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A fuzzy group decision making model with trapezoidal fuzzy preference relations based on compatibility measure and COWGA operator

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Abstract This paper proposes a fuzzy group decision-making model based on a logarithm compatibility measure with multiplicative trapezoidal fuzzy preference relations (MTFPRs) based on a continuous ordered weighted geometric averaging (COWGA) operator. New concepts are presented to measure deviation between MTFPR and its expected fuzzy preference relation. Then, an iterative algorithm is developed to help individual MTFPR reach acceptable compatibility. To determine the weights of decision makers, an optimal model is constructed using group logarithm compatibility index COWGA operator. Finally, we illustrate an example to show how it works and compare it with the existing methods. The main advantages of the proposed approach are the following: (1) The COWGA operator makes decision making more flexible; (2) an iterative and convergent algorithm is proposed to improve the compatibility of MTFPR; (3) decision makers' weights in group decision making are determined by an optimal model based on a logarithm compatibility measure.

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1 Introduction

Due to the limitation of the decision maker's (DM's) values, attitude and background, it is difficult for a single DM to take all possible aspects into consideration in the decision-making process. Group decision making (GDM) with preference relations is a common tool in human activities that consist of determining the most reasonable alternatives as realized by a group of DMs, preference information for each pairwise comparison between different alternatives, collective preference relations by aggregating DMs' preference relations, and selection of optimal alternatives using aggregation techniques. In recent years, different preference relations have been investigated for addressing GDM problems, including multiplicative preference relations [9, 20, 30, 44, 57, 58, 76], fuzzy preference relations [8, 32, 38], linguistic preference relations [18, 31, 47], and intuitionistic fuzzy preference relations [2, 23, 33].

Satty [44] proposed the multiplicative preference relation. However, because of the complexity of decision making and the lack of knowledge of the DMs, preference relations consisting of exact values do not satisfy decision-making requests. Orlovsky [42] proposed the fuzzy preference relation to show a DM's opinion in decision processes. Satty and Vargas [46] put forward interval multiplicative preference relation in the decision making process. Xu [59] defined the interval fuzzy preference relation. Buckley [3] extended the analytic hierarchy process (AHP) to fuzzy environment and introduced the MTFPR.

The key to solving GDM problems with preference relations is to effectively aggregate all of the individual preference relations, which are divided into two aspects. The first is to determine whether all of the individual preference relations are aggregated, which takes into account consistency and compatibility. Consistency is to ensure that preference relations are neither random nor illogical in pairwise comparisons. Lack of consistency in decision making with preference relations causes inconsistent conclusions. Compatibility guarantees that all individual preference relations can be aggregated effectively. It can be used to measure the consensus of ranking between the group and individual.

Satty [45] was the first to discuss the compatibility of preference relations. Studies that address different compatibilities of difference preference relations are demonstrated in Table 1. From Table 1, we can see that few studies have addressed the compatibility of two MTFPRs.

The second crucial procedure in GDM is aggregation. During the aggregation phase, the weighting method is of great importance because the weighting vector is able to affect the final aggregation results directly. At present, many methods have been developed to obtain the weighting vector in GDM. These methods are summarized in Table 2. From Table 2, we can see that no optimization model is utilized to obtain the weighting vector in GDM with MTFPRs.

However, it is necessary to accurately take into account the risk attitudes of DMs. To address this issue, continuous interval information aggregation operators are employed to address the interval values. Based on the ordered weighted averaging (OWA) operator [64], Yager developed the continuous ordered weighted averaging (COWA) operator [65], which is a famous continuous interval information aggregation operator. In addition, encouraged by the COWA operator and geometric mean, Yager and Xu [66] introduced the continuous ordered weighted geometric (COWGA) operator. From Table 1, we see that Refs. [76, 77] consider the risk attitude of DMs based on the COWGA operator. However, few studies have addressed GDM while taking into account MTFPRs.

Table 1 Discussions of related works addressing the compatibilities for different preference relations

Types of preference relations (PRs)	References
Multiplicative PRs	Chen and Chen [10]; Satty [45]
Interval multiplicative PRs	Satty and Vargas [46]; Wang et al. [50]; Wu et al. [51]; Xu [63]; Zhou et al. [75]; Zhou et al. [76]
Multiplicative linguistic PRs	Wu et al. [52]; Zhou et al. [72, 74]
Interval fuzzy linguistic PRs	Chen et al. [12]; Xu [63]; Zhou and Chen [73];
Interval fuzzy PRs	Xu [59, 63]
Triangular Fuzzy PRs	Xu [63]; Yao and Xu [67]
Intuitionistic multiplicative PRs	Jiang et al. [33, 34]; Xu [60]

Table 2 Discussions of different weighting methods

Weighting method	References
Optimization models	Chen et al. [12]; Dong et al. [19]; Wang et al. [50]; Xu and Wu [56]; Zhou et al. [75]; Zhou et al. [76]; Zhou and Chen [73]
Straightforward construction methods	Wu et al. [51]; Wu et al.[53]; Xu [61]
The entropy weight methods	Chen and Zhou [13]; Zamri and Abdullah [70]
The Shapley value methods	Tao et al. [48]
The linguistic quantifier's methods	Dong et al. [18]; Tapia Garcia et al. [49]; Yager [65]
Intelligent optimization algorithm	Cabrerizo et al. [5]

Furthermore, Gong, Lin and Yao [24] say that “the research on preference relations of trapezoidal fuzzy numbers is of theoretical and practical significance”. Based on the discussion above, the main motivations of this paper are the following: (1) The consistency improving algorithm adjusts a pair of elements in each round, but most existing algorithms adjust the elements of the preference relations in each round, which leads to a loss of preference information; (2) An optimal model to derive the weights of DMs is constructed, but many models of consistency and consensus do not address the determination of the weighting vector; (3) we change TFPRs into ordinary preference relations by using a risk attitude parameter for the DMs.

The aim of this work is to define a logarithm compatibility to measure the MTFPRs in GDM based on the COWGA operator. By using α-cut, we obtain the expected interval value of the trapezoidal fuzzy number based on the COWGA operator. Next, we present the expected fuzzy preference relation corresponding to MTFPR. Then, a logarithm compatibility index is proposed, and some desirable properties are discussed. At the same time, a compatibility-improving algorithm is presented to guarantee that each modified MTFPR is of acceptable compatibility. Moreover, we prove the property that the collective MTFPR and its expected fuzzy preference relation are of acceptable compatibility under the condition that all MTFPRs given by DMs and their expected fuzzy preference relations are acceptably compatible. Furthermore, we construct an optimal model to determine the weights of DMs based on the criterion of minimizing the logarithm compatibility index of MTFPRs, which ensures the objectivity of GDM. Then, a new approach of GDM with MTFPRs based on the logarithm compatibility is developed, which ensures the rationality of GDM. Finally, an illustrative example shows the availability and feasibility of the new approach.

Although some existing approaches have already been successfully applied to GDM problems with different kinds of preference relations, there have some difference between the existing methods and our proposed method. The reasons are as follows: (1) Our proposed method takes the risk attitude of DMs into account, which makes the decision more reasonable in GDM but it was ignored in existing methods [5, 19, 35, 50, 53, 54, 77, 78]. (2) The mechanism to generate experts' weights and the form of compatibility measure make the proposed method more effective, which are very different from Refs. [5, 19, 35, 53].

The work is set out as follows. In Section 2, some of the basic concepts are briefly reviewed. In Section 3, the concepts of the compatibility index of the MTFPRs are presented, a compatibility-improving algorithm is developed, and the optimal model is put forward to determine the optimal DMs' weights in GDM. Section 4 is devoted to proposing a complete flow for GDM with multiplicative trapezoidal fuzzy preference relations. In Section 5, a numerical example is developed. Finally, the main conclusions of the paper are summarized in Section 6.

2 Preliminaries

2.1 Trapezoidal fuzzy number and its expected interval

To rationalize uncertainty associated with impression or vagueness, Zadeh [69] proposed the fuzzy set theory, which includes a trapezoidal fuzzy number that is defined as follows.

Definition 1 Let $\tilde{a} = (a_1, a_2, a_3, a_4)$, $a_1 \leq a_2 \leq a_3 \leq a_4$; then, $\tilde{a}$ is called a trapezoidal fuzzy number, and the membership function $\mu_{\tilde{a}(x)} : R \to [0, 1]$ is defined as follows:

$$\mu_{\tilde{a}}(x) = \begin{cases} \frac{x-a_1}{a_2-a_1}, a_1 \leq x \leq a_2, \\ 1, a_2 \leq x \leq a_3, \\ \frac{x-a_4}{a_3-a_4}, a_3 \leq x \leq a_4, \\ 0, otherwise, \end{cases} \tag{1}$$

Let $\tilde{a} = (a_1, a_2, a_3, a_4)$ and $\tilde{b} = (b_1, b_2, b_3, b_4)$ be two trapezoidal fuzzy numbers, $\beta \in R^+$; then, operational laws on trapezoidal fuzzy numbers are as follows [14].

(1) $\tilde{a} \oplus \tilde{b} = (a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4)$;
(2) $\tilde{a} \ominus \tilde{b} = (a_1 - b_4, a_2 - b_3, a_3 - b_2, a_4 - b_1)$;
(3) $\tilde{a} \otimes \tilde{b} \cong (a_1 \times b_1, a_2 \times b_2, a_3 \times b_3, a_4 \times b_4)$;
(4) $\tilde{a} \oslash \tilde{b} = \left(\frac{a_1}{b_4}, \frac{a_2}{b_3}, \frac{a_3}{b_2}, \frac{a_4}{b_1}\right)$;
(5) $\beta\tilde{a} = (\beta a_1, \beta a_2, \beta a_3, \beta a_4)$;
(6) $(\tilde{a})^\beta = ((a_1)^\beta, (a_2)^\beta, (a_3)^\beta, (a_4)^\beta)$.

The α-cut set of a trapezoidal fuzzy number $\tilde{a}$, denoted by $\tilde{a}_\alpha$, is defined as [1, 16, 68] $\tilde{a}_\alpha = \{x \in R \,|\mu_{\tilde{a}}(x) \geq \alpha\}$ for all $\alpha \in [0, 1]$. Every α-cut is a closed interval $\tilde{a}_\alpha = [a_\alpha^L, a_\alpha^U] \subset \Re$, where $a_\alpha^L = \inf\{x \in \Re \,|\mu_{\tilde{a}}(x) \geq \alpha\}$ and $a_\alpha^U = \sup\{x \in \Re \,|\mu_{\tilde{a}}(x) \geq \alpha\}$ for any $\alpha \in [0, 1]$. By (1), we obtain

$$\tilde{a}_\alpha = \left[a_\alpha^L, a_\alpha^U\right] = [\alpha a_2 + (1-\alpha)a_1, \alpha a_3 + (1-\alpha)a_4]. \tag{2}$$

The expected interval $EI(\tilde{a})$ of a trapezoidal fuzzy number $\tilde{a}$ is defined by [1]:

$$EI(\tilde{a}) = [E_*(\tilde{a}), E^*(\tilde{a})] = \left[\int_0^1 a_\alpha^L d\alpha, \int_0^1 a_\alpha^U d\alpha\right]. \tag{3}$$

Based on (2) and (3), we obtain $EI(\tilde{a}) = \left[\frac{a_1+a_2}{2}, \frac{a_3+a_4}{2}\right]$.

2.2 Multiplicative preference relation

Saaty [44] first proposed multiplicative preference relation, which is widely used. Let $X = \{x_1, x_2, ..., x_n\}$ be a finite set of alternatives. Then, a multiplicative preference relation can be defined as follows.

Definition 2 Let $A = (a_{ij})_{n\times n}$ be a matrix. If

$$a_{ij}a_{ji} = 1, \quad a_{ii} = 1, \quad a_{ij} > 0, \quad \forall\, i, j = 1, 2, ..., n,$$

then A is called a multiplicative preference relation, where a_{ij} denotes the preference degree of alternative x_i over x_j.

In particular, $a_{ij} = 1$ indicates indifference between x_i and x_j, $a_{ij} > 1$ indicates that x_i is preferred over x_j, and $a_{ij} < 1$ indicates that x_j is preferred over x_i.

2.3 Multiplicative trapezoidal fuzzy preference relation

Due to the complexity of the decision-making environment, Buckley [3] extended the analytic hierarchy process (AHP) to the fuzzy environment and introduced the MTFPR, which is defined as follows:

Definition 3 A MTFPR $\tilde{A}$ is defined as $\tilde{A} = (\tilde{a}_{ij})_{n\times n}$, which satisfies

$$a_{ij1} \times a_{ji4} = a_{ij2} \times a_{ji3} = a_{ij3} \times a_{ji2} = a_{ij4} \times a_{ji1} = 1, \quad \tilde{a}_{ii} = (1, 1, 1, 1),$$

for all $i, j = 1, 2, \cdots, n$, where $\tilde{a}_{ij} = (a_{ij1}, a_{ij2}, a_{ij3}, a_{ij4})$ indicates the trapezoidal fuzzy preference degree of the ith alternative over the jth alternative.

For simplicity, in the following, we take M_n as the set of all $n \times n$ MTFPRs.

In [55], Xia and Chen defined the consistency of MTFPR, which is an important axiom in the construction of preference. It can be defined as follows:

Definition 4 A MTFPR $\tilde{A} = (\tilde{a}_{ij})_{n\times n}$ is completely consistent if and only if it satisfies the multiplicative transitivity:

$$\tilde{a}_{ij} \otimes \tilde{a}_{lk} = \tilde{a}_{lj} \otimes \tilde{a}_{ik}, i, j, k, l = 1, 2, ..., n. \tag{4}$$

2.4 The COWGA operator

Yager and Xu [66] developed a continuous ordered weighted geometric averaging (COWGA) operator, which is based on the continuous ordered weighted averaging (COWA) operator and geometric mean.

Definition 5 A continuous ordered weighted geometric (COWGA) operator is a mapping g: $\Omega^+ \to R^+$ which has associated with it a BUM function: $Q : [0, 1] \to [0, 1]$ having the properties: (1) $Q(0) = 0$; (2) $Q(1) = 1$; and (3) $Q(x) \geq Q(y)$ if $x > y$, such that

$$g_Q([a, b]) = b\left(\frac{a}{b}\right)^{\int_0^1 (dQ(y)/dy)ydy}, \tag{5}$$

where Ω^+ is the set of closed intervals, in which the lower limits of all closed intervals are positive, R^+ is the set of positive real numbers, and $[a, b]$ is a closed interval in Ω^+.

If $\lambda = \int_0^1 Q(y)dy$ is the attitudinal character of Q, a general formulation of $g_Q([a, b])$ can be obtained as follows:

$$g_\lambda([a, b]) = a^{1-\lambda} b^\lambda.$$

As we can see from the above, the COWGA operator g_λ is a linear convex exponential combination of a and b based on the attitudinal character.

3 The logarithm compatibility measure for MTFPRs

3.1 The logarithm compatibility measure

To begin with, we change the MTFPR into an expected interval multiplicative preference relation via the α-cut. Next, the expected interval multiplicative preference relation is transformed into the expected multiplicative preference relation based on the COWGA operator.

Definition 6 Let $\tilde{A} = (\tilde{a}_{ij})_{n\times n} \in M_n$ be a MTFPR, where $\tilde{a}_{ij} = (a_{ij1}, a_{ij2}, a_{ij3}, a_{ij4})$. If

$$a_{ij}^L = \begin{cases} E_*(\tilde{a}_{ij}), i \leq j \\ 1/E^*(\tilde{a}_{ji}), i > j \end{cases}, a_{ij}^U = \begin{cases} E^*(\tilde{a}_{ij}), i \leq j \\ 1/E_*(\tilde{a}_{ji}), i > j \end{cases}, \tag{6}$$

then $\tilde{\hat{A}} = (\tilde{\hat{a}}_{ij})_{n\times n} = \left(\left[a_{ij}^L, a_{ij}^U\right]\right)_{n\times n}$ is called the expected interval matrix corresponding to $\tilde{A}$, where $E_*(\tilde{a}_{ij}) = \frac{a_{ij1}+a_{ij2}}{2}$, $E^*(\tilde{a}_{ij}) = \frac{a_{ij3}+a_{ij4}}{2}$.

It can be easily found that the expected interval matrix satisfies $a_{ij}^L \times a_{ji}^U = a_{ij}^U \times a_{ji}^L = 1$, so the expected interval matrix is also called the expected interval multiplicative preference relation (EIMPR).

Based on Definition 6, we get the expected matrix $\hat{A}$ corresponding to the EIMPR $\tilde{\hat{A}}$.

Definition 7 Let $\tilde{A} = (\tilde{a}_{ij})_{n\times n} \in M_n$ be a MTFPR, and $\tilde{\hat{A}} = (\tilde{\hat{a}}_{ij})_{n\times n} = \left(\left[a_{ij}^L, a_{ij}^U\right]\right)_{n\times n}$ is the EIMPR of $\tilde{A}$. If $\hat{a}_{ij} = g_Q\left(\left[E_*\left(\tilde{a}_{ij}\right), E^*\left(\tilde{a}_{ij}\right)\right]\right) = E^*(\tilde{a}_{ij})\left(\frac{E_*(\tilde{a}_{ij})}{E^*(\tilde{a}_{ij})}\right)^{\int_0^1 (dQ(y)/dy)ydy}$, $\hat{a}_{ji} = 1/\hat{a}_{ij}$, for all $i \leq j$, then $\hat{A} = (\hat{a}_{ij})_{n\times n}$ is called the expected matrix corresponding to $\tilde{A}$. Obviously, the expected matrix is the expected multiplicative preference relation (EMPR).

Next, the expected fuzzy preference relation corresponding to MTFPR is defined as follows.

Definition 8 Let $\tilde{A} = (\tilde{a}_{ij})_{n\times n} \in M_n$ be a MTFPR, where $\tilde{a}_{ij} = (a_{ij1}, a_{ij2}, a_{ij3}, a_{ij4})$. If for $t = 1, 2, 3, 4$,

$$f_{ijt} = \begin{cases} \sqrt[n^2]{\prod_{k=1}^{n}\prod_{l=1}^{n}\left(\frac{a_{ikt}\times a_{ljt}}{a_{lkt}}\right)^{\frac{1}{3}}}, i < j, \\ 1, i = j, \\ f_{ji(5-t)}^{-1}, i > j, \end{cases} \tag{7}$$

then $\tilde{F} = (\tilde{f}_{ij})_{n\times n}$ is called the expected fuzzy preference relation of $\tilde{A}$, where $\tilde{f}_{ij} = (f_{ij1}, f_{ij2}, f_{ij3}, f_{ij4})$.

Theorem 1 *If $\tilde{F}$ is the expected fuzzy preference relation of MTFPR $\tilde{A}$, then*

(1) $\tilde{F} = (\tilde{f}_{ij})_{n\times n} \in M_n$;
(2) $\tilde{F} = (\tilde{f}_{ij})_{n\times n}$ *is consistent.*

Proof (1) Based on Definition 8, we can easily find that $\tilde{F} = (\tilde{f}_{ij})_{n\times n} \in M_n$;

(2) If $i > j$ for all t, by (7), we obtain

$$\begin{aligned} f_{ijt} &= 1/f_{ji(5-t)} = 1 \Big/ \sqrt[n^2]{\prod_{k=1}^{n}\prod_{l=1}^{n}\left(\frac{a_{jk(5-t)} \times a_{li(5-t)}}{a_{lk(5-t)}}\right)^{\frac{1}{3}}} \\ &= \sqrt[n^2]{\prod_{k=1}^{n}\prod_{l=1}^{n}\{1/[(a_{jk(5-t)} \times a_{li(5-t)})/a_{lk(5-t)}]\}^{\frac{1}{3}}} \\ &= \sqrt[n^2]{\prod_{k=1}^{n}\prod_{l=1}^{n}\left(\frac{a_{kjt} \times a_{ilt}}{a_{klt}}\right)^{\frac{1}{3}}} \\ &= \sqrt[n^2]{\prod_{k=1}^{n}\prod_{l=1}^{n}\left(\frac{a_{ilt} \times a_{kjt}}{a_{klt}}\right)^{\frac{1}{3}}} = \sqrt[n^2]{\prod_{k=1}^{n}\prod_{l=1}^{n}\left(\frac{a_{ikt} \times a_{ljt}}{a_{lkt}}\right)^{\frac{1}{3}}}. \end{aligned}$$

Then, we have

$$f_{ijt} = \sqrt[n^2]{\prod_{k=1}^{n}\prod_{l=1}^{n}\left(\frac{a_{ikt} \times a_{ljt}}{a_{lkt}}\right)^{\frac{1}{3}}},$$

$$f_{rst} = \sqrt[n^2]{\prod_{k=1}^{n}\prod_{l=1}^{n}\left(\frac{a_{rkt} \times a_{lst}}{a_{lkt}}\right)^{\frac{1}{3}}}.$$

It follows that

$$\begin{aligned} f_{ijt} \times f_{rst} &= \sqrt[n^2]{\prod_{k=1}^{n}\prod_{l=1}^{n}\left(\frac{a_{ikt} \times a_{ljt}}{a_{lkt}}\right)^{\frac{1}{3}}} \times \sqrt[n^2]{\prod_{k=1}^{n}\prod_{l=1}^{n}\left(\frac{a_{rkt} \times a_{lst}}{a_{lkt}}\right)^{\frac{1}{3}}} \\ &= \sqrt[n^2]{\prod_{k=1}^{n}\prod_{l=1}^{n}\left(\frac{a_{ikt} \times a_{ljt}}{a_{lkt}}\right)^{\frac{1}{3}}\left(\frac{a_{rkt} \times a_{lst}}{a_{lkt}}\right)^{\frac{1}{3}}} \\ &= \sqrt[n^2]{\prod_{k=1}^{n}\prod_{l=1}^{n}\left(\frac{a_{ikt} \times a_{ljt}}{a_{lkt}} \times \frac{a_{rkt} \times a_{lst}}{a_{lkt}}\right)^{\frac{1}{3}}} \\ &= \sqrt[n^2]{\prod_{k=1}^{n}\prod_{l=1}^{n}\left(\frac{a_{ikt} \times a_{lst}}{a_{lkt}}\right)^{\frac{1}{3}}\left(\frac{a_{rkt} \times a_{ljt}}{a_{lkt}}\right)^{\frac{1}{3}}} \\ &= \sqrt[n^2]{\prod_{k=1}^{n}\prod_{l=1}^{n}\left(\frac{a_{rkt} \times a_{ljt}}{a_{lkt}}\right)^{\frac{1}{3}}} \times \sqrt[n^2]{\prod_{k=1}^{n}\prod_{l=1}^{n}\left(\frac{a_{ikt} \times a_{lst}}{a_{lkt}}\right)^{\frac{1}{3}}} \\ &= f_{rjt} \times f_{ist}. \end{aligned}$$

Therefore, based on Definition 3, $\tilde{F} = (\tilde{f}_{ij})_{n\times n}$ is consistent. □

Based on theorem 1, we clearly know the expected fuzzy preference relation is a consistent MTFPR. For all MTFPRs, we are able to obtain their expected fuzzy preference relation, so we measure the consistency of MTFPR by its expected fuzzy preference relation.

Definition 9 Let $\tilde{A} = (\tilde{a}_{ij})_{n\times n} \in M_n$, $\tilde{F} = (\tilde{f}_{ij})_{n\times n}$ be the expected fuzzy preference relation; then,

$$L - CI(\tilde{A}, \tilde{F}) = \frac{2}{n(n-1)} \sum_{i<j} \left(\log \hat{a}_{ij} - \log \hat{f}_{ij}\right)^2, \quad (8)$$

is called the logarithm compatibility index of $\tilde{A}$ and $\tilde{F}$, where $\hat{A} = (\hat{a}_{ij})_{n\times n} = (g_\lambda([E_*(\tilde{a}_{ij}), E^*(\tilde{a}_{ij})]))_{n\times n}$, $\tilde{F} = (\tilde{f}_{ij})_{n\times n} = (g_\lambda([E_*(\tilde{f}_{ij}), E^*(\tilde{f}_{ij})]))_{n\times n}$.

It can be seen that the compatibility index $L - CI(\tilde{A}, \tilde{F})$ measures the average difference between $\tilde{A}$ and $\tilde{F}$, which considers the risk attitude of a DM based on the COWGA operator. The DM can choose different parameter λ according to his/her risk attitude. When $0 < \lambda < 0.5$, the DM's attitude is pessimistic. When $0.5 < \lambda < 1$, the DM's attitude is optimistic. When $\lambda = 0.5$, the DM's attitude is neutral. Obviously, the logarithm compatibility index satisfies the following properties.

Theorem 2 *Let $\tilde{A} = (\tilde{a}_{ij})_{n\times n} \in M_n$, $\tilde{F} = (\tilde{f}_{ij})_{n\times n}$ be as before. Then,*

(1) $L - CI(\tilde{A}, \tilde{F}) \geq 0$,
(2) $L - CI(\tilde{A}, \tilde{A}) = 0$,
(3) $L - CI(\tilde{A}, \tilde{F}) = L - CI(\tilde{F}, \tilde{A})$.

Theorem 2 indicates that the logarithm compatibility index is nonnegative, reflexive, and commutative.

Definition 10 Let $\tilde{A} = (\tilde{a}_{ij})_{n\times n} \in M_n$, $\tilde{F} = (\tilde{f}_{ij})_{n\times n}$ be as before. If $L - CI(\tilde{A}, \tilde{F}) = 0$, then $\tilde{A}$ and $\tilde{F}$ are perfectly compatible.

Definition 11 Let $\tilde{A} = (\tilde{a}_{ij})_{n\times n} \in M_n$, $\tilde{F} = (\tilde{f}_{ij})_{n\times n}$ be as before. If

$$L - CI(\tilde{A}, \tilde{F}) \leq \upsilon, \quad (9)$$

then $\tilde{A}$ and $\tilde{F}$ are of acceptable compatibility, where υ is the threshold of acceptable compatibility. Based on Ref. [11], we know that a lack of acceptable compatibility results in an unsatisfied decision.

As illustrated in [45], based on a different number of alternatives, we take υ with different values as the threshold of acceptable compatibility.

Theorem 3 *If $\tilde{A} = (\tilde{a}_{ij})_{n\times n} \in M_n$ and $\tilde{F} = (\tilde{f}_{ij})_{n\times n}$ are perfectly compatible, then $\tilde{A}$ and $\tilde{F}$ are of acceptable compatibility.*

Proof The theorem can be obtained by Definitions 10 and 11 immediately. □

3.2 Compatibility improving method

In real-world GDM problems, the individual MTFPR is often of unacceptable compatibility. Thus, reaching an acceptable compatibility usually requires the decision makers to modify their initial opinion. Inspired by [4], a basic procedure for the compatibility control process is depicted in Fig. 1.

The compatibility-improving process (CIP) to reach an acceptable compatibility in GDM problems is a dynamic and iterative discussion process, which is frequently coordinated by a human moderator, who is responsible for guiding the DMs in the CIP.

Algorithm 1 Input: The initial individual MTFPR $\tilde{A} = (\tilde{a}_{ij})_{n\times n}$, the threshold α and the parameter $\theta \in (0, 1)$.

Output: The modified MTFPR $\bar{A}$ and the logarithm compatibility index.

Step 1. Let $\tilde{A}_0 = (\tilde{a}_{ij,0})_{n\times n} = (\tilde{a}_{ij})_{n\times n}$ and $z = 0$. Where z is the number of iterations.

Step 2. Compute the $\tilde{F}_z$ by (7) and the logarithm compatibility index $L-CI(\tilde{A}_z, \tilde{F}_z)$, where

$$L-CI\left(\tilde{A}_z,\tilde{F}_z\right)=\frac{2}{n(n-1)}\sum_{i<j}\Big(\log\left(g_\lambda\left(\left[E_*\left(\tilde{a}_{ij}\right),E^*\left(\tilde{a}_{ij}\right)\right]\right)\right)-\log\left(g_\lambda\left(\left[E_*\left(\tilde{f}_{ij}\right),E^*\left(\tilde{f}_{ij}\right)\right]\right)\right)\Big)^2.$$

Step 3. If $L-CI(\tilde{A}_z, \tilde{F}_z) \le \upsilon$, go to Step 5; otherwise, go to the next Step.

Step 4. Employ the following strategy to modify the last matrix $\tilde{A}_z = (\tilde{a}_{ij,z})_{n\times n}$.

$$\delta_{ij,z}=\left(\log\left(g_\lambda\left(\left[E_*\left(\tilde{a}_{ij}\right),E^*\left(\tilde{a}_{ij}\right)\right]\right)\right)-\log\left(g_\lambda\left(\left[E_*\left(\tilde{f}_{ij}\right),E^*\left(\tilde{f}_{ij}\right)\right]\right)\right)\right)^2.$$

Let $\delta_{i_0j_0,z} = \max_{i<j} \delta_{ij,z}$; if $i < j$, then

$$\tilde{a}_{ij,z+1}=\begin{cases}(\tilde{a}_{ij,z})^{\theta}(\tilde{f}_{ij,z})^{1-\theta}, & i=i_0,\ j=j_0,\\ \tilde{a}_{ij,z}. & otherwise,\end{cases}$$

else, $\tilde{a}_{ij,z+1} = \tilde{1} \oslash \tilde{a}_{ji,z+1}$; where $\theta \in (0, 1)$. Let $z = z + 1$, and return to Step 2.

Step 5. Let $\bar{A} = \tilde{A}_z$. Output $\bar{A}$ and $L-CI(\bar{A}_z, \bar{F}_z)$.

Step 6. End.

Theorem 4 *Algorithm 1 is convergent. Thus, assume that $\tilde{A}$ is a MTFPR, $\theta \in (0, 1)$ is the adjusted parameter, and $\tilde{A}_z$ is the modified MTFPR obtained by Algorithm 1; then, we have $L-CI(\tilde{A}_{z+1}, \tilde{F}_{z+1}) < L-CI(\tilde{A}_z, \tilde{F}_z)$ for each z, and $\lim_{z\to+\infty} L-CI(\tilde{A}_z, \tilde{F}_z) < \upsilon$.*

Proof By (5) and (6), we only need to prove that $L-CI(\tilde{A}_{z+1}, \tilde{F}_{z+1}) < L-CI(\tilde{A}_z, \tilde{F}_z)$. In the following, we will prove that for $i = i_0$, $j = j_0$,

$$|\log a_{ijt,z+1} - \log f_{ijt,z+1}| < |\log a_{ijt,z} - \log f_{ijt,z}|.$$

By (5), for $i < j$, we get

$$f_{ijt,z+1}=\sqrt[n^2]{\prod_{k=1}^{n}\prod_{l=1}^{n}\left(\frac{a_{ikt,z+1}\times a_{ljt,z+1}}{a_{lkt,z+1}}\right)^{\frac{1}{3}}},$$

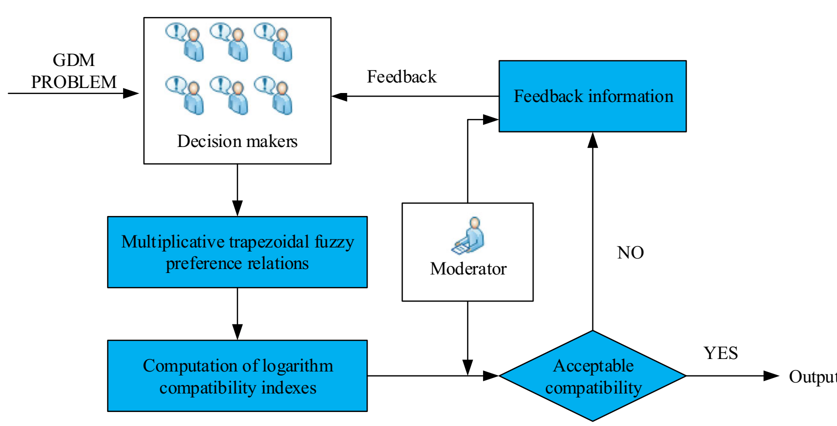
and for $i = i_0$, $j = j_0$

$$a_{ijt,z+1} = (a_{ijt,z})^{\theta}(f_{ijt,z})^{1-\theta}.$$

Then, we obtain

$$\begin{aligned}f_{i_0j_0t,z+1}&=\sqrt[n^2]{\left(\frac{a_{i_0j_0t,z+1}\times a_{i_0j_0t,z+1}}{a_{i_0j_0t,z+1}}\right)^{\frac{1}{3}}}\times\sqrt[n^2]{\prod_{l\neq i_0,k\neq j_0}^{n}\left(\frac{a_{i_0kt,z+1}\times a_{lj_0t,z+1}}{a_{lkt,z+1}}\right)^{\frac{1}{3}}}\\&=\sqrt[n^2]{\left(\left(a_{i_0j_0t,z}\right)^{\theta}\left(f_{i_0j_0t,z}\right)^{1-\theta}\right)^{\frac{1}{3}}}\end{aligned}$$

Fig. 1 Compatibility-improving process



$$\times \sqrt[n^2]{\prod_{l\neq i_0,k\neq j_0}^{n}\left(\frac{a_{i_0kt,z}\times a_{lj_0t,z}}{a_{lkt,z}}\right)^{\frac{1}{3}}}$$
$$= \sqrt[n^2]{\left(\left(a_{i_0j_0t,z}\right)^{\theta}\left(f_{i_0j_0t,z}\right)^{1-\theta}\right)^{\frac{1}{3}}}$$
$$\times \sqrt[n^2]{\prod_{k=1,l=1,}^{n}\left(\frac{a_{i_0kt,z}\times a_{lj_0t,z}}{a_{lkt,z}}\right)^{\frac{1}{3}}}$$
$$= \left(a_{i_0j_0t,z}\right)^{\frac{\theta-1}{3n^2}}\times\left(f_{i_0j_0t,z}\right)^{\frac{1-\theta}{3n^2}}\times f_{i_0j_0t,z}.$$

Thus, $\left|\log a_{i_0j_0t,z+1}-\log f_{i_0j_0t,z+1}\right|$

$$= \left|\theta\log a_{i_0j_0t,z}+(1-\theta)\log f_{i_0j_0t,z}-\frac{\theta-1}{3n^2}\log a_{i_0j_0t,z}\right.$$
$$\left.-\frac{1-\theta}{3n^2}\log f_{i_0j_0t,z}-\log f_{i_0j_0t,z}\right|$$
$$= \left|\left(\theta+\frac{1-\theta}{3n^2}\right)\left(\log a_{i_0j_0t,z}-\log f_{i_0j_0t,z}\right)\right|$$
$$< \left|\log a_{i_0j_0t,z}-\log f_{i_0j_0t,z}\right|. \tag{10}$$

□

Consequently,

$$L-CI(\tilde{A}_{z+1},\tilde{F}_{z+1})<L-CI(\tilde{A}_z,\tilde{F}_z),$$

which means that $\{L-CI(\tilde{A}_z,\tilde{F}_z)\}_z$ is monotonically decreasing with a low bound, and then there exists $\lim_{z\to+\infty} L-CI(\tilde{A}_z,\tilde{F}_z)$. Based on the proof by the contradiction and monotonicity of $\{L-CI(\tilde{A}_z,\tilde{F}_z)\}_z$, we get $\lim_{z\to+\infty} L-CI(\tilde{A}_z,\tilde{F}_z)<\upsilon$.

By algorithm 1, the individual MTFPR with maximum compatibility index has a better compatibility index.

3.3 To determine the weights of decision makers in GDM with MTFPRs

Let $D=\{d_1,d_2,...,d_m\}$ be a finite set of DMs and $\tilde{A}^{(k)}=(\tilde{a}_{ij}^{(k)})_{n\times n}\in M_n$ be the MTFPR provided by DM d_k, $k=1,2,...,m$; then the collective matrix of $\tilde{A}^{(1)},\tilde{A}^{(2)},...,\tilde{A}^{(m)}$ is defined as follows:

Definition 12 [3]. Let $\tilde{A}^{(k)}=(\tilde{a}_{ij}^{(k)})_{n\times n}\in M_n$ for $k=1,2,...,m$. If

$$\tilde{a}_{ij}=\bigotimes_{k=1}^{m}\left(\tilde{a}_{ij}^{(k)}\right)^{l_k},\ i,j=1,2,...,n. \tag{11}$$

then $\tilde{A}=(\tilde{a}_{ij})_{n\times n}$ is called the collective matrix of $\tilde{A}^{(k)}$, where $L=(l_1,l_2,...,l_m)^T$ is the weighting vector of DMs, which satisfies $l_k\geq 0$ for $k=1,2,...,m$ and $\sum_{k=1}^{m} l_k=1$.

Theorem 5 *Let* $\tilde{A}^{(k)}=\left(\tilde{a}_{ij}^{(k)}\right)_{n\times n}\in M_n, k=1,2,...,m$; *then the collective matrix* $\tilde{A}=(\tilde{a}_{ij})_{n\times n}\in M_n$.

Proof Because $\tilde{A}^{(k)}=\left(\tilde{a}_{ij}^{(k)}\right)_{n\times n}\in M_n$, $\tilde{a}_{ij}^{(k)}=\left(a_{ij1}^{(k)},a_{ij2}^{(k)},a_{ij3}^{(k)},a_{ij4}^{(k)}\right)$, we obtain

$$a_{ij1}^{(k)}\times a_{ji4}^{(k)}=a_{ij2}^{(k)}\times a_{ji3}^{(k)}=a_{ij3}^{(k)}\times a_{ji2}^{(k)}=a_{ij4}^{(k)}\times a_{ji1}^{(k)}$$
$$=1,\quad i,j=1,2,...,n,\quad i\neq j.$$

Thus, for $i\neq j$

$$\tilde{a}_{ij}=\bigotimes_{k=1}^{m}\left(\tilde{a}_{ij}^{(k)}\right)^{l_k}=\bigotimes_{k=1}^{m}\left(\left(a_{ij1}^{(k)},a_{ij2}^{(k)},a_{ij3}^{(k)},a_{ij4}^{(k)}\right)\right)^{l_k}$$
$$=\bigotimes_{k=1}^{m}\left(\left(a_{ij1}^{(k)}\right)^{l_k},\left(a_{ij2}^{(k)}\right)^{l_k},\left(a_{ij3}^{(k)}\right)^{l_k},\left(a_{ij4}^{(k)}\right)^{l_k}\right),$$
$$=\left(\prod_{k=1}^{m}\left(a_{ij1}^{(k)}\right)^{l_k},\prod_{k=1}^{m}\left(a_{ij2}^{(k)}\right)^{l_k},\prod_{k=1}^{m}\left(a_{ij3}^{(k)}\right)^{l_k},\prod_{k=1}^{m}\left(a_{ij4}^{(k)}\right)^{l_k}\right), i,j=1,2,...,n,$$

which means that

$$a_{ij1}=\prod_{k=1}^{m}\left(a_{ij1}^{(k)}\right)^{l_k},a_{ij2}=\prod_{k=1}^{m}\left(a_{ij2}^{(k)}\right)^{l_k},a_{ij3}=\prod_{k=1}^{m}\left(a_{ij3}^{(k)}\right)^{l_k},a_{ij4}$$
$$=\prod_{k=1}^{m}\left(a_{ij4}^{(k)}\right)^{l_k}.$$

Similarly,
$$\tilde{a}_{ji}=\left(\prod_{k=1}^{m}\left(a_{ji1}^{(k)}\right)^{l_k},\prod_{k=1}^{m}\left(a_{ji2}^{(k)}\right)^{l_k},\prod_{k=1}^{m}\left(a_{ji3}^{(k)}\right)^{l_k},\prod_{k=1}^{m}\left(a_{ji4}^{(k)}\right)^{l_k}\right),$$
i.e.,

$$a_{ji1}=\prod_{k=1}^{m}\left(a_{ji1}^{(k)}\right)^{l_k},a_{ji2}=\prod_{k=1}^{m}\left(a_{ji2}^{(k)}\right)^{l_k},$$
$$a_{ji3}=\prod_{k=1}^{m}\left(a_{ji3}^{(k)}\right)^{l_k},a_{ji4}=\prod_{k=1}^{m}\left(a_{ji4}^{(k)}\right)^{l_k}.$$

Therefore, for $i\neq j$

$$a_{ij1}\times a_{ji4}=\prod_{k=1}^{m}\left(a_{ij1}^{(k)}\right)^{l_k}\times\prod_{k=1}^{m}\left(a_{ji4}^{(k)}\right)^{l_k}$$
$$=\prod_{k=1}^{m}\left\{\left(a_{ij1}^{(k)}\right)^{l_k}\times\left(a_{ji4}^{(k)}\right)^{l_k}\right\}$$

$$= \prod_{k=1}^{m} \left(a_{ij1}^{(k)} \times a_{ji4}^{(k)}\right)^{l_k} = \prod_{k=1}^{m} 1^{l_k} = 1,$$

$$a_{ij2} \times a_{ji3} = \prod_{k=1}^{m} \left(a_{ij2}^{(k)}\right)^{l_k} \times \prod_{k=1}^{m} \left(a_{ji3}^{(k)}\right)^{l_k}$$

$$= \prod_{k=1}^{m} \left\{ \left(a_{ij2}^{(k)}\right)^{l_k} \times \left(a_{ji3}^{(k)}\right)^{l_k} \right\}$$

$$= \prod_{k=1}^{m} \left(a_{ij2}^{(k)} \times a_{ji3}^{(k)}\right)^{l_k} = \prod_{k=1}^{m} 1^{l_k} = 1,$$

$$a_{ij3} \times a_{ji2} = \prod_{k=1}^{m} \left(a_{ij3}^{(k)}\right)^{l_k} \times \prod_{k=1}^{m} \left(a_{ji2}^{(k)}\right)^{l_k}$$

$$= \prod_{k=1}^{m} \left\{ \left(a_{ij3}^{(k)}\right)^{l_k} \times \left(a_{ji2}^{(k)}\right)^{l_k} \right\}$$

$$= \prod_{k=1}^{m} \left(a_{ij3}^{(k)} \times a_{ji2}^{(k)}\right)^{l_k} = \prod_{k=1}^{m} 1^{l_k} = 1,$$

$$a_{ij4} \times a_{ji1} = \prod_{k=1}^{m} \left(a_{ij4}^{(k)}\right)^{l_k} \times \prod_{k=1}^{m} \left(a_{ji1}^{(k)}\right)^{l_k}$$

$$= \prod_{k=1}^{m} \left\{ \left(a_{ij4}^{(k)}\right)^{l_k} \times \left(a_{ji1}^{(k)}\right)^{l_k} \right\}$$

$$= \prod_{k=1}^{m} \left(a_{ij4}^{(k)} \times a_{ji1}^{(k)}\right)^{l_k} = \prod_{k=1}^{m} 1^{l_k} = 1.$$

□

It is easy to obtain $\tilde{a}_{ii} = (1, 1, 1, 1)$; thus, $\tilde{A} = (\tilde{a}_{ij})_{n\times n} \in M_n$.

Note that the collective matrix $\tilde{A}$ is also called the collective fuzzy preference relation of $\tilde{A}^{(k)}$.

Definition 13 Let $\tilde{A}^{(k)} = (\tilde{a}_{ij}^{(k)})_{n\times n}$ be the MTFPR of DM d_k and $L = (l_1, l_2, ..., l_m)^T$ be the weighting vector of DMs, satisfying $l_k \geq 0$, $\sum_{k=1}^{m} l_k = 1$. Assume that $\tilde{F}^{(k)} = \left(\tilde{f}_{ij}^{(k)}\right)_{n\times n} \in M_n$ is the expected fuzzy preference relations of $\tilde{A}^{(k)}$; if

$$\tilde{f}_{ij} = \bigotimes_{k=1}^{m} \left(\tilde{f}_{ij}^{(k)}\right)^{l_k}, \tag{12}$$

then $\tilde{F} = (\tilde{f}_{ij})_{n\times n}$ is called a collective expected fuzzy preference relation of $\tilde{F}^{(k)}$.

By Definition 3 and 13, the collective expected fuzzy preference relation $\tilde{F} \in M_n$.

Definition 14 Let $\tilde{A}^{(k)} \in M_n$, $\tilde{F}^{(k)} \in M_n$, $\tilde{A}$ and $\tilde{F}$ be as before.

$$L - CI(\tilde{A}^{(k)}, \tilde{F}^{(k)}) = \frac{2}{n(n-1)} \sum_{i<j} \left(\log \hat{a}_{ij}^{(k)} - \log \hat{f}_{ij}^{(k)}\right)^2$$

is called the individual logarithm compatibility index of DM d_k, and

$$L - CI(\tilde{A}, \tilde{F}) = \frac{2}{n(n-1)} \sum_{i<j} \left(\log \hat{a}_{ij} - \log \hat{f}_{ij}\right)^2$$

is called the group logarithm compatibility index.

Theorem 6 *Let $\tilde{A}^{(k)} = \left(\tilde{a}_{ij}^{(k)}\right)_{n\times n} \in M_n$, and $\tilde{F}^{(k)} = \left(\tilde{f}_{ij}^{(k)}\right)_{n\times n}$ be the expected preference relation of $\tilde{A}^{(k)}$; and let $\tilde{A} = (\tilde{a}_{ij})_{n\times n} \in M_n$ and $\tilde{F} = (\tilde{f}_{ij})_{n\times n} \in M_n$ be as before. If $L - CI(\tilde{A}^{(k)}, \tilde{F}^{(k)}) \leq \upsilon$, then*

$$L - CI(\tilde{A}, \tilde{F}) \leq \upsilon. \tag{13}$$

Proof Because $L - CI(\tilde{A}^{(k)}, \tilde{F}^{(k)}) \leq \upsilon$, we obtain

$$\begin{aligned}
L - CI(\tilde{A}, \tilde{F}) &= \frac{2}{n(n-1)} \sum_{i<j} \left(\log \hat{a}_{ij} - \log \hat{f}_{ij}\right)^2 \\
&= \frac{2}{n(n-1)} \sum_{i<j} \left(\log \left(\prod_{k=1}^{m} \hat{a}_{ij}^{(k)}\right)^{l_k} - \log \left(\prod_{k=1}^{m} \hat{f}_{ij}^{(k)}\right)^{l_k}\right)^2 \\
&= \frac{2}{n(n-1)} \sum_{i<j} \left(\sum_{k=1}^{m} l_k \log \hat{a}_{ij}^{(k)} - \sum_{k=1}^{m} l_k \log \hat{f}_{ij}^{(k)}\right)^2 \\
&= \frac{2}{n(n-1)} \sum_{i<j} \left(\sum_{k=1}^{m} l_k (\log \hat{a}_{ij}^{(k)} - \log \hat{f}_{ij}^{(k)})\right)^2 \\
&\leq \frac{2}{n(n-1)} \sum_{i<j} \sum_{k=1}^{m} l_k \left(\log \hat{a}_{ij}^{(k)} - \log \hat{f}_{ij}^{(k)}\right)^2 \\
&= \frac{2}{n(n-1)} \sum_{k=1}^{m} \sum_{i<j} l_k \left(\log \hat{a}_{ij}^{(k)} - \log \hat{f}_{ij}^{(k)}\right)^2 \\
&= \sum_{k=1}^{m} l_k \left(\frac{2}{n(n-1)} \sum_{i<j} \left(\log \hat{a}_{ij}^{(k)} - \log \hat{f}_{ij}^{(k)}\right)^2\right) \\
&\leq \sum_{k=1}^{m} l_k \upsilon = \upsilon.
\end{aligned}$$

□

Therefore, the weights of DMs may depend on the compatibility index of $\tilde{A}$ and $\tilde{F}$. To determine the weights of

DMs in group decision making with MTFPRs based on α-cut and the COWGA operator, we can minimize the compatibility index of $\tilde{A}$ and $\tilde{F}$. From Definition 7, it follows that

$$
\begin{aligned}
L-CI(\tilde{A},\tilde{F}) &= \frac{2}{n(n-1)}\sum_{i<j}\left(\log \hat{a}_{ij}-\log \hat{f}_{ij}\right)^2 \\
&= \frac{2}{n(n-1)}\sum_{i<j}\left(\log\left(\prod_{k=1}^{m}\hat{a}_{ij}^{(k)}\right)^{l_k}-\log\left(\prod_{k=1}^{m}\hat{f}_{ij}^{(k)}\right)^{l_k}\right)^2 \\
&= \frac{2}{n(n-1)}\sum_{i<j}\left(\sum_{k=1}^{m} l_k\log \hat{a}_{ij}^{(k)}-\sum_{k=1}^{m} l_k\log \hat{f}_{ij}^{(k)}\right)^2 \\
&= \frac{2}{n(n-1)}\sum_{i<j}\left(\sum_{k=1}^{m} l_k\left(\log \hat{a}_{ij}^{(k)}-\log \hat{f}_{ij}^{(k)}\right)\right)^2 \\
&= \frac{2}{n(n-1)}\sum_{i<j}\left(\sum_{k=1}^{m} l_k\left(\log g_\lambda\left(\left[\frac{a_{ij1}^{(k)}+a_{ij2}^{(k)}}{2},\frac{a_{ij3}^{(k)}+a_{ij4}^{(k)}}{2}\right]\right)\right.\right. \\
&\qquad \left.\left.-\log g_\lambda\left(\left[\frac{f_{ij1}^{(k)}+f_{ij2}^{(k)}}{2},\frac{f_{ij3}^{(k)}+f_{ij4}^{(k)}}{2}\right]\right)\right)\right)^2 \\
&= \sum_{k_1=1}^{m}\sum_{k_2=1}^{m} l_{k_1}l_{k_2}\left(\frac{2}{n(n-1)}\sum_{i<j}\left(\log\left(\left(\frac{a_{ij1}^{(k_1)}+a_{ij2}^{(k_1)}}{2}\right)^{1-\lambda}\left(\frac{a_{ij3}^{(k_1)}+a_{ij4}^{(k_1)}}{2}\right)^{\lambda}\right)\right.\right. \\
&\qquad -\log\left(\left(\frac{f_{ij1}^{(k_1)}+f_{ij2}^{(k_1)}}{2}\right)^{1-\lambda}\left(\frac{f_{ij3}^{(k_1)}+f_{ij4}^{(k_1)}}{2}\right)^{\lambda}\right)\Bigg)\left(\log\left(\left(\frac{a_{ij1}^{(k_2)}+a_{ij2}^{(k_2)}}{2}\right)^{1-\lambda}\left(\frac{a_{ij3}^{(k_2)}+a_{ij4}^{(k_2)}}{2}\right)^{\lambda}\right)\right. \\
&\qquad \left.\left.\left.-\log\left(\left(\frac{f_{ij1}^{(k_2)}+f_{ij2}^{(k_2)}}{2}\right)^{1-\lambda}\left(\frac{f_{ij3}^{(k_2)}+f_{ij4}^{(k_2)}}{2}\right)^{\lambda}\right)\right)\right)\right).
\end{aligned} \tag{14}
$$

Let $\Omega=(\delta_{k_1k_2})_{m\times m}$, where

$$
\begin{aligned}
\delta_{k_1k_2} = \frac{2}{n(n-1)}\sum_{i<j}&\left(\log\left(\left(\frac{a_{ij1}^{(k_1)}+a_{ij2}^{(k_1)}}{2}\right)^{1-\lambda}\left(\frac{a_{ij3}^{(k_1)}+a_{ij4}^{(k_1)}}{2}\right)^{\lambda}\right)-\log\left(\left(\frac{f_{ij1}^{(k_1)}+f_{ij2}^{(k_1)}}{2}\right)^{1-\lambda}\left(\frac{f_{ij3}^{(k_1)}+f_{ij4}^{(k_1)}}{2}\right)^{\lambda}\right)\right) \\
&\left(\log\left(\left(\frac{a_{ij1}^{(k_2)}+a_{ij2}^{(k_2)}}{2}\right)^{1-\lambda}\left(\frac{a_{ij3}^{(k_2)}+a_{ij4}^{(k_2)}}{2}\right)^{\lambda}\right)-\log\left(\left(\frac{f_{ij1}^{(k_2)}+f_{ij2}^{(k_2)}}{2}\right)^{1-\lambda}\left(\frac{f_{ij3}^{(k_2)}+f_{ij4}^{(k_2)}}{2}\right)^{\lambda}\right)\right)
\end{aligned}
$$

Equation (14) is then rewritten as

$$L-CI(\tilde{A},\tilde{F})=L^T\Omega L. \tag{15}$$

Thus, the optimal model for determining weights of experts based on the logarithm compatibility index of MTFPRs is expressed as follows:

$$\text{Model (1)} \qquad \min\ L-CI(\tilde{A},\tilde{F})=L^T\Omega L, \tag{16}$$

$$\text{s.t.}\begin{cases}\sum_{k=1}^{m} l_k=1,\\ l_k\ge 0,\ k=1,2,...,m.\end{cases}$$

Let $R = (1, 1, ..., 1)^T_{m\times 1}$; (15) can then be rewritten as follows:

$$\min L - CI(\tilde{A}, \tilde{F}) = L^T \Omega L \tag{17}$$

$$\text{s.t.} \begin{cases} R^T L = 1, \\ L \geq 0. \end{cases}$$

If we don't consider $\Omega \geq 0$, we have

$$\text{Model(2)} \quad \min L - CI(\tilde{A}, \tilde{F}) = L^T \Omega L$$
$$s.t. R^T L = 1. \tag{18}$$

Theorem 7 *If $\tilde{A}$ and $\tilde{F}$ are not perfectly compatible, the solution to the model (2) is*

$$L^* = \frac{\Omega^{-1} R}{R^T \Omega^{-1} R}. \tag{19}$$

Proof By (14), Ω is a symmetrical matrix. If $\tilde{A}$ and $\tilde{F}$ are not perfectly compatible, then$\tilde{A} \neq \tilde{F}$; thus, there exists $i_0, j_0 \in \{1, 2, ..., n\}$, $i_0 \neq j_0$ satisfying $\tilde{a}_{i_0 j_0} \neq \tilde{f}_{i_0 j_0}$, where

$$\tilde{a}_{i_0 j_0} = (a_{i_0 j_0 1}, a_{i_0 j_0 2}, a_{i_0 j_0 3}, a_{i_0 j_0 4}),$$
$$\tilde{f}_{i_0 j_0} = (f_{i_0 j_0 1}, f_{i_0 j_0 2}, f_{i_0 j_0 3}, f_{i_0 j_0 4}),$$

which means that

$$\left(\log g_\lambda \left(\left[\frac{a_{i_0 j_0 1} + a_{i_0 j_0 2}}{2}, \frac{a_{i_0 j_0 3} + a_{i_0 j_0 4}}{2}\right]\right)\right.$$
$$\left. - \log g_\lambda \left(\left[\frac{f_{i_0 j_0 1} + f_{i_0 j_0 2}}{2}, \frac{f_{i_0 j_0 3} + f_{i_0 j_0 4}}{2}\right]\right)\right)^2 > 0,$$

it follows that $(\log \hat{a}_{i_0 j_0} - \log \hat{f}_{i_0 j_0})^2 > 0$. Thus, $L - CI(\tilde{A}, \tilde{F}) > 0$, which means that Ω is a positive definite and invertible matrix, and Ω^{-1} is also a positive definite matrix. Then, we construct the Lagrange function as follows.

$$J(L, \lambda) = L^T \Omega L + \mu (R^T L - 1), \tag{20}$$

where μ is a Lagrange multiplier. According to the necessary conditions of the existence of extremum, by taking partial derivatives equal to zero with respect to L and μ, we can get

$$\begin{cases} \frac{\partial J(L,\mu)}{\partial L} = 0, \\ \frac{\partial J(L,\mu)}{\partial \mu} = 0. \end{cases}$$

Then,

$$\begin{cases} 2\Omega L + \mu R = 0, \\ R^T L - 1 = 0. \end{cases} \tag{21}$$

By solving (19), we have

$$L^* = \frac{\Omega^{-1} R}{R^T \Omega^{-1} R}.$$

Because $\frac{\partial^2 J(L,\mu)}{\partial L^2} = 2\Omega$ is a positive definite matrix, $J(L, \mu)$ is a strictly convex function. Thus, $L^* = \frac{\Omega^{-1} R}{R^T \Omega^{-1} R}$ is the unique optimal solution to model (2). □

4 GDM with MTFPRs based on compatibility measure

In this section, the compatibility proposed in this paper will be applied to GDM problems with MTFPRs, as clearly shown in Fig. 2.

Consider a GDM problem. Let $D = \{d_1, d_2, ..., d_m\}$ be the set of DMs and $S = \{s_1, s_2, ..., s_n\}$ be the set of alternatives. Each DM provides his/her own decision matrix $\tilde{A}^{(k)} = (\tilde{a}_{ij}^{(k)})_{n\times n}$, which is a MTFPR provided by the d_k.

This method is shown as follows.

Step 1. Calculate the expected fuzzy preference relations $\tilde{F}^{(k)}$corresponding to MTFPRs $\tilde{A}^{(k)}$.

Step 2. Compute the logarithm compatibility index $L - CI(\tilde{A}^{(k)}, \tilde{F}^{(k)})$, where

$$L-CI\left(\tilde{A}_z, \tilde{F}_z\right) = \frac{2}{n(n-1)} \sum_{i<j} \left(\log\left(g_\lambda\left(\left[E_*\left(\tilde{a}_{ij}^{(k)}\right), E^*\left(\tilde{a}_{ij}^{(k)}\right)\right]\right)\right)\right.$$
$$\left. - \log\left(g_\lambda\left(\left[E_*\left(\tilde{f}_{ij}^{(k)}\right), E^*\left(\tilde{f}_{ij}^{(k)}\right)\right]\right)\right)\right)^2$$

If the logarithm compatibility index $L - CI(\tilde{A}^{(k)}, \tilde{F}^{(k)}) > \upsilon$, we use Algorithm 1 to adjust the logarithm compatibility index to reach acceptable compatibility.

Step 3. Calculate the optimal weights for each DM.

Utilize model (1) to determine the optimal weights for each DM, which is denoted as $L^* = (l_1^*, l_2^*, ..., l_m^*)^T$.

Step 4. Calculate the collective multiplicative preference relation [6] $\hat{A} = (\hat{a}_{ij})_{n\times n} = \left(\prod_{k=1}^{m} (\hat{a}_{ij}^{(k)})^{l_k}\right)_{n\times n}$ based on the optimal weights of DMs.

Step 5. Calculate the expected values $\bar{a}_i$ using the following formula:

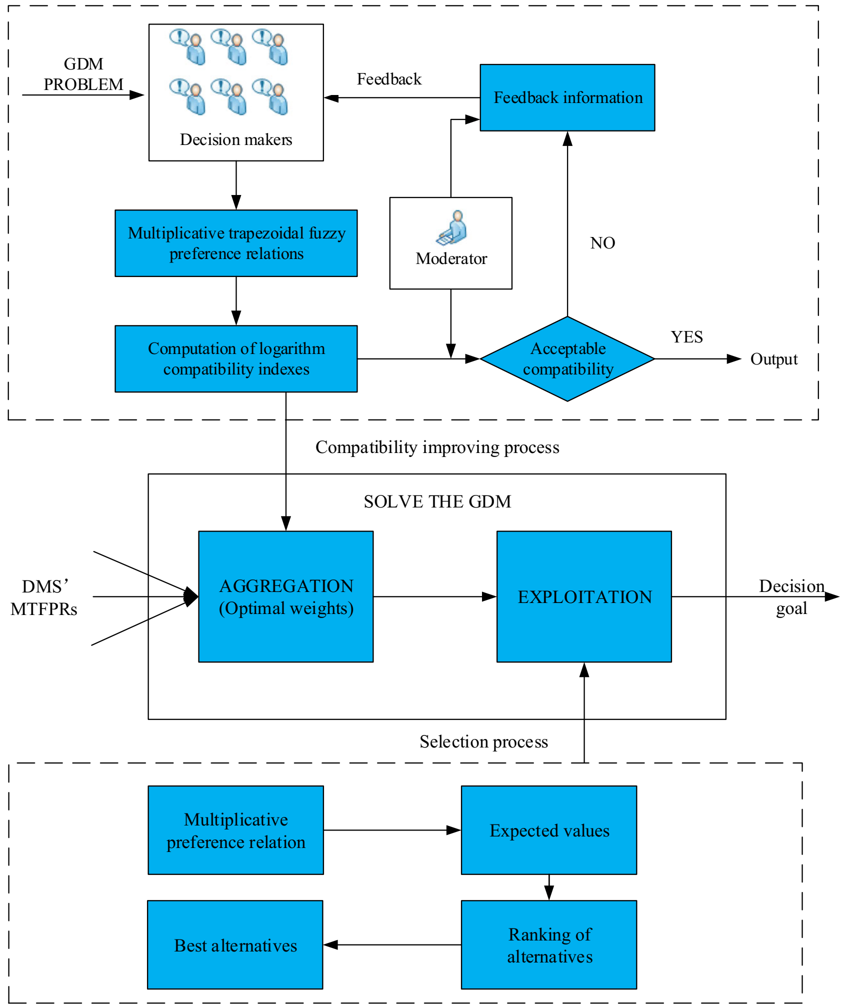
$$\bar{a}_i = \left(\prod_{j=1}^{n} a_{ij}\right)^{1/n}, \tag{22}$$

Step 6. Rank the expected values in descending order.

Step 7. Rank all the alternatives and select the best one(s) in accordance with the ranking of expected values.

Step 8. End.

Fig. 2 A GDM process with MTFPRs



5 Illustrative Example

5.1 The GDM problem

The Sing China (Zhong Guo Xin Ge Sheng), previously known as The Voice of China, is a Chinese reality television competition, airing on Zhejiang Television. One of the important premises of the show is the quality of the singing talent. Four coaches, themselves popular performing artists, train the talents in their group and occasionally perform with them. Talents are selected in blind auditions, where the coaches cannot see, but only hear the candidates' demos.

Blind auditions··

The first televised stage is the blind auditions, where competitors sing in front of the official coaches. All coaches will be sitting on a chair that is turned back from the stage. The coaches will first judge and only judge by the power, clarity, type and uniqueness of the artists singing prowess. If they like what they hear and want to mentor the artist for the next stage, they will push a button by their chair that would turn the chair around to face the stage for the first time and also see the artists for the first time after they sing to avoid any undue bias according to characteristics and personality. If more than one coach turns around, the power to pick goes to the artist who will be given the chance to pick his/her coach of choice. If no coach turns his/her chair the auditioning artist's journey ends. At the end, each of the coaches will have a certain number of artists in his or her team who will be advancing to the next round.

The Battle rounds··

The second stage, 'the Battle rounds', is where two artists are mentored and then developed by their respective coach. The coaches of the team will "dedicate themselves to developing their artists, giving them advice, and sharing the secrets of their success in the music industry". Every member of their team battles against another member from their team. They sing the same song simultaneously, while their coach decides who should continue in the competition. The coaches have to choose ten extraordinary participants from the four individual "battles", and take them to the live round.

Table 3 The linguistic variables and trapezoidal fuzzy numbers for the evaluation

Linguistic variables	Abbreviation	Trapezoidal fuzzy number
Reciprocal Absolutely important	RA	(1/9,1/9,2/17,1/8)
Intermediate RA and RV	IRARV	(1/9,2/17,2/15,1/7)
Reciprocal Very strongly important	RV	(1/8,2/15,2/13,1/6)
Intermediate RV and RE	IRVRE	(1/7,2/13,2/11,1/5)
Reciprocal Essentially important	RE	(1/6,2/11,2/9,1/4)
Intermediate RE and RW	IRERW	(1/5,2/9,2/7,1/3)
Reciprocal Weakly important	RW	(1/4,2/7,2/5,1/2)
Intermediate RW and EI	IRWEI	(1/3,2/5,2/3,1)
Equally important	EI	(1,1,1,1)
Intermediate WI and EI	IWIEI	(1,3/2,5/2,3)
Weakly important	WI	(2,5/2,7/2,4)
Intermediate WI and ES	IWIES	(3, 7/2,9/2,5)
Essentially important	ES	(4,9/2,11/2,6)
Intermediate ES and VS	IESVS	(5,11/2,13/2,7)
Very strongly important	VS	(6,13/2,15/2,8)
Intermediate VS and AI	IVSAI	(7,15/2,17/2,9)
Absolutely important	AI	(8,17/2,9,9)

Live shows··

The final stage, dubbed as the 'Live shows', is where the surviving combatants perform in front of the coaches, audience and broadcast live, once held in the National Stadium in Beijing. Each coach has four artists in their team to begin with and the artists will go head-to-head in the competition to win the public votes. These will determine which artist advances to the final eight. The remaining three artists' future in the show will be determined by the coaches, choosing who will progress.

The final eight artists will compete in a live broadcast. However, the coaches will have a 50/50 say with the audience and the public in deciding which artists move on to the 'final four' phase. In the latter, each coach will have one member who will continue. The final (the winner round) will be decided upon by the public vote. Throughout the final the coaches will frequently perform with their artists. The winner will be crowned The Sing China.

Table 4 The linguistic assessments presented by d_1

	C_1	C_2	C_3	C_4
C_1	EI	RE	WI	IRERW
C_2	ES	EI	VS	WI
C_3	RW	RV	EI	IRVRE
C_4	IWIES	RW	IESVS	EI

Table 5 The linguistic assessments presented by d_2

	C_1	C_2	C_3	C_4
C_1	EI	ES	WI	IESVS
C_2	RE	EI	ES	RW
C_3	RW	RE	EI	WI
C_4	IRVRE	WI	RW	EI

In a blind audition, there are four coaches turned around to the same competitor, so this competitor should select his/her best coach. Based on his/her friends and his/her own opinion, he/she would choose the best coach from Jay Chou, Ying Na, Feng Wang, and Harlem Yu. Let C_1 be Jay Chou, C_2 be Ying Na, C_3 be Feng Wang, and C_4 be Harlem Yu. To make the result more objective and more reasonable, three friends (d_1, d_2, d_3) of competitor (d_4) and competitor himself/herself use the linguistic variables (shown in Table 3 [71]) to construct their linguistic fuzzy preference relations, which are shown in Tables 4, 5, 6 and 7.

The linguistic assessments given by DMs are transformed into MTFPRs $\tilde{A}^{(k)} (k = 1, 2, 3, 4)$. They are shown as follows:

$$\tilde{A}^{(1)} = \begin{bmatrix} (1,1,1,1) & \left(\frac{1}{6},\frac{2}{11},\frac{2}{9},\frac{1}{4}\right) & \left(2,\frac{5}{2},\frac{7}{2},4\right) & \left(\frac{1}{5},\frac{2}{9},\frac{2}{7},\frac{1}{3}\right) \\ \left(4,\frac{9}{2},\frac{11}{2},6\right) & (1,1,1,1) & \left(6,\frac{13}{2},\frac{15}{2},8\right) & \left(2,\frac{5}{2},\frac{7}{2},4\right) \\ \left(\frac{1}{4},\frac{2}{7},\frac{2}{5},\frac{1}{2}\right) & \left(\frac{1}{8},\frac{2}{15},\frac{2}{13},\frac{1}{6}\right) & (1,1,1,1) & \left(\frac{1}{7},\frac{2}{13},\frac{2}{11},\frac{1}{5}\right) \\ \left(3,\frac{7}{2},\frac{9}{2},5\right) & \left(\frac{1}{4},\frac{2}{7},\frac{2}{5},\frac{1}{2}\right) & \left(5,\frac{11}{2},\frac{13}{2},7\right) & (1,1,1,1) \end{bmatrix},$$

$$\tilde{A}^{(2)} = \begin{bmatrix} (1,1,1,1) & \left(4,\frac{9}{2},\frac{11}{2},6\right) & \left(2,\frac{5}{2},\frac{7}{2},4\right) & \left(5,\frac{11}{2},\frac{13}{2},7\right) \\ \left(\frac{1}{6},\frac{2}{11},\frac{2}{9},\frac{1}{4}\right) & (1,1,1,1) & \left(4,\frac{9}{2},\frac{11}{2},6\right) & \left(\frac{1}{4},\frac{2}{7},\frac{2}{5},\frac{1}{2}\right) \\ \left(\frac{1}{4},\frac{2}{7},\frac{2}{5},\frac{1}{2}\right) & \left(\frac{1}{6},\frac{2}{11},\frac{2}{9},\frac{1}{4}\right) & (1,1,1,1) & \left(2,\frac{5}{2},\frac{7}{2},4\right) \\ \left(\frac{1}{7},\frac{2}{13},\frac{2}{11},\frac{1}{5}\right) & \left(2,\frac{5}{2},\frac{7}{2},4\right) & \left(\frac{1}{4},\frac{2}{7},\frac{2}{5},\frac{1}{2}\right) & (1,1,1,1) \end{bmatrix},$$

$$\tilde{A}^{(3)} = \begin{bmatrix} (1,1,1,1) & \left(2,\frac{5}{2},\frac{7}{2},4\right) & \left(\frac{1}{4},\frac{2}{7},\frac{2}{5},\frac{1}{2}\right) & \left(\frac{1}{3},\frac{2}{5},\frac{2}{3},1\right) \\ \left(\frac{1}{4},\frac{2}{7},\frac{2}{5},\frac{1}{2}\right) & (1,1,1,1) & \left(\frac{1}{6},\frac{2}{11},\frac{2}{9},\frac{1}{4}\right) & \left(4,\frac{9}{2},\frac{11}{2},6\right) \\ \left(2,\frac{5}{2},\frac{7}{2},4\right) & \left(4,\frac{9}{2},\frac{11}{2},6\right) & (1,1,1,1) & \left(\frac{1}{7},\frac{2}{13},\frac{2}{11},\frac{1}{5}\right) \\ \left(1,\frac{3}{2},\frac{5}{2},3\right) & \left(\frac{1}{6},\frac{2}{11},\frac{2}{9},\frac{1}{4}\right) & \left(5,\frac{11}{2},\frac{13}{2},7\right) & (1,1,1,1) \end{bmatrix},$$

$$\tilde{A}^{(4)} = \begin{bmatrix} (1,1,1,1) & \left(\frac{1}{5},\frac{2}{9},\frac{2}{7},\frac{1}{3}\right) & \left(3,\frac{7}{2},\frac{9}{2},5\right) & \left(4,\frac{9}{2},\frac{11}{2},6\right) \\ \left(3,\frac{7}{2},\frac{9}{2},5\right) & (1,1,1,1) & \left(\frac{1}{3},\frac{2}{5},\frac{2}{3},1\right) & \left(4,\frac{9}{2},\frac{11}{2},6\right) \\ \left(\frac{1}{5},\frac{2}{9},\frac{2}{7},\frac{1}{3}\right) & \left(1,\frac{3}{2},\frac{5}{2},3\right) & (1,1,1,1) & \left(\frac{1}{5},\frac{2}{9},\frac{2}{7},\frac{1}{3}\right) \\ \left(\frac{1}{6},\frac{2}{11},\frac{2}{9},\frac{1}{4}\right) & \left(\frac{1}{6},\frac{2}{11},\frac{2}{9},\frac{1}{4}\right) & \left(3,\frac{7}{2},\frac{9}{2},5\right) & (1,1,1,1) \end{bmatrix}.$$

Step 1. Calculate the initial expected fuzzy preference relations of $\tilde{A}^{(1)}$, $\tilde{A}^{(2)}$, $\tilde{A}^{(3)}$and $\tilde{A}^{(4)}$ by (7); then, we obtain $\tilde{F}^{(k)}(k = 1, 2, 3, 4)$ as follows:

$$\tilde{F}^{(1)} = \begin{bmatrix} (1,1,1,1) & (0.5192, 0.5326, 0.5623, 0.5796) & (1.1641, 1.2137, 1.2995, 1.3386) & (1.5577, 1.5920, 1.6660, 1.7108) \\ (1.7255, 1.7785, 1.8775, 1.9260) & (1,1,1,1) & (0.9135, 0.9281, 0.9585, 0.9736) & (2.1739, 2.2437, 2.3473, 2.3819) \\ (0.7471, 0.7695, 0.8239, 0.8591) & (0.4198, 0.4260, 0.4457, 0.4600) & (1,1,1,1) & (0.5872, 0.5989, 0.6322, 0.6554) \\ (1.2110, 1.2538, 1.3355, 1.3771) & (0.6805, 0.6941, 0.7224, 0.7374) & (1.5258, 1.5817, 1.6697, 1.7031) & (1,1,1,1) \end{bmatrix},$$

$$\tilde{F}^{(2)} = \begin{bmatrix} (1,1,1,1) & (1.4897, 1.5469, 1.6334, 1.6638) & (1.5409, 1.6063, 1.7154, 1.7627) & (1.5699, 1.6334, 1.7395, 1.7855) \\ (0.6011, 0.6122, 0.6465, 0.6713) & (1,1,1,1) & (0.9760, 1.0091, 1.0807, 1.1229) & (0.9943, 1.0261, 1.0958, 1.1374) \\ (0.5673, 0.5829, 0.6226, 0.6490) & (0.8906, 0.9253, 0.9910, 1.0246) & (1,1,1,1) & (0.9385, 0.9771, 1.0553, 1.0996) \\ (0.5601, 0.5749, 0.6122, 0.6370) & (0.8792, 0.9126, 0.9746, 1.0057) & (0.9094, 0.9476, 1.0235, 1.0656) & (1,1,1,1) \end{bmatrix},$$

$$\tilde{F}^{(3)} = \begin{bmatrix} (1,1,1,1) & (0.9529, 0.9925, 1.0813, 1.1388) & (0.8013, 0.8284, 0.9025, 0.9576) & (0.8208, 0.8519, 0.9418, 1.0145) \\ (0.8781, 0.9248, 1.0076, 1.0494) & (1,1,1,1) & (0.8013, 0.8146, 0.8552, 0.8825) & (0.8208, 0.8378, 0.8923, 0.9349) \\ (1.0443, 1.1080, 1.2072, 1.2480) & (1.1332, 1.1694, 1.2276, 1.2480) & (1,1,1,1) & (0.9760, 1.0038, 1.0691, 1.1118) \\ (0.9857, 1.0619, 1.1738, 1.2184) & (1.0696, 1.1206, 1.1936, 1.2184) & (0.8994, 0.9353, 0.9963, 1.0246) & (1,1,1,1) \end{bmatrix}.$$

$$\tilde{F}^{(3)} = \begin{bmatrix} (1,1,1,1) & (0.8704, 0.9098, 0.9752, 1.0047) & (1.2664, 1.3120, 1.4123, 1.4746) & (1.2733, 1.3027, 1.3607, 1.3871) \\ (0.9954, 1.0254, 1.0991, 1.1489) & (1,1,1,1) & (1.3215, 1.3779, 1.5156, 1.6160) & (1.3286, 1.3681, 1.4603, 1.5201) \\ (0.6781, 0.7081, 0.7622, 0.7896) & (0.6188, 0.6598, 0.7258, 0.7567) & (1,1,1,1) & (0.9052, 0.9447, 1.0127, 1.0448) \\ (0.7209, 0.7349, 0.7676, 0.7854) & (0.6578, 0.6848, 0.7309, 0.7526) & (0.9571, 0.9875, 1.0585, 1.1047) & (1,1,1,1) \end{bmatrix}.$$

Step 2. Without loss of generality, we take $Q(y) = y$, and then $\lambda = \int_0^1 Q(y)dy = \frac{1}{2}$. By (8), we calculate the logarithm compatibility index $L-CI(\tilde{A}^{(k)}, \tilde{F}^{(k)})$ of each individual MTFPRs $\tilde{A}^{(k)}$ and its expected fuzzy preference relations $\tilde{F}^{(k)}$:

$$L-CI\left(\tilde{A}^{(1)}, \tilde{F}^{(1)}\right) = 1.0660, L-CI\left(\tilde{A}^{(2)}, \tilde{F}^{(2)}\right) = 1.3303,$$

$$L-CI\left(\tilde{A}^{(3)}, \tilde{F}^{(3)}\right) = 1.7477, L-CI\left(\tilde{A}^{(4)}, \tilde{F}^{(4)}\right) = 1.4705.$$

We take the threshold value $\upsilon = 1.053$, and we can then see that all the MTFPRs and their expected fuzzy preference relations are not of acceptable compatibility. Thus, we need to carry out Algorithm 1 to adjust each individual MTFPR $\tilde{A}^{(k)}$ to satisfy $L-CI(\tilde{A}^{(k)}, \tilde{F}^{(k)}) < \upsilon$. By setting parameter $\theta = 0.6$, the result of the iterative process is shown in Table 8, and the final logarithm compatibility indexes are as follows:

$$L-CI\left(\tilde{A}^{(1)}, \tilde{F}^{(1)}\right) = 0.9607,$$
$$\times \quad L-CI\left(\tilde{A}^{(2)}, \tilde{F}^{(2)}\right) = 0.9373,$$

$$L-CI\left(\tilde{A}^{(3)}, \tilde{F}^{(3)}\right) = 1.0403,$$
$$\times \quad L-CI\left(\tilde{A}^{(4)}, \tilde{F}^{(4)}\right) = 0.9449.$$

Step 3. Utilize model (1) to determine the optimal weights of each DM, which are shown in the following:

$$l_1^* = 0.06, \quad l_2^* = 0.47, \quad l_3^* = 0.10, \quad l_4^* = 0.37.$$

Step 4. Calculate the collective multiplicative preference relation $\hat{A} = (\hat{a}_{ij})_{n\times n} = \left(\prod_{k=1}^{m} (\hat{a}_{ij}^{(k)})^{l_k}\right)_{n\times n}$ based on the optimal weights of DMs.

$$\hat{A} = \begin{bmatrix} 1.0000 & 1.5679 & 2.6266 & 2.8597 \\ 0.6378 & 1.0000 & 1.2172 & 1.0676 \\ 0.3807 & 0.8216 & 1.0000 & 1.0101 \\ 0.3497 & 0.9367 & 0.9900 & 1.0000 \end{bmatrix}.$$

Step 5. From the collective multiplicative preference relation $\hat{A}$, we obtain the expected values of the preference degree by (22).

$$\bar{a}_1 = 1.8525, \quad \bar{a}_2 = 0.9541, \quad \bar{a}_3 = 0.7497, \quad \bar{a}_4 = 0.7546.$$

Step 6. The results of $\bar{a}_i(i = 1, 2, 3, 4)$ are ranked in descending order as follows:

$$\bar{a}_1 > \bar{a}_2 > \bar{a}_4 > \bar{a}_3.$$

Step 7. Rank all the alternatives $s_i(i = 1, 2, 3, 4)$ in accordance with the $\bar{a}_i(i = 1, 2, 3, 4)$, and we have

$$C_1 \succ C_2 \succ C_4 \succ C_3.$$

Note that "$\succ$" means preferred to.

Table 6 The linguistic assessments presented by d_3

	C_1	C_2	C_3	C_4
C_1	EI	WI	RW	IRWEI
C_2	RW	EI	RE	ES
C_3	WI	ES	EI	IRVRE
C_4	IWIEI	RE	IESVS	EI

Thus, the most desirable coach is Jay Chou (C_1) in this GDM problem. The group logarithm compatibility index is $L - CI(\tilde{A}, \tilde{F}) = 0.1791$, and the individual logarithm compatibility index are shown as follows:

$$L - CI(\tilde{A}^{(1)}, \tilde{F}^{(1)}) = 0.9607,$$
$$L - CI(\tilde{A}^{(2)}, \tilde{F}^{(2)}) = 0.9373,$$

$$L - CI(\tilde{A}^{(3)}, \tilde{F}^{(3)}) = 1.0403,$$
$$L - CI(\tilde{A}^{(4)}, \tilde{F}^{(4)}) = 0.9449.$$

Then, we have $L - CI(\tilde{A}^{(k)}, \tilde{F}^{(k)}) \leq \upsilon = 1.053$ for $k = 1, 2, 3, 4$. We can see that the group logarithm compatibility index is less than each individual logarithm compatibility index, i.e., $L - CI(\tilde{A}, \tilde{F}) \leq L - CI(\tilde{A}^{(k)}, \tilde{F}^{(k)})$ for all k. Therefore, we obtain that the group logarithm compatibility index is superior to the individual logarithm compatibility index.

Afterward, we perform an analysis to determine how the different weights of DMs may affect the compatibility index and what role the different attitude character λ plays in the problem.

Furthermore, to analyze the role of the attitude character λ in this GDM problem, we consider $\lambda \in [0, 1]$ with different values given by the DMs. The weights of DMs and different values λ are shown in Fig. 3, and the results $\bar{a}_i (i = 1, 2, 3, 4)$ are shown in Fig. 4.

It is apparent from Fig. 3 that l_1, l_2 and l_3 always increase as λ increases when $\lambda \in [0, 0.4]$, but l_4 decreases when $\lambda \in [0, 0.4]$. This tendency also shows when $\lambda \in [0.5, 0.9]$. When $\lambda \in [0.4, 0.5]$, l_4 is monotonically increasing, l_1, l_2 and l_3 all are monotonically decreasing as λ increases. And when $\lambda \in [0.9, 1]$, l_1 and l_4 decrease and l_2 and l_3 increase as λ increases. Moreover, Fig. 4 indicates that the final choice depends on the attitude parameter λ that is used. However, it seems that the coach C_1 is the best choice.

Table 7 The linguistic assessments presented by d_4

	C_1	C_2	C_3	C_4
C_1	EI	IRERW	IWIES	ES
C_2	IWIES	EI	IRWEI	ES
C_3	IRERW	IWIEI	EI	IRERW
C_4	RE	RE	IWIES	EI

Table 8 Logarithm compatibility index (L-CI) and the number of iterations (z) for $\tilde{A}^{(k)} (k = 1, 2, 3, 4)$

$\tilde{A}^{(1)}$		$\tilde{A}^{(2)}$		$\tilde{A}^{(4)}$		$\tilde{A}^{(4)}$	
z	L-CI	z	L-CI	z	L-CI	z	L-CI
1	0.9607	1	1.3812	1	1.2749	1	1.2691
		2	0.9373	2	1.0403	2	1.0836
						3	0.9449

It is also noteworthy to take into account the DM weights determined under different λ and θ. The concrete results are shown in Figs. 5, 6, 7 and 8.

Based on Figs. 5, 6, 7 and 8, l_1, l_2, l_3 and l_4 all show different degrees of fluctuation with different λ and θ. It shows that the logarithm compatibility based on the COWGA operator are effective at deriving the optimal DMs' weights, which will be used in the aggregation phase.

5.2 Comparison with other methods

In this subsection, we will make comparisons to validate the feasibility of the proposed GDM method with MTFPRs.

5.2.1 Comparison analysis with the expected value of the fuzzy number

Based on the expected interval of the fuzzy number, Heilpren [29] proposed the expected value of the fuzzy number, which is denoted by $EV(\tilde{a})$, i.e. $EV(\tilde{a}) = [E_*(\tilde{a}) + E^*(\tilde{a})]/2$, where $E_*(\tilde{a})$ and $E^*(\tilde{a})$ are defined as before. To perform a comparison, we use the expected value of the trapezoidal fuzzy number, and then we obtain the ordinary multiplicative preference relations as follows:

$$\widehat{A}^{(1)} = \begin{bmatrix} 1.0000 & 0.2052 & 3.0000 & 0.2603 \\ 4.8738 & 1.0000 & 7.0000 & 3.0000 \\ 0.3333 & 0.1429 & 1.0000 & 0.1696 \\ 3.8415 & 0.3333 & 5.8952 & 1.0000 \end{bmatrix},$$

$$\widehat{A}^{(2)} = \begin{bmatrix} 1.0000 & 5.0000 & 3.0000 & 6.0000 \\ 0.2000 & 1.0000 & 5.0000 & 0.3589 \\ 0.3333 & 0.2000 & 1.0000 & 3.0000 \\ 0.1667 & 2.7861 & 0.3333 & 1.0000 \end{bmatrix},$$

$$\widehat{A}^{(3)} = \begin{bmatrix} 1.0000 & 3.0000 & 0.3589 & 0.6000 \\ 0.3333 & 1.0000 & 0.2052 & 5.0000 \\ 2.7861 & 4.8738 & 1.0000 & 0.1696 \\ 1.6667 & 0.2000 & 5.8952 & 1.0000 \end{bmatrix},$$

$$\widehat{A}^{(4)} = \begin{bmatrix} 1.0000 & 0.2603 & 4.0000 & 5.0000 \\ 3.8415 & 1.0000 & 0.6000 & 5.0000 \\ 0.2500 & 1.6667 & 1.0000 & 0.2603 \\ 0.2000 & 0.2000 & 3.8415 & 1.0000 \end{bmatrix}.$$

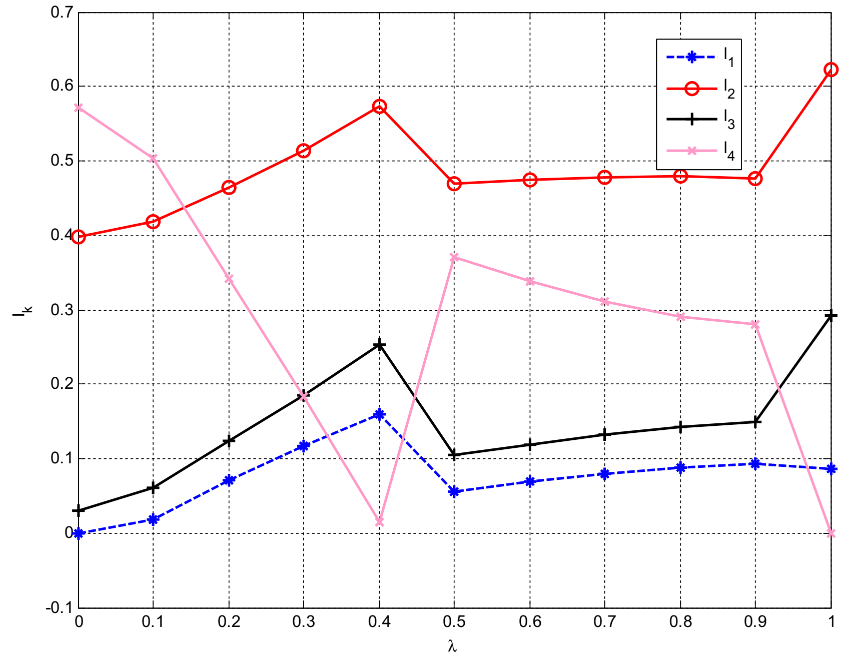


Fig. 3 Weights of DMs with different λ

Here, we omit the characteristic preference relations $\widehat{W}^{(k)}$. By using Model (1) or (19), we obtain the optimal DM weights as follows:

$$l_1 = 0.2146, \quad l_2 = 0.5295, \quad l_3 = 0.1534, \quad l_4 = 0.1025.$$

Then we have the collective multiplicative preference relation, which is omitted here. Using (22), we obtain:

$$\bar{a}_1 = 1.6870, \quad \bar{a}_2 = 1.1436, \quad \bar{a}_3 = 0.6087, \quad \bar{a}_4 = 0.8516.$$

Thus, we have $C_1 \succ C_2 \succ C_4 \succ C_3$.

As we can see, the decision is the same as the proposed method. Compared to the method with the expected values of the fuzzy number, we observe that:

(1) The proposed method uses the COWGA operator rather than the expected value of the fuzzy number. The former is more effective in aggregating information. The latter is a simple mean, which may cause a loss of information in the aggregation process.

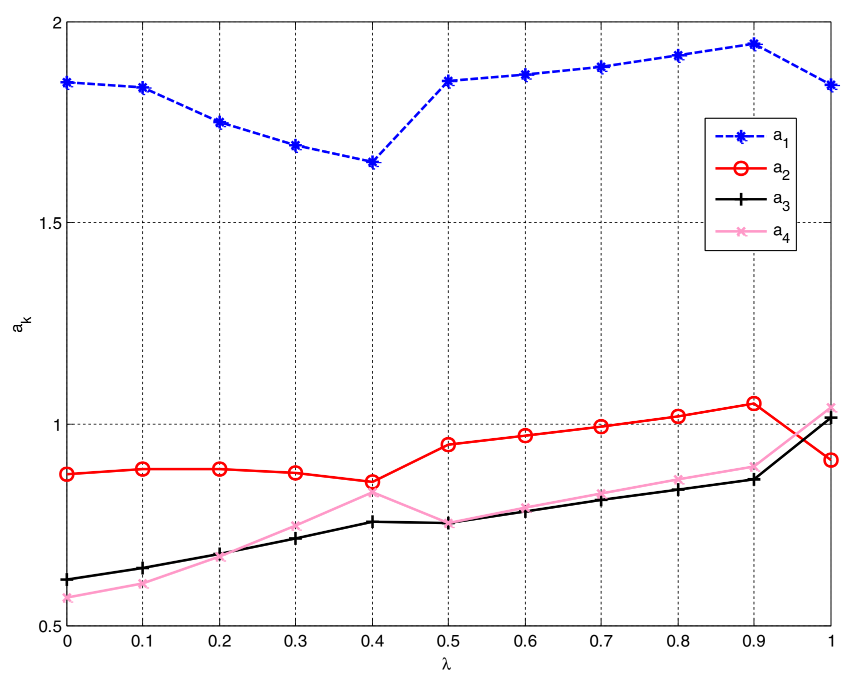
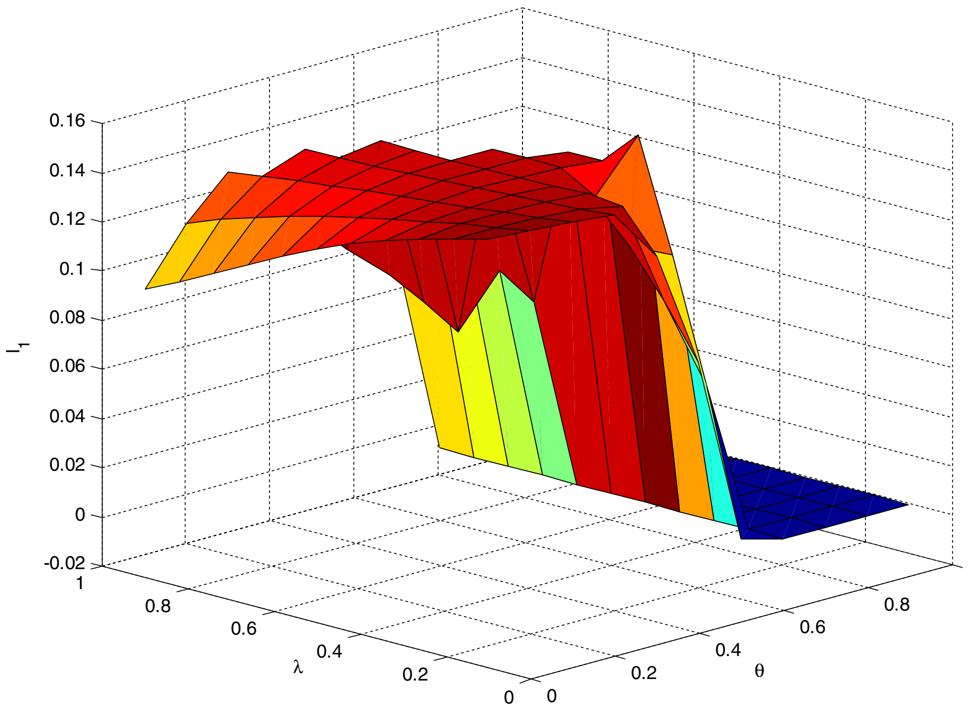


Fig. 4 Values $\bar{a}_i$ with different λ

Fig. 5 The weight of d_1 determined by different λ and θ



(2) The proposed method considers the risk attitudes of DMs, which makes the decision more reasonable and humanized in the GDM problem.

(3) We can obtain the optimal DM weights, which is objective and rational by using the optimal model based on the criterion of minimizing the group logarithm compatibility index.

5.2.2 Comparison analysis with the existing method using the interval multiplicative fuzzy preference relation

To further illustrate the effectiveness of the proposed method in this paper, trapezoidal fuzzy numbers are transformed into interval fuzzy numbers by combining their lower and upper bounds. The values of the illustrated example that have been converted are shown as follows.

$$\tilde{\tilde{A}}^{(1)} = \begin{bmatrix} [1,1] & \left[\frac{1}{6},\frac{1}{4}\right] & [2,4] & \left[\frac{1}{5},\frac{1}{3}\right] \\ [4,6] & [1,1] & [6,8] & [2,4] \\ \left[\frac{1}{4},\frac{1}{2}\right] & \left[\frac{1}{8},\frac{1}{6}\right] & [1,1] & \left[\frac{1}{7},\frac{1}{5}\right] \\ [3,5] & \left[\frac{1}{7},\frac{1}{5}\right] & [5,7] & [1,1] \end{bmatrix},$$

$$\tilde{\tilde{A}}^{(2)} = \begin{bmatrix} [1,1] & [4,6] & [2,4] & [5,7] \\ \left[\frac{1}{6},\frac{1}{4}\right] & [1,1] & [4,6] & \left[\frac{1}{4},\frac{1}{2}\right] \\ \left[\frac{1}{4},\frac{1}{2}\right] & \left[\frac{1}{6},\frac{1}{4}\right] & [1,1] & [2,4] \\ \left[\frac{1}{7},\frac{1}{5}\right] & [2,4] & \left[\frac{1}{4},\frac{1}{2}\right] & [1,1] \end{bmatrix},$$

$$\tilde{\tilde{A}}^{(3)} = \begin{bmatrix} [1,1] & [2,4] & \left[\frac{1}{4},\frac{1}{2}\right] & \left[\frac{1}{3},1\right] \\ \left[\frac{1}{4},\frac{1}{2}\right] & [1,1] & \left[\frac{1}{6},\frac{1}{4}\right] & [4,6] \\ [2,4] & [4,6] & [1,1] & \left[\frac{1}{7},\frac{1}{5}\right] \\ [1,3] & \left[\frac{1}{6},\frac{1}{4}\right] & [5,7] & [1,1] \end{bmatrix},$$

$$\tilde{\tilde{A}}^{(4)} = \begin{bmatrix} [1,1] & \left[\frac{1}{5},\frac{1}{3}\right] & [3,5] & [4,6] \\ [3,5] & [1,1] & \left[\frac{1}{3},1\right] & [4,6] \\ \left[\frac{1}{5},\frac{1}{3}\right] & [1,3] & [1,1] & \left[\frac{1}{5},\frac{1}{3}\right] \\ \left[\frac{1}{6},\frac{1}{4}\right] & \left[\frac{1}{6},\frac{1}{4}\right] & [3,5] & [1,1] \end{bmatrix}.$$

The priority vectors of the $\tilde{\tilde{A}}^{(k)}$ for $k = 1, 2, 3, 4$ can be generated using method [3], and the method of ranking interval numbers is given by literature [62]. Then, we utilize the method proposed by Wang, Chen and Zhou [50]:

$$CI^{Wang}(\tilde{\tilde{A}}, \tilde{\tilde{W}}) = \frac{1}{2n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\left|\log a_{ij}^{L} - \log w_{ij}^{L}\right| + \left|\log a_{ij}^{U} - \log w_{ij}^{U}\right|\right).$$

Finally, we get $C_2 \succ C_4 \succ C_1 \succ C_3$. This is different from the results of our method. Compared to the method developed in [50], we observe that:

(1) An iterative and convergent algorithm is proposed to improve the compatibility of MTFPR in this paper, but it was ignored in [50].

(2) CI^{Wang} can be seen as the sum of the logarithm absolute deviation of preference relations, which is based

on the two endpoints of each interval. That is to say, CI^{Wang} only depends on the two simple endpoints of each interval rather than the whole interval, which is very different from the approach developed in this paper.

(3) The proposed method considers the risk attitude of DMs, which makes the decision more reasonable in the GDM problem.

On the other hand, Wu et al. [53] used induced continuous ordered weighted geometric operators to solve GDM problems with interval multiplicative preference relations. By using this approach, the alternative weights are generated by the geometric mean, and the global preference relation degrees $z_i (i = 1, 2, 3, 4)$ of the alternatives are calculated as follows:

$$z_1 = 0.8105; \quad z_2 = 2.5259; \quad z_3 = 0.3596; \quad z_4 = 1.3583.$$

Therefore, $z_2 > z_4 > z_1 > z_3$, we get $C_2 \succ C_4 \succ C_1 \succ C_3$.

It can be seen that the ranking based on the method proposed by Wu et al. is different from the result by using the method proposed in this paper. Compared to the method in [53], we observe that:

(1) The logarithm compatibility index of MTFPRs used in this paper are more suitable for expressing the evaluation information of DMs because they can be extended to other environments and are more flexible.

(2) The proposed method in this paper starts with linguistic variables rather than intervals, which is very different from the approach developed in [53].

5.2.3 Comparison analysis with the existing method using the multiplicative trapezoidal fuzzy preference relations

In order to verify our method, we will use the method presented by Wu et al. [54]. All parameters are the same as the literature [54], and the main results are shown as follows.

(1) Compute the adjusted multiplicative trapezoidal fuzzy preference relations $\bar{A}^{(k)} (k = 1, 2, 3, 4)$ of $\tilde{A}^{(1)}$, $\tilde{A}^{(2)}$, $\tilde{A}^{(3)}$ and $\tilde{A}^{(4)}$.

$$\bar{A}^{(1)} = \begin{bmatrix} (1.0000, 1.0000, 1.0000, 1.0000) & (0.1667,0.1818,0.2222,0.2500) & (2.0000,2.5000,3.500,4.0000) & (0.2000,0.2222,0.2857,0.3333) \\ (4.0000,4.5000, 5.5000,6.0000) & (1.0000, 1.0000, 1.0000, 1.0000) & (6.0000,6.5000,7.5000,8.0000) & (2.0000,2.5000,3.5000,4.0000) \\ (0.2500,0.2857,0.4000,0.5000) & (0.1250,0.1333,0.1538,0.1667) & (1.0000, 1.0000, 1.0000, 1.0000) & (0.1429,0.1539,0.1818,0.2000) \\ (3.0000,3.5000, 4.5000,5.5000) & (0.2500, 0.2857,0.4000, 0.5000) & (5.0000,5.5000, 6.5000,7.0000) & (1.0000, 1.0000, 1.0000, 1.0000) \end{bmatrix},$$

$$\bar{A}^{(4)} = \begin{bmatrix} (1.0000, 1.0000, 1.0000, 1.0000) & (4.0000,4.5000, 5.5000,6.0000) & (2.0000,2.5000,3.5000,4.0000) & (5.0000,5.5000,6.5000,7.0000) \\ (0.1667,0.1818,0.2222,0.2500) & (1.0000, 1.0000, 1.0000, 1.0000) & (2.3842,2.6007,3.0290,3.2462) & (0.2500,0.2857,0.4000,0.5000) \\ (0.2500,0.2857,0.4000,0.5000) & (0.3081,0.3301,0.3845,0.4194) & (1.0000, 1.0000, 1.0000, 1.0000) & (2.0000,2.5000,3.5000,4.0000) \\ (0.1429,0.1538, 0.1818, 0.2000) & (2.0000,2.5000,3.5000,4.0000) & (0.2500,0.2857,0.4000,0.5000) & (1.0000, 1.0000, 1.0000, 1.0000) \end{bmatrix},$$

$$\bar{A}^{(3)} = \begin{bmatrix} (1.0000, 1.0000, 1.0000, 1.0000) & (2.0000,2.5000, 3.5000,4.0000) & (0.2500,0.2857, 0.4000,0.5000) & (0.3333,0.4000,0.6667,1.0000) \\ (0.2500,0.2857, 0.4000,0.5000) & (1.0000, 1.0000, 1.0000, 1.0000) & (0.1667,0.1818,0.2222,0.2500) & (2.2622,2.2554,2.8512,3.0629) \\ (0.2000,0.2500,0.3500,0.4000) & (4.0000,4.5000,5.5000,6.0000) & (1.0000, 1.0000, 1.0000, 1.0000) & (0.2845,0.3011,0.3420,0.3681) \\ (1.0000,1.5000, 2.5000,3.0000) & (0.3265,0.3507,0.4073,0.4420) & (2.7164,2.9241,3.3212,3.5145) & (1.0000, 1.0000, 1.0000, 1.0000) \end{bmatrix},$$

$$\bar{A}^{(4)} = \begin{bmatrix} (1.0000, 1.0000, 1.0000, 1.0000) & (0.2683,0.2495, 0.3653,0.4158) & (3.0000,3.5000, 4.5000,5.0000) & (3.2113,3.5444,4.1955,4.5130) \\ (2.4053,2.7376,3.3950,3.7270) & (1.0000, 1.0000, 1.0000, 1.0000) & (0.3333,0.4000,0.6667,1.0000) & (3.1985,3.5334,4.1955,4.5368) \\ (0.2000,0.2222,0.2857,0.3333) & (1.0000,1.5000,2.5000,3.0000) & (1.0000, 1.0000, 1.0000, 1.0000) & (0.2705,0.2968,0.3680,0.4189) \\ (0.2216,0.2383, 0.2821, 0.3114) & (0.2204,0.2381,0.2830,0.3126) & (2.3872,2.7175,3.3690,3.6968) & (1.0000, 1.0000, 1.0000, 1.0000) \end{bmatrix},$$

(2) The optimal weighting vector of DMs is obtained in the following:

$$V^* = (v_1^*, v_2^*, v_3^*, v_4^*)^T = (0.0000, 0.3381, 0.0000, 0.6619)^T.$$

(3) Based on the optimal weights of DMs, we obtain the synthetic fuzzy preference relation as follows:

$$\tilde{A}^s = \begin{bmatrix} (1, 1, 1, 1) & (0.6689,0.7404,0.9137,1.0252) & (2.6157,3.1236,4.1334,4.6367) & (3.7299,4.1121,4.8648,5.2350) \\ (0.9754,1.0944,1.3506,1.4950) & (1, 1, 1, 1) & (0.6483,0.7533,1.1122,1.4890) & (1.3510,1.5097,1.8965,2.1524) \\ (0.2157,0.2419,0.3201,0.3823) & (0.6716,0.8991,1.3276,1.5425) & (1, 1, 1, 1) & (0.5320,0.6101,0.7881,0.8983) \\ (0.1910,0.2056,0.2432,0.2681) & (0.4646,0.5273,0.6624,0.7402) & (1.1132,1.2689,1.6392,1.8797) & (1,1,1,1) \end{bmatrix}$$

(4) The fuzzy priority vectors of synthetic fuzzy preference relation $\tilde{A}^s$ are calculated below:

$$\tilde{\omega}_1 = (0.4382,0.4692,0.5026,0.5026); \quad \tilde{\omega}_2 = (0.2636,0.2822,0.3154,0.3330);$$

$$\tilde{\omega}_3 = (0.1445,0.1613,0.1847,0.1920); \quad \tilde{\omega}_4 = (0.1537,0.1627,0.1740,0.1759).$$

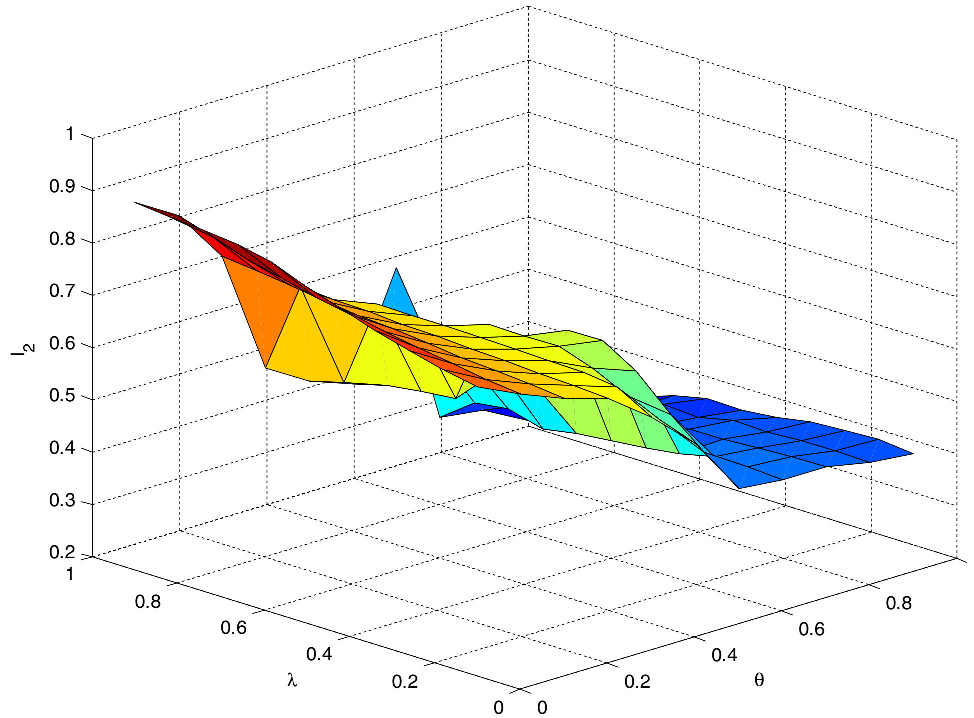


Fig. 6 The weight of d_2 determined by different λ and θ

(5) By the fuzzy priority vectors, we get:

$$R(\tilde{\omega}_1) = 0.1689, \quad R(\tilde{\omega}_2) = 0.1091,$$
$$R(\tilde{\omega}_3) = 0.0631, \quad R(\tilde{\omega}_4) = 0.0584.$$

Therefore, we have $C_1 \succ C_2 \succ C_3 \succ C_4$, which means that Jay Zhou is the best choice for the competitor and it is the same as the result computed by our proposed method.

Compared to the method developed in [54], we observe that the risk attitude of DMs is taken into account proposed in this paper, but Ref. [54] ignores it. And the compatibility index based on the COWGA operator can be used to deal with the multiplicative trapezoidal fuzzy preference relations with more flexibility due to the fact that the decision maker can choose a different value of the parameter λ according to his/her own opinion.

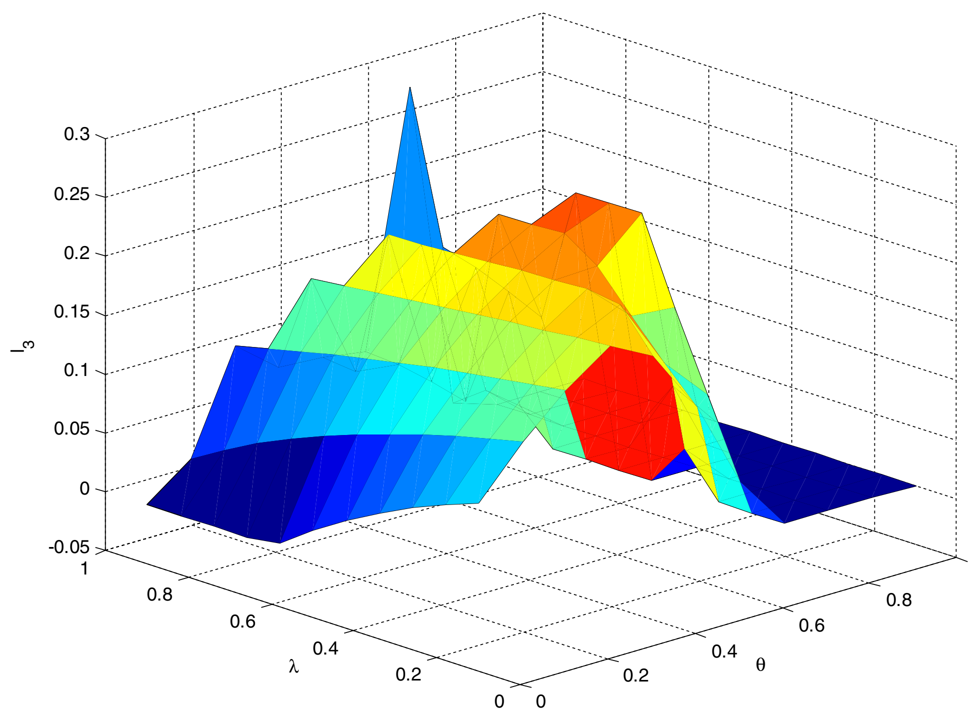


Fig. 7 The weight of d_3 determined by different λ and θ

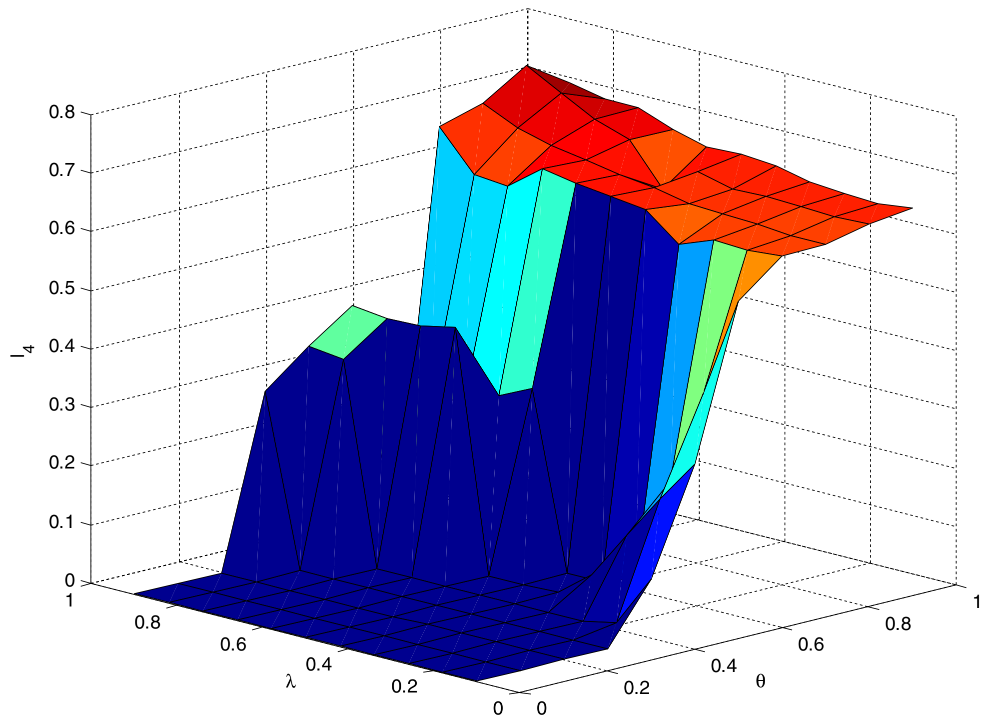


Fig. 8 The weight of d_4 determined by different λ and θ

In addition, the proposed method is very different from the approach developed in [35]. In [35], Li et al. proposed a personalized individual semantics model to derive consistency and consensus of 2-tuple linguistic preference relations. They personalized individual semantics directly rather than translating the linguistic terms into trapezoidal fuzzy numbers. And they put forward a consistency-driven optimization-based model, but the optimal model presented in this paper is based on the compatibility measure with COWGA operator. Moreover, the additional emphasis of the proposed method is to derive experts' weights by means of compatibility-driven optimization-based model.

Moreover, Dong, Zhang and Herrera-Viedma [19] presented a self-management mechanism to generate experts' weights, in which the experts' weights are dynamically derived from the multi-attribute mutual evaluation matrices. The approach developed in [19] is different from the proposed method in this paper, the reasons are as follows: (1) The mechanisms to generate experts' weights are different, Ref.[19] derived weights of experts by using a self-management mechanism, but this paper focuses on optimization-based model by means of compatibility measure with the COWGA operator. (2) The proposed method emphasizes the GDM with trapezoidal fuzzy preference relations rather than the GDM with [0,1] fuzzy preference relations.

In [5], Cabrerizo, Herrera-Viedma and Pedrycz put forward an approach to deriving weights of experts with heterogeneous linguistic contexts in which the experts have associated importance degrees reflecting their ability to handle the problem, and the weights are obtained by using the particle swarm optimization. But the characteristic of the proposed method is to derive experts' weights on the basis of compatibility-driven model.

In [77], Zhou et al. developed a new compatibility between additive trapezoidal fuzzy preference relation and its characteristic preference relation, and priority vectors are derived utilizing a least deviation model. However, the proposed method concentrates on GDM with MTFPRs rather than the GDM with additive trapezoidal fuzzy preference relations. Moreover, the proposed approach uses the COWGA operator to transform the MTFPRs into ordinary MPRs. And a compatibility improving algorithm makes MTFPRs acceptably compatible but it has not been considered in Ref.[77].

In [78], Zhou et al. presented an approach to deal with the GDM with additive trapezoidal fuzzy preference relations by using compatibility measure and a compatibility improving algorithm. Compared to [78], we find that the FPRs discussed in this paper are two different PRs, and the proposed method takes the attitude of DMs into account by using the COWGA operator.

6 Conclusion

In this paper, we develop the logarithm compatibility measure with MTFPRs based on the COWGA operator. The main work of this paper is summarized as follows:

(1) By the α-cut, we obtain the expected interval value from the trapezoidal fuzzy number. We get a real number from the expected interval value based on the COWGA operator. Thus, we can translate the MTFPR into an MPR via the α-cut and the COWGA operator. Based on the above, we presented the logarithm compatibility measure with MTFPRs. At the same time, we investigated some desirable properties of the compatibility index.
(2) An iterative and convergent algorithm to adjust each MTFPR automatically guarantees that all the adjusted MTFPRs are of acceptable compatibility.
(3) We have further proposed the optimal model to determine the DMs' weights by minimizing the logarithm compatibility index in GDM with the collective fuzzy preference relation.
(4) A numerical example is developed to ensure the validity of the proposed method in the whole GDM process with MTFPRs via α-cut and the COWGA operator.

The main contribution of this paper is to offer a new approach to GDM problems with MTFPRs based on the logarithm compatibility measure. The contributions of this paper are the following:

(1) The new approach is more flexible and reasonable because it not only utilizes the trapezoidal fuzzy numbers but also considers the DM's risk attitude.
(2) The proposed method uses an iterative and convergent algorithm to help each DM's preference relation achieve acceptable compatibility.
(3) Using the optimal model based on the criterion for the minimization of the group logarithm compatibility index, we are able to obtain the DMs' weights, which is objective and rational.

Based on the linguistic models [35, 40], future research may be performed to extend our compatibility measure and compatibility improving process to other type preference relations, including linguistic preference relation [18, 31, 47], interval-valued fuzzy preference relation [17], heterogeneous linguistic contexts [5], etc. Additional research on application of proposed approach should be implemented, for example, the proposed approach can be combined with data envelopment analysis (DEA) [7, 43], analytic network process (ANP) [36, 41], Dempster-Shafer theory [7, 21], utility theory [25, 26] and fuzzy set qualitative comparative analysis (fsQCA) [22, 37] and can be applied to performance evaluations [27, 28], management information systems [15], computing with words [35, 39], etc.

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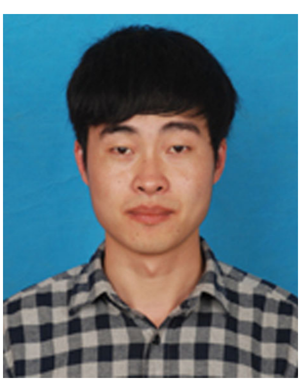

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