Appl Intell (2014) 40:229–243
DOI 10.1007/s10489-013-0454-4

# On compatibility of uncertain multiplicative linguistic preference relations based on the linguistic COWGA

**Ligang Zhou · Yingdong He · Huayou Chen · Jinpei Liu**

Published online: 5 July 2013


**Abstract** The aim of this work is to develop a new compatibility for the uncertain multiplicative linguistic preference relations and utilize it to determine the optimal weights of experts in the group decision making (GDM). First, the compatibility degree and compatibility index for the two multiplicative linguistic preference relations are proposed. Then, based on the linguistic continuous ordered weighted geometric averaging (LCOWGA) operator, some concepts of the compatibility degree and compatibility index for the two uncertain multiplicative linguistic preference relations are presented. We prove the property that the synthetic uncertain linguistic preference relation is of acceptable compatibility under the condition that the uncertain multiplicative linguistic preference relations given by experts are all of acceptable compatibility with the ideal uncertain multiplicative linguistic preference relation, which provides a theoretic basis for the application of the uncertain multiplicative linguistic preference relations in GDM. Next, an optimal model is constructed to determine the weights of experts based on the criterion of minimizing the compatibility index in GDM. Moreover, an approach to GDM with uncertain multiplicative linguistic preference relations is developed, and finally, an application of the approach to supplier selection problem with uncertain multiplicative linguistic preference relations is pointed out.

**Keywords** Group decision making · Uncertain multiplicative linguistic preference relation · Compatibility · LCOWGA operator

L. Zhou · Y. He · H. Chen (✉)
School of Mathematical Science, Anhui University, Hefei, Anhui 230601, China
e-mail: huayouc@126.com

L. Zhou
e-mail: shuiqiaozlg@126.com

Y. He
e-mail: 812256307@qq.com

J. Liu
School of Business, Anhui University, Hefei, Anhui 230601, China
e-mail: liujinpei2009@gmail.com

## 1 Introduction

Group decision making (GDM) is one of the most significant and omnipresent human activities in business, service, manufacturing, selection of products, etc. In a GDM problem, experts have to express their preferences by means of a set of evaluations over a set of alternatives and find the best alternative(s) by using a proper aggregation technique [1]. According to the nature of the information expressed for every pair of alternatives there exist many different representation formats of preference relations: the multiplicative preference relation [6, 19, 31, 32, 49], the fuzzy preference relation [7, 20, 43], the multiplicative linguistic preference relation [15, 47, 51, 53, 59], the additive linguistic preference relation [1, 11, 14, 18, 51, 55], the intuitionistic preference relation [16, 60–62, 65].

However, sometimes, experts are only willing or able to provide uncertain information because of time pressure, lack of knowledge, or data, and their limited expertise related to the problem domain. For example, many researchers pay attention to GDM with the uncertain fuzzy preference relations [8, 50, 57, 64], the uncertain multiplicative preference relations [9, 36, 37, 46, 47, 74], the uncertain additive linguistic preference relations [12, 35, 54], the uncertain multiplicative linguistic preference relations [56] and the interval-valued intuitionistic fuzzy preference relations [63].

A crucial problem in GDM with all kinds of preference relations is the compatibility degree. The lack of acceptable compatibility can lead to unsatisfied decision making with preference relations because there is a significant difference among the preference relations given by experts in GDM [12]. In [7], Chen and Zhao presented the compatibility degree of two fuzzy preference relations. Xu [50] extended the compatibility degree to the uncertain environment and developed the compatibility degree of two uncertain fuzzy preference relations. In [32], Saaty and Vargas presented the compatibility to judge the difference between the two multiplicative preference relations. Chen, Zhou and Han [12] proposed a new compatibility degree for the uncertain additive linguistic preference relations and utilized it to determine the optimal weights of experts in GDM. Therefore, it is necessary to investigate this issue.

Another important issue of GDM is to find the proper way to aggregate experts' preferences. The ordered weighted averaging (OWA) operator introduced by Yager [68] is a useful tool for aggregating the exact arguments that lie between the max and the min operators. Since it has appeared, the OWA operator has been studied in a wide range of applications and extensions [4, 5, 13, 21–30, 39–42, 44, 48, 52, 54, 56, 58, 62, 66, 67, 69–73, 75–80, 82–84, 86–89]. A very practical extension of the OWA operator is the continuous ordered weighted averaging (COWA) operator [70] in which the arguments to be aggregated are interval numbers [33]. Recently, several authors have developed different extensions and applications of the COWA operator [8–10, 45, 46, 51, 74, 85]. For example, Yager and Xu [74] extended the COWA operator and obtained the continuous ordered weighted geometric averaging (COWGA) operator. Zhang and Xu [81] extended the COWGA operator to the linguistic environment and obtained the linguistic COWGA (LCOWGA) operator.

The aim of this paper is to develop a new compatibility for the uncertain multiplicative linguistic preference relations based on the LCOWGA operator and utilize it to determine the optimal weights of experts in GDM. To do that, we define the compatibility degree and compatibility index of two multiplicative linguistic preference relations, and some concepts of the compatibility degree and compatibility index of two uncertain multiplicative linguistic preference relations based on the LCOWGA operator. Some properties of new concepts, which are theoretic bases for the application of the uncertain multiplicative linguistic preference relations in GDM, are studied. We also developed a nonlinear model to determine experts' weights based on the criterion of minimizing the compatibility index in GDM. Furthermore, the expected multiplicative linguistic preference relation is proposed and the applicability of the new approach is analyzed in a supplier selection problem.

In order to do so, this paper is organized as follows. In Sect. 2, we briefly review some basic concepts. Section 3 presents the compatibility of multiplicative linguistic preference relations. Section 4 proposes the concepts of compatibility degree of uncertain multiplicative linguistic preference relations based on the LCOWGA operator and build up the optimal model to determine the optimal experts' weights in GDM. In Sect. 5, we present the expected multiplicative linguistic preference relation and propose a new approach for uncertain multiplicative linguistic preference relations. In Sect. 6, we developed an illustrative example of the new approach focusing on the supplier selection. Finally, in Sect. 7 we summarize the main conclusions of the paper.

## 2 Preliminaries

In this section, we briefly review the uncertain multiplicative linguistic variable, the uncertain multiplicative linguistic preference relation, the OWA operator, the COWGA operator and the LCOWGA operator.

### 2.1 Uncertain multiplicative linguistic variable and operational laws

Let $S = \{s_\alpha \mid \alpha = 1/t, \dots, 1/2, 1, 2, \dots, t\}$ be a multiplicative linguistic label set with odd cardinality, which requires that the multiplicative linguistic label set should satisfy the following characteristics [56]:

(1) The set $S$ is ordered: if $s_\alpha, s_\beta \in S$ and $\alpha > \beta$, then $s_\alpha > s_\beta$.
(2) There exists the reciprocal operator: $\mathrm{rec}(s_\alpha) = s_\beta$ such that $\alpha\beta = 1$,

where $s_\alpha$ and $s_\beta$ represent possible values for the linguistic variables and $t$ is a positive integer.

The multiplicative linguistic label set $S$ is called the multiplicative linguistic scale. For example, a set of nine labels $S$ can be defined as:

$$S = \{s_{1/5} = EL, s_{1/4} = VL, s_{1/3} = L, s_{1/2} = SL, s_1 = M,\\ s_2 = SH, s_3 = H, s_4 = VH, s_5 = EH\}.$$

Note that $EL = Extremely\ low$, $VL = Very\ low$, $L = Low$, $SL = Slightly\ low$, $M = Medium$, $SH = Slightly\ high$, $H = High$, $VH = Very\ high$, $EH = Extremely\ high$.

To preserve all the given information, we can extend the discrete linguistic term set $S$ to a continuous linguistic term set $\tilde{S} = \{s_\alpha \mid \alpha \in [1/q, q]\}$, where $q$ $(q > t)$ is a sufficiently large positive integer. If $s_\alpha \in S$, we call $s_\alpha$ the original multiplicative linguistic term, which is provided to evaluate alternatives by the decision makers, otherwise, we call $s_\alpha$ the virtual multiplicative linguistic term, which can only appear in operations.

**Definition 1** [56] Let $\tilde{s} = [s_\alpha, s_\beta] = \{x | s_\alpha \le x \le s_\beta\}$, then $\tilde{s}$ is called the uncertain multiplicative linguistic variable, where $s_\alpha, s_\beta \in \tilde{S}$, $s_\alpha$, $s_\beta$ are the lower and upper limits, respectively. Especially, $\tilde{s}$ is called the multiplicative linguistic variable if $s_\alpha = s_\beta$.

Suppose that $\tilde{s} = [s_\alpha, s_\beta]$, $\tilde{s}_1 = [s_{\alpha_1}, s_{\beta_1}]$ and $\tilde{s}_2 = [s_{\alpha_2}, s_{\beta_2}]$ are any three uncertain multiplicative linguistic variables, and $\mu, \mu_1, \mu_2 \in [0, 1]$. Xu [56] defined some operational laws as follows:

1. $\tilde{s}_1 \otimes \tilde{s}_2 = [s_{\alpha_1}, s_{\beta_1}] \otimes [s_{\alpha_2}, s_{\beta_2}] = [s_{\alpha_1} \otimes s_{\alpha_2}, s_{\beta_1} \otimes s_{\beta_2}] = [s_{\alpha_1\alpha_2}, s_{\beta_1\beta_2}]$.
2. $\tilde{s}_1 \otimes \tilde{s}_2 = \tilde{s}_2 \otimes \tilde{s}_1$.
3. $\tilde{s}^\mu = [s_\alpha, s_\beta]^\mu = [s_\alpha^\mu, s_\beta^\mu] = [s_{\alpha^\mu}, s_{\beta^\mu}]$.
4. $\tilde{s}^{\mu_1} \otimes \tilde{s}^{\mu_2} = \tilde{s}^{\mu_1+\mu_2}$.
5. $(\tilde{s}_1 \otimes \tilde{s}_2)^\mu = \tilde{s}_1^\mu \otimes \tilde{s}_2^\mu$.

### 2.2 Uncertain multiplicative linguistic preference relation

In a GDM problem, let $X = \{x_1, x_2, \dots, x_n\}$ be a finite set of alternatives. When an expert makes pairwise comparisons using the multiplicative linguistic term set $S$, he/she can express his/her own opinions by a multiplicative linguistic preference relation [51] on $X$. The multiplicative linguistic preference relation can be defined as follows:

**Definition 2** [51] A multiplicative linguistic preference relation $A = (a_{ij})_{n\times n}$ on the set $X$ is denoted by a linguistic decision matrix $A = (a_{ij})_{n\times n} \subset X \times X$, such that

$$\begin{aligned} &a_{ij} \in S, \qquad a_{ij} \otimes a_{ji} = s_1, \\ &a_{ii} = s_1, \quad \forall i, j = 1, 2, \dots, n, \end{aligned} \tag{1}$$

where $a_{ij}$ represents the preference degree of the alternative $x_i$ over $x_j$. Especially, $a_{ij} = s_1$ indicates that $x_i$ is equivalent to $x_j$, $a_{ij} > s_1$ indicates that $x_i$ is preferred to $x_j$, and $a_{ij} < s_1$ indicates that $x_j$ is preferred to $x_i$. For convenience, throughout this paper, let $Z_n$ be the set of all $n \times n$ multiplicative linguistic preference relations.

However, experts may only be able to provide uncertain multiplicative linguistic preference relations [56] because of time pressure, lack of knowledge or data and their limited expertise related to the problem domain. The uncertain multiplicative linguistic preference relation can be defined as follows.

**Definition 3** [56] An uncertain multiplicative linguistic preference relation on the set $X$ is defined as matrix $\tilde{A} = (\tilde{a}_{ij})_{n\times n} \subset X \times X$ satisfying

$$\begin{aligned} &\tilde{a}_{ij}^U \otimes \tilde{a}_{ji}^L = s_1, \qquad \tilde{a}_{ij}^L \otimes \tilde{a}_{ji}^U = s_1, \\ &\tilde{a}_{ij}^U = \tilde{a}_{ij}^L = s_1, \quad \forall i, j = 1, 2, \dots, n, \end{aligned} \tag{2}$$

where $\tilde{a}_{ij} = [\tilde{a}_{ij}^L, \tilde{a}_{ij}^U]$ indicates the multiplicative linguistic preference relation degree of the alternative $x_i$ over $x_j$, $\tilde{a}_{ij}^L, \tilde{a}_{ij}^U \in \tilde{S}$, $\tilde{a}_{ij}^U \ge \tilde{a}_{ij}^L$, $\tilde{a}_{ij}^L$ and $\tilde{a}_{ij}^U$ are the lower and upper bounds of uncertain multiplicative linguistic variables $\tilde{a}_{ij}$, respectively.

Note that throughout this paper, let $M_n$ be the set of all $n \times n$ uncertain multiplicative linguistic preference relations. For convenience, in [59], Xu given a mark for the linguistic term with a index, i.e., suppose that $s_\alpha \in \tilde{S}$, $I(s_\alpha)$ denotes the lower index of multiplicative linguistic term $s_\alpha$, then we have $I(s_\alpha) = \alpha > 0$.

### 2.3 The OWA operator and the COWGA operator

The OWA operator [68] is an aggregation operator that provides a parameterized family of aggregation operators between the minimum and the maximum, which can be defined as follows:

**Definition 4** An OWA operator of dimension $n$ is a mapping $OWA : R^n \to R$ that has an associated weighting vector $w$ with $\sum_{j=1}^n w_j = 1$ and $w_j \in [0, 1]$, such that

$$OWA(a_1, a_2, \dots, a_n) = \sum_{j=1}^n w_j b_j, \tag{3}$$

where $b_j$ is the $j$th largest of the arguments $a_1, a_2, \dots, a_n$.

The OWA operator is monotonic, commutative, bounded and idempotent. Other properties could be studied such as different families of the OWA operators [4, 68, 71, 73, 75].

The COWGA operator was developed by Yager and Xu [74], which can be defined as follows:

**Definition 5** A COWGA operator is a mapping $G : \Sigma^+ \to R^+$ associated with a basic unit interval monotonic (BUM) function $Q$, such that

$$G_Q(a) = G_Q([a^L, a^U]) = a^U \left(\frac{a^L}{a^U}\right)^{\int_0^1 \frac{dQ(y)}{dy} y dy}, \tag{4}$$

where $\Sigma^+$ is the set of closed intervals, in which the lower limits of all closed intervals are positive, $R^+$ is the set of positive real numbers, the BUM function $Q : [0, 1] \to [0, 1]$ is monotonic, and $Q(0) = 0$, $Q(1) = 1$.

If $\lambda = \int_0^1 Q(y)dy$ is the attitudinal character of $Q$, then a general formulation of $G_Q(a)$ can be obtained as follows:

$$G_Q(a) = G_Q([a^L, a^U]) = (a^U)^\lambda (a^L)^{1-\lambda}. \tag{5}$$

As we can see, the COWGA operator $G_Q(a)$ is always the weighted geometric mean of end points based on the attitudinal character. That is to say, the interval number $a$ can be replaced by $G_Q(a)$.

### 2.4 Linguistic COWGA operator

Zhang and Xu [81] extended the COWGA operator to linguistic environment and obtained the linguistic COWGA operator, which can be defined as follows.

**Definition 6** Let $\tilde{s} = [s_\alpha, s_\beta]$ be an uncertain multiplicative linguistic variable. If

$$g_Q(\tilde{s}) = g_Q\big([s_\alpha, s_\beta]\big) = s_\gamma, \tag{6}$$

and

$$\gamma = G_Q\big(\big[I(s_\alpha), I(s_\beta)\big]\big) = G_Q\big([\alpha, \beta]\big), \tag{7}$$

then $g$ is called the linguistic COWGA (LCOWGA) operator, where $s_\alpha, s_\beta \in \tilde{S}$ and $Q$ is the BUM function.

If $\lambda = \int_0^1 Q(y)dy$ is the attitudinal character of $Q$, then the LCOWGA operator can be written as follows:

$$g_Q(\tilde{s}) = g_Q\big([s_\alpha, s_\beta]\big) = s_{\beta^\lambda \times \alpha^{1-\lambda}} = (s_\beta)^\lambda \otimes (s_\alpha)^{1-\lambda}. \tag{8}$$

It can be seen from Eq. (8) that the LCOWGA operator may be determined by the attitudinal character $\lambda$. For convenience, in this paper, we assume that $g_\lambda(\tilde{s})$ denotes $g_Q(\tilde{s})$, i.e.,

$$g_\lambda(\tilde{s}) = g_Q(\tilde{s}) = (s_\beta)^\lambda \otimes (s_\alpha)^{1-\lambda} = s_{\beta^\lambda \times \alpha^{1-\lambda}}. \tag{9}$$

## 3 The compatibility of multiplicative linguistic preference relations

In this section, a new compatibility index of multiplicative linguistic preference relations will be introduced and some desired properties of the compatibility index will be investigated.

**Definition 7** Let $A = (a_{ij})_{n\times n} \in Z_n$ and $B = (b_{ij})_{n\times n} \in Z_n$ be two multiplicative linguistic preference relations, then

$$C(A, B) = \frac{1}{2t}\sum_{i=1}^{n}\sum_{j=1}^{n}\big(I(a_{ij})I(b_{ji}) + I(b_{ij})I(a_{ji}) - 2\big), \tag{10}$$

is called the compatibility degree of $A$ and $B$.

It can be seen that the compatibility degree $C(A, B)$ reflects the total reciprocal difference between the multiplicative linguistic preference relations $A$ and $B$.

**Lemma 1** *If $s_{\alpha_i} \in \tilde{S}$ and $\mu_i \in [0, 1]$ for $i = 1, 2, \ldots, n$, then*

$$I\big((s_{\alpha_1})^{\mu_1} \otimes (s_{\alpha_2})^{\mu_2} \otimes \cdots \otimes (s_{\alpha_n})^{\mu_n}\big) = \prod_{i=1}^{n}\big(I(s_{\alpha_i})\big)^{\mu_i}. \tag{11}$$

Specially, by Lemma 1, if $n = 2$, then we have

$$I\big((s_{\alpha_1})^{\mu_1} \otimes (s_{\alpha_2})^{\mu_2}\big) = \big(I(s_{\alpha_1})\big)^{\mu_1} \times \big(I(s_{\alpha_2})\big)^{\mu_2}. \tag{12}$$

The compatibility degree of any two multiplicative linguistic preference relations is nonnegative, reflexive and commutative, which can be expressed as follows:

**Theorem 1** *Assume that $A = (a_{ij})_{n\times n} \in Z_n$, $B = (b_{ij})_{n\times n} \in Z_n$, then*

(1) $C(A, B) \geq 0$.
(2) $C(A, A) = 0$.
(3) $C(A, B) = C(B, A)$.

**Lemma 2** *Let $s_\alpha, s_\beta \in \tilde{S}$. Then $I(s_\alpha) = I(s_\beta)$ if and only if $s_\alpha = s_\beta$.*

**Lemma 3** *Assume that $A = (a_{ij})_{n\times n} \in Z_n$, $B = (b_{ij})_{n\times n} \in Z_n$, then $C(A, B) = 0$ if and only if $a_{ij} = b_{ij}$ for $i, j = 1, 2, \ldots, n$.*

**Theorem 2** *Assume that $A = (a_{ij})_{n\times n} \in Z_n$, $B = (b_{ij})_{n\times n} \in Z_n$ and $D = (d_{ij})_{n\times n} \in Z_n$. If $C(A, B) = 0$ and $C(B, D) = 0$, then $C(A, D) = 0$.*

Theorem 2 indicates that the compatibility degree of multiplicative linguistic preference relations is transitive.

**Definition 8** Let $A = (a_{ij})_{n\times n} \in Z_n$, $B = (b_{ij})_{n\times n} \in Z_n$ be two multiplicative linguistic preference relations. If $C(A, B) = 0$, then $A$ and $B$ are perfectly compatible.

Obviously, the following theorem can be obtained from Definition 8.

**Theorem 3** *Let $A = (a_{ij})_{n\times n} \in Z_n$ and $B = (b_{ij})_{n\times n} \in Z_n$ be two multiplicative linguistic preference relations, then $A$ and $B$ are perfectly compatible if and only if $a_{ij} = b_{ij}$, $i, j = 1, 2, \ldots, n$.*

**Definition 9** If $A = (a_{ij})_{n\times n} \in Z_n$ and $B = (b_{ij})_{n\times n} \in Z_n$, then

$$CI(A, B) = \frac{1}{n^2}C(A, B), \tag{13}$$

is called the compatibility index of the multiplicative linguistic preference relations $A$ and $B$.

By Definition 9, Theorems 1, 3 and Lemma 3, the following conclusion is obvious.

**Theorem 4** *If* $A=(a_{ij})_{n\times n}\in Z_n$ *and* $B=(b_{ij})_{n\times n}\in Z_n$, *then*

(1) $CI(A,B)\geq 0$.
(2) $CI(A,B)=0$ *if and only if* $A$ *and* $B$ *are perfectly compatible.*

**Definition 10** Let $A=(a_{ij})_{n\times n}\in Z_n$ and $B=(b_{ij})_{n\times n}\in Z_n$. If

$$CI(A,B)\leq\alpha, \tag{14}$$

then $A$ and $B$ are of acceptable compatibility, where $\alpha$ is the threshold of acceptable compatibility.

Similar to [12], we can take $\alpha=0.2$ as the threshold of acceptable compatibility.

Let $E=\{e_1,e_2,\ldots,e_m\}$ be a finite set of experts and $A^{(k)}=(a_{ij}^{(k)})_{n\times n}\in Z_n$ be the multiplicative linguistic preference relation provided by expert $e_k$, $k=1,2,\ldots,m$. In [51], Xu defined that the synthetic linguistic preference relation developed some properties of the synthetic linguistic preference relations, which can be expressed as follows:

**Definition 11** Let $A^{(k)}=(a_{ij}^{(k)})_{n\times n}\in Z_n$ for $k=1,2,\ldots,m$. If

$$a_{ij}=\bigotimes_{k=1}^{m}\left(a_{ij}^{(k)}\right)^{\omega_k}, \tag{15}$$

then the matrix $A=(a_{ij})_{n\times n}$ is called the synthetic linguistic preference relation of all experts, where $\Omega=(\omega_1,\omega_2,\ldots,\omega_m)$ is the weighting vector of experts, which satisfies that $\omega_k\geq 0$ for all $j=1,2,\ldots,m$ and $\sum_{k=1}^m\omega_k=1$.

**Theorem 5** [51] *If* $A^{(k)}=(a_{ij}^{(k)})_{n\times n}\in Z_n$ *for all* $k=1,2,\ldots,m$, *then the synthetic linguistic preference relation* $A=(a_{ij})_{n\times n}\in Z_n$.

**Lemma 4** [49] *Assume that* $x_i>0$ *and* $w_i>0$ *for* $i=1,2,\ldots,l$. *Then*

$$\prod_{i=1}^{l}x_i^{w_i}\leq\sum_{i=1}^{l}w_ix_i, \tag{16}$$

*with equality if and only if* $x_1=x_2=\cdots=x_l$, *where* $\sum_{i=1}^n w_i=1$.

**Theorem 6** *Let* $A^{(k)}=(a_{ij}^{(k)})_{n\times n}\in Z_n$ *for* $k=1,2,\ldots,m$, *and* $B=(b_{ij})_{n\times n}\in Z_n$. *If* $CI(A^{(k)},B)\leq\alpha$, *for all* $k=1,2,\ldots,m$, *then*

$$CI(A,B)\leq\alpha, \tag{17}$$

*where* $A=(a_{ij})_{n\times n}$ *is the synthetic linguistic preference relation and* $\alpha$ *is the threshold of acceptable compatibility.*

*Proof* By Definition 11, we have that

$$a_{ji}=\bigotimes_{k=1}^{m}\left(a_{ji}^{(k)}\right)^{\omega_k}\quad\text{and}\quad a_{ji}=\bigotimes_{k=1}^{m}\left(a_{ji}^{(k)}\right)^{\omega_k},$$

where $\Omega=(\omega_1,\omega_2,\ldots,\omega_m)$ is the weighting vector of $m$ experts, which satisfies that $\omega_k\geq 0$ for all $j=1,2,\ldots,m$ and $\sum_{k=1}^m\omega_k=1$.

By Lemma 1, we get $I(a_{ij})=\prod_{k=1}^m(I(a_{ij}^{(k)}))^{\omega_k}$, $I(a_{ji})=\prod_{k=1}^m(I(a_{ji}^{(k)}))^{\omega_k}$.

Since $CI(A^{(k)},B)\leq\alpha$ for all $k$, then

$$\begin{aligned}CI\big(A^{(k)},B\big)&=\frac{1}{2tn^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\Big(I\big(a_{ij}^{(k)}\big)I(b_{ji})+I(\tilde{b}_{ij})I\big(\tilde{a}_{ji}^{(k)}\big)-2\Big)\\&\leq\alpha.\end{aligned}$$

Thus, by Definition 9 and Lemma 4, it can be obtained that

$$\begin{aligned}CI(A,B)&=\frac{1}{2tn^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\big(I(a_{ij})I(\tilde{b}_{ji})+I(\tilde{b}_{ij})I(\tilde{a}_{ji})-2\big)\\&=\frac{1}{2tn^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\Bigg(\prod_{k=1}^{m}\Big(I\big(a_{ij}^{(k)}\big)\Big)^{\omega_k}I(b_{ji})\\&\quad+I(b_{ij})\prod_{k=1}^{m}\Big(I\big(a_{ji}^{(k)}\big)\Big)^{\omega_k}-2\Bigg)\\&\leq\frac{1}{2tn^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\Bigg(\sum_{k=1}^{m}\omega_kI\big(a_{ij}^{(k)}\big)I(b_{ji})\\&\quad+\sum_{k=1}^{m}\omega_kI\big(a_{ji}^{(k)}\big)I(b_{ij})-2\Bigg)\\&=\frac{1}{2tn^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\Bigg(\sum_{k=1}^{m}\omega_k\Big(I\big(a_{ij}^{(k)}\big)I(b_{ji})\\&\quad+I\big(a_{ji}^{(k)}\big)I(b_{ij})-2\Big)\Bigg)\\&=\sum_{k=1}^{m}\omega_k\Bigg(\frac{1}{2tn^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\Big(I\big(a_{ij}^{(k)}\big)I(b_{ji})\\&\quad+I\big(a_{ji}^{(k)}\big)I(b_{ij})-2\Big)\Bigg)\\&=\sum_{k=1}^{m}\omega_kCI\big(A^{(k)},B\big)\leq\sum_{k=1}^{m}\omega_k\alpha=\alpha.\end{aligned}$$

□

It is obvious that the synthetic linguistic preference relation is of acceptable compatibility under the condition that the multiplicative linguistic preference relations given by experts are all of acceptable compatibility with the ideal multiplicative linguistic preference relation.

**Corollary 1** *Let $A^{(k)} = (a_{ij}^{(k)})_{n\times n} \in Z_n$ for $k = 1, 2, \dots, m$, and $B = (b_{ij})_{n\times n} \in Z_n$. If $CI(A^{(k)}, B) = 0$, for all $\lambda$, $\forall k = 1, 2, \dots, m$, then*

$$CI(A, B) = 0, \tag{18}$$

*where $A = (a_{ij})_{n\times n}$ is the synthetic linguistic preference relation.*

It can be seen from Corollary 1 that if $A^{(k)}$ and $B$ are perfectly compatible for all $k = 1, 2, \dots, m$, then $A$ and $B$ are perfectly compatible.

## 4 The compatibility index of uncertain multiplicative linguistic preference relations based on the LCOWGA operator

In this section, we shall present a compatibility index of uncertain multiplicative linguistic preference relations based on the LCOWGA operator. Then some desirable properties of the compatibility index will be studied.

**Definition 12** Let $\tilde{A} = (\tilde{a}_{ij})_{n\times n} \in M_n$ and $\tilde{B} = (\tilde{b}_{ij})_{n\times n} \in M_n$ be two uncertain multiplicative linguistic preference relations, then

$$C_\lambda(\tilde{A}, \tilde{B}) = \frac{1}{2t}\sum_{i=1}^{n}\sum_{j=1}^{n}\Big(I\big(g_\lambda(\tilde{a}_{ij})\big)I\big(g_{1-\lambda}(\tilde{b}_{ji})\big) + I\big(g_\lambda(\tilde{b}_{ij})\big)I\big(g_{1-\lambda}(\tilde{a}_{ji})\big) - 2\Big), \tag{19}$$

is called the compatibility degree of $\tilde{A}$ and $\tilde{B}$ based on the LCOWGA operator, where $g_\lambda(\tilde{a}_{ij})$, $g_\lambda(\tilde{b}_{ij})$ are determined by Eqs. (6) and (7), and $\tilde{a}_{ij} = [\tilde{a}_{ij}^L, \tilde{a}_{ij}^U]$, $\tilde{b}_{ij} = [\tilde{b}_{ij}^L, \tilde{b}_{ij}^U]$, for all $i, j = 1, 2, \dots, n$.

As we can see, the compatibility degree $C_\lambda(\tilde{A}, \tilde{B})$ based on the LCOWGA operator reflects the total reciprocal difference between the uncertain multiplicative linguistic preference relations $\tilde{A}$ and $\tilde{B}$, in which all the corresponding elements are aggregated by the LCOWGA operator.

Specially, if $\tilde{A}$ and $\tilde{B}$ are ordinary multiplicative linguistic preference relations, then $C_\lambda(\tilde{A}, \tilde{B})$ reduces to the compatibility degree in Definition 7.

**Theorem 7** *Assume that $\tilde{A} = (\tilde{a}_{ij})_{n\times n} \in M_n$, $\tilde{B} = (\tilde{b}_{ij})_{n\times n} \in M_n$, then we have*

(1) $C_\lambda(\tilde{A}, \tilde{B}) \geq 0$.
(2) $C_\lambda(\tilde{A}, \tilde{A}) = 0$.
(3) $C_\lambda(\tilde{A}, \tilde{B}) = C_\lambda(\tilde{B}, \tilde{A})$.

Theorem 7 indicates that the compatibility degree of any two uncertain multiplicative linguistic preference relations, based on the LCOWGA operator, is nonnegative, reflexive and commutative.

**Lemma 5** *Assume that $\tilde{A} = (\tilde{a}_{ij})_{n\times n} \in M_n$, $\tilde{B} = (\tilde{b}_{ij})_{n\times n} \in M_n$, then $C_\lambda(\tilde{A}, \tilde{B}) = 0$ if and only if $g_\lambda(\tilde{a}_{ij}) = g_\lambda(\tilde{b}_{ij})$.*

**Theorem 8** *Assume that $\tilde{A} = (\tilde{a}_{ij})_{n\times n} \in M_n$, $\tilde{B} = (\tilde{b}_{ij})_{n\times n} \in M_n$ and $\tilde{D} = (\tilde{d}_{ij})_{n\times n} \in M_n$. If $C_\lambda(\tilde{A}, \tilde{B}) = 0$ and $C_\lambda(\tilde{B}, \tilde{D}) = 0$, then $C_\lambda(\tilde{A}, \tilde{D}) = 0$.*

Theorem 8 indicates that the compatibility degree of uncertain multiplicative linguistic preference relations based on the LCOWGA operator is transitive.

**Definition 13** Let $\tilde{A} = (\tilde{a}_{ij})_{n\times n} \in M_n$ and $\tilde{B} = (\tilde{b}_{ij})_{n\times n} \in M_n$ be two uncertain multiplicative linguistic preference relations. If $C_\lambda(\tilde{A}, \tilde{B}) = 0$ for any attitudinal character $\lambda$, then $\tilde{A}$ and $\tilde{B}$ are perfectly compatible.

It is obvious that we can obtain the following theorem from Definition 13.

**Theorem 9** *Let $\tilde{A} = (\tilde{a}_{ij})_{n\times n} \in M_n$ and $\tilde{B} = (\tilde{b}_{ij})_{n\times n} \in M_n$ be two uncertain multiplicative linguistic preference relations, then $\tilde{A}$ and $\tilde{B}$ are perfectly compatible if and only if $\tilde{a}_{ij} = \tilde{b}_{ij}$, $i, j = 1, 2, \dots, n$.*

**Definition 14** If $\tilde{A} = (\tilde{a}_{ij})_{n\times n} \in M_n$ and $\tilde{B} = (\tilde{b}_{ij})_{n\times n} \in M_n$, then

$$CI_\lambda(\tilde{A}, \tilde{B}) = \frac{1}{n^2} C_\lambda(\tilde{A}, \tilde{B}), \tag{20}$$

is called the compatibility index of the uncertain multiplicative linguistic preference relations $\tilde{A}$ and $\tilde{B}$ based on the LCOWGA operator.

By Definition 14, Theorems 7, 9 and Lemma 5, we can get the following conclusions.

**Theorem 10** *If $\tilde{A} = (\tilde{a}_{ij})_{n\times n} \in M_n$ and $\tilde{B} = (\tilde{b}_{ij})_{n\times n} \in M_n$, then*

(1) $CI_\lambda(\tilde{A}, \tilde{B}) \geq 0$.
(2) $CI_\lambda(\tilde{A}, \tilde{B}) = 0$ *if and only if $\tilde{A}$ and $\tilde{B}$ are perfectly compatible.*

**Definition 15** Let $\tilde{A} = (\tilde{a}_{ij})_{n\times n} \in M_n$ and $\tilde{B} = (\tilde{b}_{ij})_{n\times n} \in M_n$. If

$$CI_\lambda(\tilde{A}, \tilde{B}) \leq \alpha, \tag{21}$$

then $\tilde{A}$ and $\tilde{B}$ are of acceptable compatibility, where $\alpha$ is the threshold of acceptable compatibility.

Let $E=\{e_1,e_2,\dots,e_m\}$ be a finite set of experts and $\tilde{A}^{(k)}=(\tilde{a}_{ij}^{(k)})_{n\times n}\in M_n$ be the uncertain multiplicative linguistic preference relation provided by expert $e_k$, $k=1,2,\dots,m$. In [56], Xu defined the synthetic uncertain linguistic preference relation based on the induced uncertain linguistic preference relations and investigated some properties of the synthetic uncertain linguistic preference relations, which can be expressed as follows:

**Definition 16** Let $\tilde{A}^{(k)}=(\tilde{a}_{ij}^{(k)})_{n\times n}\in M_n$ for $k=1,2,\dots,m$. If

$$\tilde{a}_{ij}=\bigotimes_{k=1}^{m}(\tilde{a}_{ij}^{(k)})^{\omega_k}, \tag{22}$$

then the matrix $\tilde{A}=(\tilde{a}_{ij})_{n\times n}$ is called the synthetic uncertain linguistic preference relation of all experts, where $\Omega=(\omega_1,\omega_2,\dots,\omega_m)$ is the weighting vector of experts, which satisfies that $\omega_k\ge 0$ for all $j=1,2,\dots,m$ and $\sum_{k=1}^m\omega_k=1$.

**Theorem 11** *If $\tilde{A}^{(k)}=(\tilde{a}_{ij}^{(k)})_{n\times n}\in M_n$ for all $k=1,2,\dots,m$, then the synthetic uncertain linguistic preference relation $\tilde{A}=(\tilde{a}_{ij})_{n\times n}\in M_n$.*

**Theorem 12** *Let $\tilde{A}^{(k)}=(\tilde{a}_{ij}^{(k)})_{n\times n}\in M_n$ for $k=1,2,\dots,m$, and $\tilde{B}=(\tilde{b}_{ij})_{n\times n}\in M_n$. If $CI_\lambda(\tilde{A}^{(k)},\tilde{B})\le\alpha$, for all $k=1,2,\dots,m$, then*

$$CI_\lambda(\tilde{A},\tilde{B})\le\alpha, \tag{23}$$

*where $\tilde{A}=(\tilde{a}_{ij})_{n\times n}$ is the synthetic uncertain linguistic preference relation, $\tilde{a}_{ij}=[\tilde{a}_{ij}^L,\tilde{a}_{ij}^U]$, and $\alpha$ is the threshold of acceptable compatibility.*

*Proof* By Definition 16, it follows that

$$\tilde{a}_{ij}^L=\bigotimes_{k=1}^{m}(\tilde{a}_{ij}^{(k)L})^{\omega_k},\qquad \tilde{a}_{ij}^U=\bigotimes_{k=1}^{m}(\tilde{a}_{ij}^{(k)U})^{\omega_k},$$

and

$$\tilde{a}_{ji}^L=\bigotimes_{k=1}^{m}(\tilde{a}_{ji}^{(k)L})^{\omega_k},\qquad \tilde{a}_{ji}^U=\bigotimes_{k=1}^{m}(\tilde{a}_{ji}^{(k)U})^{\omega_k},$$

where $\Omega=(\omega_1,\omega_2,\dots,\omega_m)$ is the weighting vector of $m$ experts, which satisfies that $\omega_k\ge 0$ for all $j=1,2,\dots,m$ and $\sum_{k=1}^m\omega_k=1$.

By Lemma 1, we get $I(\tilde{a}_{ij}^U)=\prod_{k=1}^m(I(\tilde{a}_{ij}^{(k)U}))^{\omega_k}$, $I(\tilde{a}_{ij}^L)=\prod_{k=1}^m(I(\tilde{a}_{ij}^{(k)L}))^{\omega_k}$.

Since $CI_\lambda(\tilde{A}^{(k)},\tilde{B})\le\alpha$ for all $k$, then

$$CI_\lambda(\tilde{A}^{(k)},\tilde{B})=\frac{1}{2tn^2}\sum_{i=1}^n\sum_{j=1}^n\big(I\big(g_\lambda(\tilde{a}_{ij}^{(k)})\big)I\big(g_{1-\lambda}(\tilde{b}_{ji})\big) + I\big(g_\lambda(\tilde{b}_{ij})\big)I\big(g_{1-\lambda}(\tilde{a}_{ji}^{(k)})\big)-2\big)\le\alpha.$$

Thus, it can be obtained that

$$\begin{aligned}
&CI_\lambda(\tilde{A},\tilde{B})\\
&=\frac{1}{2tn^2}\sum_{i=1}^n\sum_{j=1}^n\big(I\big(g_\lambda(\tilde{a}_{ij})\big)I\big(g_{1-\lambda}(\tilde{b}_{ji})\big)\\
&\quad+I\big(g_\lambda(\tilde{b}_{ij})\big)I\big(g_{1-\lambda}(\tilde{a}_{ji})\big)-2\big)\\
&=\frac{1}{2tn^2}\sum_{i=1}^n\sum_{j=1}^n\big(\big(I\big(\tilde{a}_{ij}^U\big)\big)^\lambda\big(I\big(\tilde{a}_{ij}^L\big)\big)^{1-\lambda}\\
&\quad\times\big(I\big(\tilde{b}_{ji}^U\big)\big)^{1-\lambda}\big(I\big(\tilde{b}_{ji}^L\big)\big)^\lambda\\
&\quad+\big(I\big(\tilde{b}_{ij}^U\big)\big)^\lambda\big(I\big(\tilde{b}_{ij}^L\big)\big)^{1-\lambda}\times\big(I\big(\tilde{a}_{ji}^U\big)\big)^{1-\lambda}\big(I\big(\tilde{a}_{ji}^L\big)\big)^\lambda-2\big)\\
&=\frac{1}{2tn^2}\sum_{i=1}^n\sum_{j=1}^n\left(\left(\prod_{k=1}^m\big(I\big(\tilde{a}_{ij}^{(k)U}\big)\big)^{\omega_k}\right)^\lambda\right.\\
&\quad\times\left(\prod_{k=1}^m\big(I\big(\tilde{a}_{ij}^{(k)L}\big)\big)^{\omega_k}\right)^{1-\lambda}\times\big(I\big(\tilde{b}_{ji}^U\big)\big)^{1-\lambda}\big(I\big(\tilde{b}_{ji}^L\big)\big)^\lambda\\
&\quad+\big(I\big(\tilde{b}_{ij}^U\big)\big)^\lambda\big(I\big(\tilde{b}_{ij}^L\big)\big)^{1-\lambda}\times\left(\prod_{k=1}^m\big(I\big(\tilde{a}_{ji}^{(k)U}\big)\big)^{\omega_k}\right)^{1-\lambda}\\
&\quad\left.\times\left(\prod_{k=1}^m\big(I\big(\tilde{a}_{ji}^{(k)L}\big)\big)^{\omega_k}\right)^\lambda-2\right)\\
&=\frac{1}{2tn^2}\sum_{i=1}^n\sum_{j=1}^n\left(\prod_{k=1}^m\big(\big(I\big(\tilde{a}_{ij}^{(k)U}\big)\big)^\lambda\big(I\big(\tilde{a}_{ij}^{(k)L}\big)\big)^{1-\lambda}\big)^{\omega_k}\right.\\
&\quad\times\big(I\big(\tilde{b}_{ji}^U\big)\big)^{1-\lambda}\big(I\big(\tilde{b}_{ji}^L\big)\big)^\lambda\\
&\quad+\big(I\big(\tilde{b}_{ij}^U\big)\big)^\lambda\big(I\big(\tilde{b}_{ij}^L\big)\big)^{1-\lambda}\\
&\quad\left.\times\prod_{k=1}^m\big(\big(I\big(\tilde{a}_{ji}^{(k)U}\big)\big)^{1-\lambda}\big(I\big(\tilde{a}_{ji}^{(k)L}\big)\big)^\lambda\big)^{\omega_k}-2\right)\\
&\le\frac{1}{2tn^2}\sum_{i=1}^n\sum_{j=1}^n\left(\sum_{k=1}^m\omega_k\big(\big(I\big(\tilde{a}_{ij}^{(k)U}\big)\big)^\lambda\big(I\big(\tilde{a}_{ij}^{(k)L}\big)\big)^{1-\lambda}\big)\right.\\
&\quad\times\big(I\big(\tilde{b}_{ji}^U\big)\big)^{1-\lambda}\big(I\big(\tilde{b}_{ji}^L\big)\big)^\lambda\\
&\quad+\big(I\big(\tilde{b}_{ij}^U\big)\big)^\lambda\big(I\big(\tilde{b}_{ij}^L\big)\big)^{1-\lambda}\\
&\quad\left.\times\sum_{k=1}^m\omega_k\big(\big(I\big(\tilde{a}_{ji}^{(k)U}\big)\big)^{1-\lambda}\big(I\big(\tilde{a}_{ji}^{(k)L}\big)\big)^\lambda\big)-2\right)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=1}^{m}\omega_k\Bigg(\frac{1}{2tn^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\big((I(\tilde{a}_{ij}^{(k)U}))^{\lambda}(I(\tilde{a}_{ij}^{(k)L}))^{1-\lambda}\\
&\quad\times (I(\tilde{b}_{ji}^{U}))^{1-\lambda}(I(\tilde{b}_{ji}^{L}))^{\lambda}\\
&\quad+(I(\tilde{b}_{ij}^{U}))^{\lambda}(I(\tilde{b}_{ij}^{L}))^{1-\lambda}(I(\tilde{a}_{ji}^{(k)U}))^{1-\lambda}\\
&\quad\times (I(\tilde{a}_{ji}^{(k)L}))^{\lambda}-2\big)\Bigg)\\
&= \sum_{k=1}^{m}\omega_k\Bigg(\frac{1}{2tn^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\big(I(g_{\lambda}(\tilde{a}_{ij}^{(k)}))I(g_{1-\lambda}(\tilde{b}_{ji}))\\
&\quad+I(g_{\lambda}(\tilde{b}_{ij}))I(g_{1-\lambda}(\tilde{a}_{ji}^{(k)}))-2\big)\Bigg)\\
&= \sum_{k=1}^{m}\omega_k CI_{\lambda}(\tilde{A}^{(k)},\tilde{B})\le\sum_{k=1}^{m}\omega_k\alpha=\alpha. \qquad \square
\end{aligned}
$$

Theorem 12 indicates that the synthetic uncertain linguistic preference relation is of acceptable compatibility under the condition that the uncertain multiplicative linguistic preference relations given by experts are all of acceptable compatibility with the ideal uncertain multiplicative linguistic preference relation.

**Corollary 2** *Let $\tilde{A}^{(k)}=(\tilde{a}_{ij}^{(k)})_{n\times n}\in M_n$ for $k=1,2,\dots,m$, and $\tilde{B}=(\tilde{b}_{ij})_{n\times n}\in M_n$. If $CI_{\lambda}(\tilde{A}^{(k)},\tilde{B})=0$, for all $\lambda$, $\forall k=1,2,\dots,m$, then*

$$CI_{\lambda}(\tilde{A},\tilde{B})=0, \tag{24}$$

*where $\tilde{A}=(\tilde{a}_{ij})_{n\times n}$ is the synthetic uncertain linguistic preference relation and $\tilde{a}_{ij}=[\tilde{a}_{ij}^{L},\tilde{a}_{ij}^{U}]$.*

Corollary 2 indicates that if $\tilde{A}^{(k)}$ and $\tilde{B}$ are perfectly compatible for all $k=1,2,\dots,m$, then $\tilde{A}$ and $\tilde{B}$ are perfectly compatible.

As we can see, in GDM, less compatibility index of uncertain multiplicative linguistic preference relations provided by expert $e_k$, means more reliable information given by $e_k$. Therefore, the aggregation weight of $e_k$ may depend on the compatibility index of the uncertain multiplicative linguistic preference relations. Let $\tilde{A}^{(k)}=(\tilde{a}_{ij}^{(k)})_{n\times n}\in M_n$ be the uncertain multiplicative linguistic preference relation provided by expert $e_k$ $(k=1,2,\dots,m)$ and $\tilde{A}=(\tilde{a}_{ij})_{n\times n}$ be the synthetic uncertain linguistic preference relation of all experts. In order to determine the weights of experts, we can minimize the compatibility index of the synthetic uncertain linguistic preference relation $\tilde{A}$ and the ideal uncertain multiplicative linguistic preference relation $\tilde{B}$. Based on the proof of Theorem 12, the compatibility index of $\tilde{A}$ and $\tilde{B}$ can be rewritten as follows:

$$
\begin{aligned}
&CI_{\lambda}(\tilde{A},\tilde{B})\\
&= \frac{1}{2tn^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\Bigg(\bigg(\prod_{k=1}^{m}(I(\tilde{a}_{ij}^{(k)U}))^{\omega_k}\bigg)^{\lambda}\\
&\quad\times\bigg(\prod_{k=1}^{m}(I(\tilde{a}_{ij}^{(k)L}))^{\omega_k}\bigg)^{1-\lambda}\times(I(\tilde{b}_{ji}^{U}))^{1-\lambda}(I(\tilde{b}_{ji}^{L}))^{\lambda}\\
&\quad+(I(\tilde{b}_{ij}^{U}))^{\lambda}(I(\tilde{b}_{ij}^{L}))^{1-\lambda}\times\bigg(\prod_{k=1}^{m}(I(\tilde{a}_{ji}^{(k)U}))^{\omega_k}\bigg)^{1-\lambda}\\
&\quad\times\bigg(\prod_{k=1}^{m}(I(\tilde{a}_{ji}^{(k)L}))^{\omega_k}\bigg)^{\lambda}-2\Bigg)\\
&= \frac{1}{2tn^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\prod_{k_1=1}^{m}\prod_{k_2=1}^{m}\big(((I(\tilde{a}_{ij}^{(k_1)U}))^{\omega_{k_1}})^{\lambda}\\
&\quad\times((I(\tilde{a}_{ij}^{(k_2)L}))^{\omega_{k_2}})^{1-\lambda}\times(I(\tilde{b}_{ji}^{U}))^{1-\lambda}(I(\tilde{b}_{ji}^{L}))^{\lambda}\\
&\quad+(I(\tilde{b}_{ij}^{U}))^{\lambda}(I(\tilde{b}_{ij}^{L}))^{1-\lambda}\\
&\quad\times((I(\tilde{a}_{ji}^{(k_1)U}))^{\omega_{k_1}})^{1-\lambda}((I(\tilde{a}_{ji}^{(k_2)L}))^{\omega_{k_2}})^{\lambda}-2\big).
\end{aligned}
\tag{25}
$$

Let $\Omega=(\omega_1,\omega_2,\dots,\omega_m)^T$ be the experts' weighting vector. Then the compatibility index $CI_{\lambda}(\tilde{A},\tilde{B})$ can be considered as a function of $\Omega$. Denoting $f(\Omega)=CI_{\lambda}(\tilde{A},\tilde{B})$, we can construct the following optimal model to determine experts' weights:

$$
\begin{aligned}
&\text{Min } f(\Omega)=CI_{\lambda}(\tilde{A},\tilde{B})\\
&\text{s.t.}\quad\begin{cases}\sum_{k=1}^{m}\omega_k=1,\\ \omega_k\ge 0,\quad k=1,2,\dots,m.\end{cases}
\end{aligned}
\tag{26}
$$

Note that the model (26) is nonlinear and can be solved by using Matlab or LINGO software package.

## 5 The application of the compatibility index based on the LCOWGA operator to group decision making with uncertain multiplicative linguistic preference relations

In this section, a new approach based on the compatibility index to group decision making with uncertain multiplicative linguistic preference relations will be presented. The main advantage of this method is that it gives a completely objective data process of the decision problem because the weights of experts depend on the optimal model of minimization of compatibility index. We firstly define an expected preference relation by using Eqs. (6) and (7) as follows:

**Definition 17** Let $\tilde{A} = (\tilde{a}_{ij})_{n\times n} \in M_n$, where

$$\tilde{a}_{ij}^U \otimes \tilde{a}_{ji}^L = s_1, \qquad \tilde{a}_{ij}^L \otimes \tilde{a}_{ji}^U = s_1,$$
$$\tilde{a}_{ij}^U = \tilde{a}_{ij}^L = s_1, \quad \forall i, j = 1, 2, \ldots, n,$$

then we call $g_Q(\tilde{A}) = (g_Q(\tilde{a}_{ij}))_{n\times n}$ the expected multiplicative linguistic preference relation corresponding to $\tilde{A}$, where $g_Q(\tilde{a}_{ij})$ is the expected value of preference degree $\tilde{a}_{ij}$ of the alternative $x_i$ to $x_j$, obtained by the LCOWGA operator:

$$\begin{aligned} g_Q(\tilde{a}_{ij}) &= g_Q([\tilde{a}_{ij}^L, \tilde{a}_{ij}^U]) = (\tilde{a}_{ij}^U)^\lambda \otimes (\tilde{a}_{ij}^L)^{1-\lambda}, \\ g_Q(\tilde{a}_{ji}) &= \frac{1}{g_Q(\tilde{a}_{ij})}, \quad \text{for all } i \le j, \end{aligned} \tag{27}$$

where $\lambda \in [0, 1]$ is the attitudinal character of BUM function $Q$.

Then consider a GDM problem. Let $X = \{x_1, x_2, \ldots, x_n\}$ be a set of finite alternatives and $E = \{e_1, e_2, \ldots, e_m\}$ be a finite set of experts. Each expert provides his/her own decision matrix $\tilde{A}^{(k)} = (\tilde{a}_{ij}^{(k)})_{n\times n}$, which are uncertain multiplicative linguistic preference relation given by the expert $e_k \in E$. The process of new approach can be summarized as follows:

**Step 1**: Utilize the model (26) to determine the optimal weights of experts:

$$\omega^* = (\omega_1^*, \omega_2^*, \ldots, \omega_m^*)^T.$$

**Step 2**: Utilize Eq. (22) to obtain the synthetic uncertain linguistic preference relation $\tilde{A}^* = (\tilde{a}_{ij}^*)_{n\times n}$ based on the optimal weights of experts, where

$$\tilde{a}_{ij}^* = \bigotimes_{k=1}^{m} (\tilde{a}_{ij}^{(k)})^{\omega_k^*}. \tag{28}$$

**Step 3**: Utilize Eq. (27) to construct the expected multiplicative linguistic preference relation $g_Q(\tilde{A}^*) = (g_Q(\tilde{a}_{ij}^*))_{n\times n}$ based on the LCOWGA operator:

$$\begin{aligned} g_Q(\tilde{a}_{ij}^*) &= (\tilde{a}_{ij}^{*U})^\lambda \otimes (\tilde{a}_{ij}^{*L})^{1-\lambda}, \\ g_Q(\tilde{a}_{ji}^*) &= \frac{1}{g_Q(\tilde{a}_{ij}^*)}, \quad \text{for all } i \le j, \end{aligned} \tag{29}$$

where $\lambda \in [0, 1]$ is the attitudinal character of BUM function $Q$.

**Step 4**: Calculate the expected value $\tilde{a}_i^*$ of preference degree of the alternative $x_i$ to all the alternative by the following formula:

$$\tilde{a}_i^* = \left(\bigotimes_{j=1}^{n} g_Q(\tilde{a}_{ij}^*)\right)^{1/n}, \qquad i = 1, 2, \ldots, n. \tag{30}$$

**Step 5**: Rank the expected value $\tilde{a}_i^*$ $(i = 1, 2, \ldots, n)$ in descending order.

**Step 6**: Rank all the alternatives $x_i$ $(i = 1, 2, \ldots, n)$ and select the best one(s) in accordance with the expected value $\tilde{a}_i^*$ $(i = 1, 2, \ldots, n)$.

**Step 7**: End.

## 6 Illustrative example

In this section, we regard the use of the compatibility index of uncertain multiplicative linguistic preference relations in a GDM problem. Supplier selection is a very important strategic decision involving decisions balancing a number of conflicting criteria [34]. With the increase in outsourcing, offshore sourcing and various electronic businesses, supplier's service performance is becoming ever more complex in the global market. An international company established in Hefei wants to select potential partners for a collaborative project. In order to select an ideal supplier, the company formed a team of three experts $e_k$ $(k = 1, 2, 3)$ to evaluate five potential supplier partners $x_i$ $(i = 1, 2, 3, 4, 5)$. Three experts are invited to compare these five suppliers with respect to the main criterion service performance by using the multiplicative linguistic scale:

$$\begin{aligned} S = \{&s_{1/5} = EL, s_{1/4} = VL, s_{1/3} = L, s_{1/2} = SL, s_1 = M, \\ &s_2 = SH, s_3 = H, s_4 = VH, s_5 = EH\}. \end{aligned}$$

Note that $EL = \textit{Extremely low}$, $VL = \textit{Very low}$, $L = \textit{Low}$, $SL = \textit{Slightly low}$, $M = \textit{Medium}$, $SH = \textit{Slightly high}$, $H = \textit{High}$, $VH = \textit{Very high}$, $EH = \textit{Extremely high}$.

Experts constructed the uncertain multiplicative linguistic preference relations $\tilde{A}^{(k)}$ $(k = 1, 2, 3)$, respectively, which are listed as follows:

$$\tilde{A}^{(1)} = \begin{pmatrix} [s_1, s_1] & [s_2, s_3] & [s_{1/4}, s_{1/3}] & [s_4, s_5] & [s_{1/2}, s_1] \\ [s_{1/3}, s_{1/2}] & [s_1, s_1] & [s_4, s_5] & [s_2, s_3] & [s_{1/3}, s_{1/2}] \\ [s_3, s_4] & [s_{1/5}, s_{1/4}] & [s_1, s_1] & [s_3, s_4] & [s_{1/5}, s_{1/4}] \\ [s_{1/5}, s_{1/4}] & [s_{1/3}, s_{1/2}] & [s_{1/4}, s_{1/3}] & [s_1, s_1] & [s_5, s_6] \\ [s_1, s_2] & [s_2, s_3] & [s_4, s_5] & [s_{1/6}, s_{1/5}] & [s_1, s_1] \end{pmatrix};$$

$$\tilde{A}^{(2)} = \begin{pmatrix} [s_1, s_1] & [s_3, s_4] & [s_{1/3}, s_{1/2}] & [s_3, s_4] & [s_{1/4}, s_{1/3}] \\ [s_{1/4}, s_{1/3}] & [s_1, s_1] & [s_5, s_6] & [s_3, s_4] & [s_{1/3}, s_{1/2}] \\ [s_2, s_3] & [s_{1/6}, s_{1/5}] & [s_1, s_1] & [s_4, s_5] & [s_{1/4}, s_{1/3}] \\ [s_{1/4}, s_{1/3}] & [s_{1/4}, s_{1/3}] & [s_{1/5}, s_{1/4}] & [s_1, s_1] & [s_5, s_6] \\ [s_3, s_4] & [s_2, s_3] & [s_3, s_4] & [s_{1/6}, s_{1/5}] & [s_1, s_1] \end{pmatrix};$$

$$\tilde{A}^{(3)} = \begin{pmatrix} [s_1, s_1] & [s_4, s_5] & [s_{1/4}, s_{1/3}] & [s_2, s_3] & [s_{1/4}, s_{1/3}] \\ [s_{1/5}, s_{1/4}] & [s_1, s_1] & [s_4, s_5] & [s_3, s_4] & [s_{1/3}, s_{1/2}] \\ [s_3, s_4] & [s_{1/5}, s_{1/4}] & [s_1, s_1] & [s_4, s_5] & [s_{1/3}, s_{1/2}] \\ [s_{1/3}, s_{1/2}] & [s_{1/4}, s_{1/3}] & [s_{1/5}, s_{1/4}] & [s_1, s_1] & [s_5, s_6] \\ [s_3, s_4] & [s_2, s_3] & [s_2, s_3] & [s_{1/6}, s_{1/5}] & [s_1, s_1] \end{pmatrix}.$$

The president of the company provided the ideal uncertain multiplicative linguistic preference relations, which is listed as follows:

$$\tilde{B} = \begin{pmatrix} [s_1, s_1] & [s_3, s_5] & [s_{1/4}, s_{1/2}] & [s_3, s_5] & [s_{1/4}, s_{1/2}] \\ [s_{1/5}, s_{1/3}] & [s_1, s_1] & [s_4, s_6] & [s_2, s_4] & [s_{1/3}, s_1] \\ [s_2, s_4] & [s_{1/6}, s_{1/4}] & [s_1, s_1] & [s_3, s_5] & [s_{1/5}, s_{1/3}] \\ [s_{1/5}, s_{1/3}] & [s_{1/4}, s_{1/2}] & [s_{1/5}, s_{1/3}] & [s_1, s_1] & [s_4, s_6] \\ [s_2, s_4] & [s_1, s_3] & [s_3, s_5] & [s_{1/6}, s_{1/4}] & [s_1, s_1] \end{pmatrix}.$$

With this information, we can use the proposed decision making method to get the ranking of the suppliers. Note that in this case, we assume that $Q(y) = y^2$, then $\lambda = \int_0^1 Q(y)dy = 1/3$. The following steps are involved:

**Step 1**: Utilize the model (26) to determine the optimal weights of experts:

$$\omega_1^* = 0.2386, \qquad \omega_2^* = 0.6673, \qquad \omega_3^* = 0.0941.$$

**Step 2**: Utilize Eq. (27) to obtain the synthetic uncertain linguistic preference relation $\tilde{A}^* = (\tilde{a}_{ij}^*)_{n\times n}$ based on the optimal weights of experts, where

$$\tilde{A}^* = \begin{pmatrix} [s_1, s_1] & [s_{2.7982}, s_{3.8140}] & [s_{0.3029}, s_{0.4369}] & [s_{3.0928}, s_{4.1060}] & [s_{0.2950}, s_{0.4332}] \\ [s_{0.2622}, s_{0.3574}] & [s_1, s_1] & [s_{4.6422}, s_{5.6469}] & [s_{2.7234}, s_{3.7347}] & [s_{0.3333}, s_{0.5}] \\ [s_{2.2888}, s_{3.3013}] & [s_{0.1771}, s_{0.2154}] & [s_1, s_1] & [s_{3.7347}, s_{4.7408}] & [s_{0.2569}, s_{0.3233}] \\ [s_{0.2435}, s_{0.3233}] & [s_{0.2678}, s_{0.3672}] & [s_{0.2109}, s_{0.2678}] & [s_1, s_1] & [s_5, s_6] \\ [s_{2.3083}, s_{43.3904}] & [s_2, s_3] & [s_{3.0928}, s_{4.1060}] & [s_{0.1667}, s_{0.2}] & [s_1, s_1] \end{pmatrix}.$$

**Step 3**: Utilize Eq. (29) to construct the expected multiplicative linguistic preference relation $g_Q(\tilde{A}^*) = (g_Q(\tilde{a}_{ij}^*))_{n\times n}$ based on the LCOWGA operator:

$$g_Q(\tilde{A}^*) = \begin{pmatrix} s_1 & s_{3.1025} & s_{0.3422} & s_{3.3992} & s_{0.3353} \\ s_{0.3223} & s_1 & s_{4.9555} & s_{3.0257} & s_{0.3815} \\ s_{2.9221} & s_{0.2018} & s_1 & s_{4.0438} & s_{0.2773} \\ s_{0.2942} & s_{0.3305} & s_{0.2473} & s_1 & s_{5.3133} \\ s_{2.9827} & s_{2.6209} & s_{3.6057} & s_{0.1882} & s_1 \end{pmatrix}.$$

**Step 4**: Calculate the expected value $\tilde{a}_i^*$ of preference degree of the alternative $x_i$ to all the alternative by Eq. (30):

$$\tilde{a}_1^* = s_{1.0389}, \qquad \tilde{a}_2^* = s_{1.1302}, \qquad \tilde{a}_3^* = s_{0.9206},$$
$$\tilde{a}_4^* = s_{0.6626}, \qquad \tilde{a}_5^* = s_{1.3962}.$$

**Step 5**: Rank the expected value $\tilde{a}_i^*$ $(i = 1, 2, \ldots, 5)$ in descending order:

$$\tilde{a}_5^* > \tilde{a}_2^* > \tilde{a}_1^* > \tilde{a}_3^* > \tilde{a}_4^*.$$

**Step 6**: Rank all the alternatives $x_i$ $(i = 1, 2, \ldots, 5)$ in accordance with the expected value $\tilde{a}_i^*$ $(i = 1, 2, \ldots, 5)$:

$$x_5 \succ x_2 \succ x_1 \succ x_3 \succ x_4.$$

The best alternative is the supplier $x_5$.

Moreover, according to Eq. (20), we get the compatibility indexes of $\tilde{A}^{(k)}$ $(k = 1, 2, 3)$ and $\tilde{B}$:

$$CI_\lambda\left(\tilde{A}^{(1)}, \tilde{B}\right) = \frac{1}{2\times 5\times 5^2}\sum_{i=1}^{5}\sum_{j=1}^{5}\left(I\left(g_\lambda\left(\tilde{a}_{ij}^{(1)}\right)\right)I\left(g_{1-\lambda}(\tilde{b}_{ji})\right)\right.$$

$$+ I\big(g_\lambda(\tilde{b}_{ij})\big) I\big(g_{1-\lambda}(\tilde{a}_{ji}^{(1)})\big) - 2\big)$$
$$= 0.008847,$$

$$CI_\lambda\big(\tilde{A}^{(2)}, \tilde{B}\big)$$
$$= \frac{1}{2 \times 5 \times 5^2} \sum_{i=1}^{5} \sum_{j=1}^{5} \big(I\big(g_\lambda(\tilde{a}_{ij}^{(2)})\big) I\big(g_{1-\lambda}(\tilde{b}_{ji})\big)$$
$$+ I\big(g_\lambda(\tilde{b}_{ij})\big) I\big(g_{1-\lambda}(\tilde{a}_{ji}^{(2)})\big) - 2\big)$$
$$= 0.004749,$$

$$CI_\lambda\big(\tilde{A}^{(3)}, \tilde{B}\big)$$
$$= \frac{1}{2 \times 5 \times 5^2} \sum_{i=1}^{5} \sum_{j=1}^{5} \big(I\big(g_\lambda(\tilde{a}_{ij}^{(3)})\big) I\big(g_{1-\lambda}(\tilde{b}_{ji})\big)$$
$$+ I\big(g_\lambda(\tilde{b}_{ij})\big) I\big(g_{1-\lambda}(\tilde{a}_{ji}^{(3)})\big) - 2\big)$$
$$= 0.006160.$$

Letting $\alpha = 0.2$, then $CI_\lambda(A^{(k)}, B) \leq \alpha$, $k = 1, 2, 3$.

As we can see, the uncertain multiplicative linguistic preference relations $\tilde{A}^{(k)}$ and $\tilde{B}$ are of acceptable compatibility.

By Eq. (20), we also get

$$CI_\lambda\big(\tilde{A}^{*}, \tilde{B}\big)$$
$$= \frac{1}{2 \times 5 \times 5^2} \sum_{i=1}^{5} \sum_{j=1}^{5} \big(I\big(g_\lambda(\tilde{a}_{ij}^{*})\big) I\big(g_{1-\lambda}(\tilde{b}_{ji})\big)$$
$$+ I\big(g_\lambda(\tilde{b}_{ij})\big) I\big(g_{1-\lambda}(\tilde{a}_{ji}^{*})\big) - 2\big)$$
$$= 0.002103.$$

It can be seen easily that $CI_\lambda(\tilde{A}^*, \tilde{B}) < \alpha$, which means that $\tilde{A}^*$ and $\tilde{B}$ are of acceptable compatibility. And $CI_\lambda(\tilde{A}^*, B) < CI_\lambda(A^{(k)}, B)$ for $k = 1, 2, 3$, which means that the synthetic uncertain multiplicative linguistic preference relation $\tilde{A}^*$ based on the optimal weights of experts is more effective than $A^{(k)}$ for $k = 1, 2, 3$.

Furthermore, in order to analyze how the different weights of experts have affection for the compatibility index, in this example, we consider the equal experts' weights, i.e., $\omega_i = 1/3$ for $i = 1, 2, 3$. Then the synthetic uncertain multiplicative preference relation $\hat{A} = (\hat{a}_{ij})_{5\times 5}$ is calculated as follows:

$$\hat{A} = \begin{pmatrix} [s_1, s_1] & [s_{2.8845}, s_{3.9149}] & [s_{0.2752}, s_{0.3815}] & [s_{2.8845}, s_{3.9149}] & [s_{0.3150}, s_{0.4807}] \\ [s_{0.2554}, s_{0.3467}] & [s_1, s_1] & [s_{4.3089}, s_{5.3133}] & [s_{2.6207}, s_{3.6342}] & [s_{0.3333}, s_{0.5}] \\ [s_{2.6207}, s_{3.6342}] & [s_{0.1882}, s_{0.2321}] & [s_1, s_1] & [s_{3.6342}, s_{4.6416}] & [s_{0.2752}, s_{0.3467}] \\ [s_{0.2554}, s_{0.3467}] & [s_{0.2752}, s_{0.3815}] & [s_{0.2154}, s_{0.2752}] & [s_1, s_1] & [s_5, s_6] \\ [s_{2.0801}, s_{3.1748}] & [s_2, s_3] & [s_{2.8845}, s_{3.9149}] & [s_{0.1667}, s_{0.2}] & [s_1, s_1] \end{pmatrix}.$$

Then,

$$CI_\lambda(\hat{A}, \tilde{B})$$
$$= \frac{1}{2 \times 5 \times 5^2} \sum_{i=1}^{5} \sum_{j=1}^{5} \big(I\big(g_\lambda(\hat{a}_{ij})\big) I\big(g_{1-\lambda}(\tilde{b}_{ji})\big)$$
$$+ I\big(g_\lambda(\tilde{b}_{ij})\big) I\big(g_{1-\lambda}(\hat{a}_{ji})\big) - 2\big)$$
$$= 0.002388.$$

Thus, we have

$$CI_\lambda\big(\tilde{A}^*, B\big) < CI_\lambda(\hat{A}, B),$$

which shows that the synthetic uncertain linguistic preference relation corresponding to the optimal experts' weights based on the minimization of compatibility index is superior to that of equal experts' weights.

It is possible to analyze how the different attitudinal character $\lambda$ plays a role in the aggregation results, in this case, we consider different value of $\lambda$: $0, 0.1, \ldots, 0.9, 1$, which are provided by the experts. The results of experts' weights by Eq. (26) are shown in Table 1 and the results of $\tilde{a}_i^*$ $(i = 1, 2, 3, 4, 5)$ are shown in Table 2.

It is observed from Table 1 that $\omega_1$ and $\omega_3$ first decrease and then increase as $\lambda$ increases, while $\omega_2$ first increases and then decreases as $\lambda$ increases. From Table 1 we also can see that the experts' weights are symmetric with respect to the attitudinal character, which are caused by Eq. (8). Moreover, from Table 2 we can see that $\tilde{a}_1^*$ and $\tilde{a}_2^*$ increase as $\lambda$ increases while $\tilde{a}_3^*$, $\tilde{a}_4^*$ and $\tilde{a}_5^*$ decrease as $\lambda$ increases.

We can establish an ordering of the suppliers for each value of $\lambda$. The results are shown in Table 3. Note that "$\succ$" means "preferred to".

As we can see, depending on the particular cases of the attitudinal character $\lambda$ used, the ordering of the companies is different, thus leading to different decisions. However, it seems that $x_5$ is the best choice when $\lambda \leq 0.8$, and $x_1$ sometimes is also the best one.

Another interesting issue is to determine the compatibility indexes with different $\lambda$. The results are shown in Table 4.

**Table 1** Experts' weights with different attitudinal character λ

| $\omega_i$ | λ | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | 0 | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1 |
| $\omega_1$ | 0.2572 | 0.2499 | 0.2440 | 0.2396 | 0.2370 | 0.2361 | 0.2370 | 0.2396 | 0.2440 | 0.2499 | 0.2572 |
| $\omega_2$ | 0.6398 | 0.6506 | 0.6593 | 0.6657 | 0.6696 | 0.6709 | 0.6696 | 0.6657 | 0.6593 | 0.6506 | 0.6398 |
| $\omega_3$ | 0.1030 | 0.0995 | 0.0967 | 0.0946 | 0.0934 | 0.0930 | 0.0934 | 0.0946 | 0.0967 | 0.0995 | 0.1030 |

**Table 2** Aggregation results with different attitudinal character λ

| | λ | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | 0 | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1 |
| $\tilde{a}_1^*$ | 0.9504 | 0.9760 | 1.0025 | 1.0296 | 1.0576 | 1.0863 | 1.1159 | 1.1465 | 1.1781 | 1.2134 | 1.2446 |
| $\tilde{a}_2^*$ | 1.0836 | 1.0975 | 1.1115 | 1.1255 | 1.1395 | 1.1535 | 1.1676 | 1.1816 | 1.1956 | 1.2108 | 1.2238 |
| $\tilde{a}_3^*$ | 0.9284 | 0.9259 | 0.9235 | 0.9213 | 0.9193 | 0.9175 | 0.9159 | 0.9144 | 0.9132 | 0.9120 | 0.9111 |
| $\tilde{a}_4^*$ | 0.6937 | 0.6841 | 0.6747 | 0.6657 | 0.6568 | 0.6482 | 0.6398 | 0.6316 | 0.6236 | 0.6152 | 0.6082 |
| $\tilde{a}_5^*$ | 1.5077 | 1.4739 | 1.4403 | 1.4071 | 1.3744 | 1.3419 | 1.3098 | 1.2781 | 1.2467 | 1.2131 | 1.1849 |

**Table 3** Ordering of the bidders

| λ | Ordering | λ | Ordering |
|---|---|---|---|
| 0 | $x_5 \succ x_2 \succ x_1 \succ x_3 \succ x_4$ | 0.6 | $x_5 \succ x_2 \succ x_1 \succ x_3 \succ x_4$ |
| 0.1 | $x_5 \succ x_2 \succ x_1 \succ x_3 \succ x_4$ | 0.7 | $x_5 \succ x_2 \succ x_1 \succ x_3 \succ x_4$ |
| 0.2 | $x_5 \succ x_2 \succ x_1 \succ x_3 \succ x_4$ | 0.8 | $x_5 \succ x_2 \succ x_1 \succ x_3 \succ x_4$ |
| 0.3 | $x_5 \succ x_2 \succ x_1 \succ x_3 \succ x_4$ | 0.9 | $x_1 \succ x_5 \succ x_2 \succ x_3 \succ x_4$ |
| 0.4 | $x_5 \succ x_2 \succ x_1 \succ x_3 \succ x_4$ | 1 | $x_1 \succ x_2 \succ x_5 \succ x_3 \succ x_4$ |
| 0.5 | $x_5 \succ x_2 \succ x_1 \succ x_3 \succ x_4$ | 1/3 | $x_5 \succ x_2 \succ x_1 \succ x_3 \succ x_4$ |

It can be seen from Table 4 that $CI_\lambda(\tilde{A}^*, B) < CI_\lambda(A^{(k)}, B)$ for all λ, $\forall k = 1, 2, 3$. We also can see that the compatibility indexes are symmetric with respect to the attitudinal character.

## 7 Concluding remarks

The compatibility degree of preference relations and the weights of experts play an important role in reaching a reasonable decision result for GDM. In this paper, we have presented the compatibility degree and the compatibility index of the multiplicative linguistic preference relations and some desired properties have been investigated. Furthermore, based on the LCOWGA operator, the compatibility degree and compatibility index of uncertain multiplicative linguistic preference relations has been defined. The advantage of the compatibility index based on the LCOWGA operator is that it can be used to deal with the uncertain multiplicative linguistic preference relations with more flexibility due to the fact that the decision maker can choose a different value of the parameter λ according to oneself.

Then, some desirable properties have been studied including nonegativity, reflexivity, commutativity and transitivity. Especially, we have proved that the synthetic uncertain linguistic preference relation $\tilde{A}$ and the ideal uncertain multiplicative linguistic preference relation $\tilde{B}$ are of acceptable compatibility under the condition that the uncertain multiplicative linguistic preference relation of the $k$th expert $\tilde{A}^{(k)}$ and $\tilde{B}$ are of acceptable compatibility for all $k = 1, 2, \ldots, m$, which is the scientific basis of using the uncertain multiplicative linguistic preference relations in the GDM. Moreover, the concepts of perfect compatibility and the reciprocal matrix of uncertain multiplicative linguistic preference relation have been developed and their properties have been investigated.

In order to determine the experts' weights, we further have proposed the optimal model by minimizing the compatibility index. The characteristic of this method is that it can improve the group consistency by minimizing the compatibility of the group, which is an example its superiority. At last, an illustrative example has shown the feasibility and effectiveness of the new approach.

In the future, we expect to develop the compatibility and its properties of other preference relations. For example, the compatibility of the intuitionistic fuzzy preference relations and the interval-valued intuitionistic fuzzy preference relations. We will also consider other decision making problems [3, 17, 38] and other applications [2].

**Table 4** Compatibility indices with different attitudinal character λ

| $\lambda$ | 0 | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $CI_{\lambda}(\tilde{A}^{*}, \tilde{B})$ | 0.004217 | 0.003360 | 0.002694 | 0.002219 | 0.001935 | 0.001840 | 0.001935 | 0.002219 | 0.002694 | 0.003360 | 0.004217 |

**Acknowledgements** The work was supported by National Natural Science Foundation of China (No. 71071002), Higher School Specialized Research Fund for the Doctoral Program (No. 20123401110001), The Scientific Research Foundation of the Returned Overseas Chinese Scholars, Anhui Provincial Natural Science Foundation (No. 1308085QG127), Provincial Natural Science Research Project of Anhui Colleges (No. KJ2012A026), Humanity and Social Science Youth foundation of Ministry of Education (No. 13YJC630092), Humanities and social science Research Project of Department of Education of Anhui Province (No. SK2013B041). The authors are very grateful to the anonymous reviewers and the editor for their insightful and constructive comments and suggestions that have led to an improved version of this paper.

## References

1. Alonso S, Cabrerizo FJ, Chiclana F, Herrera F, Herrera-Viedma E (2009) Group decision making with incomplete fuzzy linguistic preference relations. Int J Intell Syst 24:201–222
2. Bahrammirzaee A, Ghatari AR, Ahmada P, Madani K (2011) Hybrid credit ranking intelligent system using expert system and artificial neural networks. Appl Intell 34:28–46
3. Bahrammirzaee A, Chohra A, Madani K (2013) An adaptive approach for decision making tactics in automated negotiation. Appl Intell. doi:10.1017/s10489-013-0434-8
4. Calvo T, Mayor G, Mesiar R (2002) Aggregation operators: new trends and applications. Physica-Verlag, New York
5. Chen SY, Chen SM (2005) A prioritized information fusion method for handling fuzzy decision-making problems. Appl Intell 22:219–232
6. Chen HY, Chen C (2009) Research on compatibility and consistency of combination judgment matrices based on I-IOWG operators. Syst Eng Electron 31:2137–2140
7. Chen HY, Zhao JB (2004) Research on compatibility of fuzzy judgement matrices. Oper Res Manag Sci 13:44–47
8. Chen HY, Zhou LG (2011) An approach to group decision making with interval fuzzy preference relations based on induced generalized continuous ordered weighted averaging operator. Expert Syst Appl 38:13432–13440
9. Chen HY, Zhou LG (2012) A relative entropy approach to group decision making with interval reciprocal relations based on COWA operator. Group Decis Negot 21:585–599
10. Chen HY, Liu JP, Wang H (2008) A class of continuous ordered weighted harmonic (C-OWHA) averaging operators for interval argument and its applications. Syst Eng-Theory Pract 28:86–92
11. Chen YH, Wang TC, Wu CY (2011) Multi-criteria decision making with fuzzy linguistic preference relations. Appl Math Model 35:1322–1330
12. Chen HY, Zhou LG, Han B (2011) On compatibility of uncertain additive linguistic preference relations and its application in the group decision making. Knowl-Based Syst 24:816–823
13. Chen SM, Lee LW, Liu HC, Yang SW (2012) Multiattribute decision making based on interval-valued intuitionistic fuzzy values. Expert Syst Appl 39:10343–10351
14. Dong YC, Xu YF, Li HY (2008) On consistency measures of linguistic preference relations. Eur J Oper Res 189:430–444
15. Dong YC, Xu YF, Yu S (2009) Linguistic multiperson decision making based on the use of multiple preference relations. Fuzzy Sets Syst 160:603–623
16. Gong ZW, Li LS, Zhou FX, Yao TX (2009) Goal programming approaches to obtain the priority vectors from the intuitionistic fuzzy preference relations. Comput Ind Eng 57:1187–1193
17. Granmo O, Glimsdal S (2013) Accelerated Bayesian learning for decentralized two-armed bandit based decision making with applications to the Goore game. Appl Intell 38:479–488
18. Herrera F, Herrera-Viedma E, Verdegay JL (1996) Direct approach processes in group decision making using linguistic OWA operators. Fuzzy Sets Syst 79:175–190
19. Herrera F, Herrera-Viedma E, Chiclana F (2001) Multiperson decision-making based on multiplicative preference relations. Eur J Oper Res 129:372–385
20. Herrera-Viedma E, Herrera F, Chiclana F, Luque M (2004) Some issues on consistency of fuzzy preference relations. Eur J Oper Res 154:98–109
21. Li DF (2011) The GOWA operator based approach to multiattribute decision making using intuitionistic fuzzy sets. Math Comput Model 53:1182–1196
22. Liu PD (2011) A weighted aggregation operators multi-attribute group decision-making method based on interval-valued trapezoidal fuzzy numbers. Expert Syst Appl 38:1053–1060
23. Merigó JM (2008) New extensions to the OWA operator and its application in business decision making. PhD thesis, Department of Business Administration, University of Barcelona
24. Merigó JM (2011) A unified model between the weighted average and the induced OWA operator. Expert Syst Appl 38:11560–11572
25. Merigó JM, Casanovas M (2011) Induced aggregation operators in the Euclidean distance and its application in financial decision making. Expert Syst Appl 38:7603–7608
26. Merigó JM, Casanovas M (2011) Induced and uncertain heavy OWA operators. Comput Ind Eng 60:106–116
27. Merigó JM, Gil-Lafuente AM (2011) Fuzzy induced generalized aggregation operators and its application in multi-person decision making. Expert Syst Appl 38:9761–9772
28. Merigó JM, Gil-Lafuente AM (2011) Decision-making in sport management based on the OWA operator. Expert Syst Appl 38:10408–10413
29. Merigó JM, Gil-Lafuente AM, Zhou LG, Chen HY (2011) A generalization of the linguistic aggregation operators and its application in decision making. J Syst Eng Electron 22:1–5
30. Merigó JM, Gil-Lafuente AM, Zhou LG, Chen HY (2012) Induced and linguistic generalized aggregation operators and their application in linguistic group decision making. Group Decis Negot 21:531–549
31. Saaty TL (1980) The analytic hierarchy process. McGraw-Hill, New York
32. Saaty TL, Vargas LG (2007) Dispersion of group judgments. Math Comput Model 46:918–925
33. Sengupta A, Pal TK (2009) Fuzzy preference ordering of interval numbers in decision problems. Springer, Berlin
34. Tan CQ, Wu DD, Ma BJ (2011) Group decision making with linguistic preference relations with application to supplier selection. Expert Syst Appl 38:14382–14389
35. Wang TC, Chen YH (2010) Incomplete fuzzy linguistic preference relations under uncertain environments. Inf Fusion 11:201–207

36. Wang YM, Elhag TMS (2007) A goal programming method for obtaining interval weights from an interval comparison matrix. Eur J Oper Res 177:458–471
37. Wang YM, Yang JB, Xu DL (2005) A two-stage logarithmic goal programming method for generating weights from interval comparison matrices. Fuzzy Sets Syst 152:475–498
38. Wang Z, Wang HW, Qi C, Wang J (2013) A resource enhanced HTN planning approach for emergency decision-making. Appl Intell 38:226–238
39. Wei GW (2010) Some induced geometric aggregation operators with intuitionistic fuzzy information and their application to group decision making. Appl Soft Comput 10:423–431
40. Wei GW (2010) A method for multiple attribute group decision making based on the ET-WG and ET-OWG operators with 2-tuple linguistic information. Expert Syst Appl 37:7895–7900
41. Wei GW, Zhao XF (2012) Some induced correlated aggregating operators with intuitionistic fuzzy information and their application to multiple attribute group decision making. Expert Syst Appl 39:2026–2034
42. Wei GW, Zhao XF (2012) Some dependent aggregation operators with 2-tuple linguistic information and their application to multiple attribute group decision making. Expert Syst Appl 39:5881–5886
43. Wu J, Cao QW (2011) Some issues on properties of the extended IOWA operators in fuzzy group decision making. Expert Syst Appl 38:7059–7066
44. Wu J, Cao QW (2013) Same families of geometric aggregation operators with intuitionistic trapezoidal fuzzy numbers. Appl Math Model 37:318–327
45. Wu J, Li JC, Li H, Duan WQ (2009) The induced continuous ordered weighted geometric operators and their application in group decision making. Comput Ind Eng 58:1545–1552
46. Wu J, Cao QW, Zhang JL (2010) Some properties of the induced continuous ordered weighted geometric operators in group decision making. Comput Ind Eng 59:100–106
47. Wu J, Cao QW, Zhang JL (2011) An ILOWG operator based group decision making method and its application to evaluate the supplier criteria. Math Comput Model 54:19–34
48. Xia MM, Xu ZS, Zhu B (2012) Some issues on intuitionistic fuzzy aggregation operators based on Archimedean t-conorm and t-norm. Knowl-Based Syst 31:78–88
49. Xu ZS (2000) On consistency of the weighted geometric mean complex judgement matrix in AHP. Eur J Oper Res 170:683–687
50. Xu ZS (2004) On compatibility of interval fuzzy preference relations. Fuzzy Optim Decis Mak 3:217–225
51. Xu ZS (2004) EOWA and EOWG operators for aggregating linguistic labels based on linguistic preference relations. Int J Uncertain Fuzziness Knowl-Based Syst 12:91–810
52. Xu ZS (2004) Uncertain multiple attribute decision making: methods and applications. Tsinghua University Press, Beijing
53. Xu ZS (2004) A method based on linguistic aggregation operators for group decision making with linguistic preference relations. Inf Sci 166:19–30
54. Xu ZS (2004) Uncertain linguistic aggregation operators based approach to multiple attribute group decision making under uncertain linguistic environment. Inf Sci 168:171–184
55. Xu ZS (2005) Deviation measures of linguistic preference relations in group decision making. Omega 33:249–254
56. Xu ZS (2006) An approach based on the uncertain LOWG and the induced uncertain LOWG operators to group decision making with uncertain multiplicative linguistic preference relations. Decis Support Syst 41:488–499
57. Xu ZS (2006) A C-OWA operator based approach to decision making with interval fuzzy preference relation. Int J Intell Syst 21:1289–1298
58. Xu ZS (2006) Induced uncertain linguistic OWA operators applied to group decision making. Inf Fusion 7:231–238
59. Xu ZS (2006) A practical procedure for group decision making under incomplete multiplicative linguistic preference relations. Group Decis Negot 15:593–604
60. Xu ZS (2007) A survey of preference relations. Int J Gen Syst 36:179–203
61. Xu ZS (2007) Intuitionistic preference relations and their application in group decision making. Inf Sci 177:2363–2379
62. Xu ZS (2008) Intuitionistic fuzzy information: aggregation theory and applications. Science Press, Beijing
63. Xu ZS (2010) A method based on distance measure for interval-valued intuitionistic fuzzy group decision making. Inf Sci 180:181–190
64. Xu ZS (2011) Consistency of interval fuzzy preference relations in group decision making. Appl Soft Comput 11:3898–3909
65. Xu ZS, Chen J (2008) An overview of distance and similarity measures of intuitionistic fuzzy sets. Int J Uncertain Fuzziness Knowl-Based Syst 16:529–555
66. Xu YJ, Wang HM (2012) The induced generalized aggregation operators for intuitionistic fuzzy sets and their application in group decision making. Appl Soft Comput 12:1168–1179
67. Xu ZS, Xia MM (2011) Induced generalized intuitionistic fuzzy operators. Knowl-Based Syst 24:197–209
68. Yager RR (1988) On ordered weighted averaging aggregation operators in multi-criteria decision making. IEEE Trans Syst Man Cybern, Part B, Cybern 18:183–190
69. Yager RR (2003) Induced aggregation operators. Fuzzy Sets Syst 137:59–69
70. Yager RR (2004) OWA aggregation over a continuous interval argument with applications to decision making. IEEE Trans Syst Man Cybern, Part B, Cybern 34:1952–1963
71. Yager RR (2004) Generalized OWA aggregation operators. Fuzzy Optim Decis Mak 3:93–107
72. Yager RR, Filev DP (1999) Induced ordered weighted averaging operators. IEEE Trans Syst Man Cybern, Part B, Cybern 29:141–150
73. Yager RR, Kacprzyk J (1997) The ordered weighted averaging operators: theory and applications. Kluwer Academic, Norwell
74. Yager RR, Xu ZS (2006) The continuous ordered weighted geometric operator and its application to decision making. Fuzzy Sets Syst 157:1393–1402
75. Yager RR, Kacprzyk J, Beliakov G (2011) Recent developments in the ordered weighted averaging operators: theory and practice. Springer, Berlin
76. Yang W, Chen ZP (2012) The quasi-arithmetic intuitionistic fuzzy OWA operators. Knowl-Based Syst 27:219–233
77. Yu XH, Xu ZS (2013) Prioritized intuitionistic fuzzy aggregation operators. Inf Fusion 14:108–116
78. Yu DJ, Wu YY, Lu T (2012) Interval-valued intuitionistic fuzzy prioritized operators and their application in group decision making. Knowl-Based Syst 30:57–66
79. Yue ZL (2011) Deriving decision maker's weights based on distance measure for interval-valued intuitionistic fuzzy group decision making. Expert Syst Appl 38:11665–11670
80. Zeng SZ, Su WH (2011) Intuitionistic fuzzy ordered weighted distance operator. Knowl-Based Syst 24:1224–1232
81. Zhang HM, Xu ZS (2005) Uncertain linguistic information based COWA and COWG operators and their applications. J PLA Univ Sci Technol 6:604–608
82. Zhang QS, Jiang SY, Jia BG, Luo SH (2011) Some information measures for interval-valued intuitionistic fuzzy sets. Inf Sci 180:5130–5145
83. Zhao H, Xu ZS, Ni MF, Liu SS (2010) Generalized aggregation operators for intuitionistic fuzzy sets. Int J Intell Syst 25:1–30
84. Zhou LG, Chen HY (2010) Generalized ordered weighted logarithm aggregation operators and their applications to group decision making. Int J Intell Syst 25:683–707

85. Zhou LG, Chen HY (2011) Continuous generalized OWA operator and its application to decision making. Fuzzy Sets Syst 168:18–34
86. Zhou LG, Chen HY (2012) A generalization of the power aggregation operators for linguistic environment and its application in group decision making. Knowl-Based Syst 26:216–224
87. Zhou LG, Chen HY, Liu JP (2012) Generalized power aggregation operators and their applications in group decision making. Comput Ind Eng 62:989–999
88. Zhou LG, Chen HY, Merigó JM, Gil-Lafuente AM (2012) Uncertain generalized aggregation operators. Expert Syst Appl 39:1105–1117
89. Zhou LG, Chen HY, Liu JP (2013) Generalized multiple averaging operators and their applications to group decision making. Group Decis Negot 22:331–358



**Ligang Zhou** is an associate professor of School of Mathematical Sciences, Anhui University, Hefei, China. He received a Ph.D. degree in Operational Research from Anhui University in 2013. He has contributed over 40 journal articles to professional journals, such as Fuzzy Sets and Systems, Applied Mathematical Modelling, Applied Soft Computing, Group Decision and Negotiation, Expert Systems with Applications etc. He is also a paper reviewer of many international journals. His current research interests include information fusion, group decision making, aggregation operators and combined forecasting.

**Yingdong He** is a postgraduate of School of Mathematical Sciences, Anhui University, Hefei, China. He has contributed several journal articles to professional journals, such as Information Sciences, Fuzzy Systems and Mathematics etc. His current research interests include fuzzy mathematics, information fusion, group decision making, aggregation operators and combined forecasting.

**Huayou Chen** is a Professor of School of Mathematical Sciences, Anhui University, Hefei, China. He received a Ph.D. degree in Operational Research from University of Science Technology of China in 2002. He graduated from Nanjing University for 2 years postdoctoral research work in 2005. He has published a book: The Efficient Theory of Combined Forecasting and Applications (Science Press, Beijing, 2008) and has contributed over 120 journal articles to professional journals, such as Fuzzy Sets and Systems, Group Decision and Negotiation, Expert Systems with Applications, Knowledge-Based Systems etc. He is also a paper reviewer of many international professional journals. His current research interests include information fusion, multi-criteria decision making, aggregation operators and combined forecasting. For further information, see his webpage: http://www1.ahu.edu.cn/math/mathweb2/jiaoshou/chenghuayou.htm.

**Jinpei Liu** is a lecturer of School of Business, Anhui University, Hefei, China. He received a Ph.D. degree in Management Science and Engineering from Tianjin University in 2012. His current research interests include information fusion, group decision making, pattern recognition and combined forecasting.