right)\right] e^{-\left[\tilde{R}^+(\beta)-\tilde{R}^-(\beta)\right]^2} \end{array} \right\} dxdyd\alpha d\beta \tag{135}$$

where $\tilde{A}$, $\tilde{R}$ are two regular fuzzy numbers, which represent the "constraint" and "reliability" of a Z-number, $-1 \leq \tilde{A} \leq 1$, $0 \leq \tilde{R} \leq 1$. $[\tilde{A}^-(\alpha), \tilde{A}^+(\alpha)]$ is the α-cut set of fuzzy number $\tilde{A}$ ($\alpha \in [0, 1]$), $[\tilde{R}^-(\beta), \tilde{R}^+(\beta)]$ is the β-cut set of fuzzy number $\tilde{R}$ ($\beta \in [0, 1]$), which are shown in Fig. 4.

6.2.5 Priority weighting of each alternative

The priority weight of each alternative can be defined as follows:

$$priority = \sum TU\left(Z_a\right) \times TU\left(Z_f\right) \tag{136}$$

where Z_a is the weight of the criteria, and Z_f is the value of each criteria.

Here we give an example of the selection of a specific vehicles for journey in order to illustrate the procedure of the proposed approach. There are three different choices, namely car, taxi and train. The three main criteria, price, journey time, and comfort, are taken into consideration. For each vehicle, according to the particular case, the cost is the most significant element, and can be described using the linguistic variable "Very High", and the reliability of the cost is also very strong, described using the linguistic variable "Very High". Similarly, the journey time and the comfort can be also be described using linguistic under the notion of Z-numbers. The linguistic criteria evaluation of the three vehicles can be described based on the information in Table 3.

According to the membership function denoted by (127) to (131) and described by Fig. 18, the linguistic variable can be converted to a numerical value, which is described in Table 4.

The third step is to normalize the fuzzy data to avoid complexity of mathematical operations in the decision-making process according to the (133) and (134). The criteria of price and journey time are categorized as cost criteria, and the comfort criteria is categorized as a benefit. The normalized decision matrix is denoted by the Table 5.

The fourth step is to convert Z-number to a crisp number according to the proposed total utility of Z-number (135). The resulting normalized matrix result is shown in Table 6.

Finally, after normalizing the weights of each criteria, according to (136). the final priority weights of the three vehicles can be achieved, and are shown in Table 7. The results are shown in Fig. 19.

Table 14 Values of potential failure modes (PFMs)

Failure mode	FM1	FM2	FM3	FM4	FM5	FM6	FM7	FM8	FM9	FM10
Occurrence	0.2674	0.7185	0.497	0.3232	0.7509	0.8122	0.8561	0.8342	0.8116	0.7349
Severity	0.7509	0.2157	0.328	0.4592	0.4894	0.264	0.4653	0.8122	0.834	0.3995
Detectability	0.2933	0.2981	0.7377	0.3574	0.8007	0.2592	0.1822	0.241	0.1822	0.5065
	FM11	FM12	FM13	FM14	FM15	FM16	FM17	FM18	FM19	FM20
Occurrence	0.7599	0.2276	0.335	0.2593	0.6871	0.8146	0.2138	0.3167	0.209	0.4622
Severity	0.3289	0.6871	0.7308	0.4042	0.3726	0.4563	0.4256	0.7702	0.6406	0.4894
Detectability	0.4657	0.3523	0.286	0.56	0.6721	0.7053	0.6547	0.2909	0.2722	0.1983
	FM21	FM22	FM23	FM24	FM25					
Occurrence	0.8783	0.7785	0.4184	0.8148	0.2981					
Severity	0.3559	0.2324	0.3671	0.6164	0.4739					
Detectability	0.6345	0.5465	0.2065	0.7677	0.46					

Table 15 Entropy measure, divergence, objective weights of risk factors, subjective weights of risk factors, and total weights of risk factors

	Occurrence	Severity	Detection
e_j	0.9680	0.9794	0.9671
div_j	0.0320	0.0206	0.0329
w_j^o	0.3744	0.2405	0.3850
w_j^s	0.441	0.301	0.258
w	0.4077	0.2708	0.3215

6.3 Application of total utility of Z-number in the failure modes risk assessment of the geothermal power plant (a case study)

In this paper, an application of the total utility of Z-number in the failure modes risk assessment of the geothermal power plant (GPP) (a case study in [67])) is used to illustrate the effectiveness of the proposed notion of total utility of Z-number. The FMEA (failure modes and effect analysis) adopts three parameters of severity (S), occurrence (O), and detection (D) as risk factors is used to calculate a risk priority number (R.P.N) [63, 68]. One of the most critical steps in the application of the FMEA is to decompose a system into its individual components. In this paper, we use an equipment block diagram (EBD) of the GPP (geothermal power plant), which refers to [67]) as Fig. 20. The EBD accounted for a number of different systems including generator and electrical systems, turbine and auxiliaries, production and transmission, and cooling and gas extraction systems. The explanation of the EDB is summarized in Table 8 (taken from [67]).

The linguistic variables for the importance weight of risk factors and the fuzzy rates of failure modes are shown in Tables 9 and 10 respectively.

After the evaluation of the domain experts, the decision matrix including factors as severity (S), occurrence (O), and detection (D) are established and presented in Table 11 (taken from [67]).

Next, we will use the utility of Z-numbers and FMEA to rank the risk factors and get their RPNs.

Firstly, the linguistic variables are normalized using the equation from (109) to (113), then the fuzzy rates of failure modes (PFMs) can be normalized as Table 12.

Secondly, we use the (5) to get the evaluation of the PFMs with regard to the risk factors using the utility of Z-numbers, the results are shown in Table 13.

Thirdly, the knowledge of five DMs is combined and an average is calculated, assuming the weight of the five DMs are equal (weights = 1/5). The values of potential failure modes (PFMs) are shown in Table 14.

Fourthly, we use the method of entropy (refers to [67]) to get the entropy measure (e_j), divergence (div_j), and objective weights of risk factors (w_j^o), which are denoted in Table 15. At the same time, we use the subjective weight (w_j^s) of risk factors in [67] directly. At this point, the comprehensive weight (w) can be obtained through combining the average of the subjective weights and objective weights of risk factors. The comprehensive weights for O, S, and D are shown in Table 15.

Lastly, the values and ranking of potential failure modes are achieved by using the averaging method of the comprehensive weights w, as shown in Table 15. The results are shown in Table 16.

Table 16 Values and ranking of potential failure modes (PFMs)

Failure mode	FM1	FM2	FM3	FM4	FM5	FM6	FM7	FM8	FM9	FM10
RPN ($*10^{-3}$)	0.0167	0.0131	0.0342	0.0151	0.0836	0.0158	0.0206	0.0464	0.0350	0.0422
Ranking	17	22	9	21	2	19	12	6	8	7
Ranking [67]	16	17	8	11	2	14	12	4	9	8
	FM11	FM12	FM13	FM14	FM15	FM16	FM17	FM18	FM19	FM20
RPN ($*10^{-3}$)	0.0330	0.0156	0.0199	0.0167	0.0489	0.0744	0.0169	0.0202	0.0103	0.0127
Ranking	10	20	14	18	5	3	16	13	24	23
Ranking [67]	10	13	13	16	6	3	10	16	17	18
	FM21	FM22	FM23	FM24	FM25					
RPN ($*10^{-3}$)	0.0563	0.0281	0.0090	0.1095	0.0185					
Ranking	4	11	25	1	15					
Ranking [67]	5	7	19	1	15					

Compared with [67], the results achieved by the methods proposed in this paper are the same as the results achieved in [67] for the first three options. These results are emphasized by the colour grey in Table 16. We conclude that the newly-proposed method is useful for identifying top potential failure modes. Other rankings are not the same as the results of [67] because the authors of this paper attribute this difference in rankings to a parameter v in [67] used to get the final ranking of potential failure modes and the parameter v can be generated in an arbitrary way. Compared with [67], the advantage of the newly proposed method of the utility of Z-numbers is simple and easily understood. We attribute this simplicity to the lack of complex defuzzification procedures in the newly proposed method, except for the first step. The majority of time and complexity using the proposed method is spent on the calculation of total utility of Z-number in the first step. Time is saved using the proposed method because there is no need to scan the fuzzy decision matrices, as is necessary in [67].

7 Conclusions

Z-numbers have been introduced by Zadeh in 2011, and are considered as a powerful tool in describing human knowledge. In this paper, we developed a new notion of the total utility of Z-number to measure the comprehensive effects of a Z-number, which is potentially useful in determining the ordering of Z-numbers and to simplify the Z-number based applications in fuzzy decision making. The function of the total utility of Z-numbers is absolutely derived from the format of Z-numbers without subjective judgment. The proposed total utility of Z-numbers is a general framework to deal with arbitrary kinds of Z-numbers (e.g., triangular fuzzy number-based, Gaussian fuzzy number-based, trapezoidal fuzzy number-based, mixed types-based, etc.). The analytical solutions of the common cases of Z-numbers based on triangular fuzzy numbers and Gaussian fuzzy numbers are obtained using the proposed method in this paper. The mathematical properties of the total utility of Z-numbers are also specifically discussed. The results of the proposed method were compared with the results of previous work, and the effectiveness of the total utility of Z-number was verified. Finally, the application of determining the priority of Z-numbers, an application in multi-criteria decision making under uncertain environments, and a real-world application of the total utility of Z-number in the failure modes risk assessment of a geothermal power plant (a case study) were used to illustrate the procedure of application of the total utility of Z-numbers. From the results of the application in the failure modes risk assessment of the geothermal power plant, the proposed method was deemed useful to identify the top potential failure modes. The results of the proposed method are of great significance for dealing with emergency scenarios.

In future work, the authors will extend the application of the total utility of Z-numbers in natural language processing and fuzzy game theory, especially in the context of linguistic data-based applications.

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