



The Structure of Aisles and Co-aisles of t-Structures and Co-t-structures

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Abstract

Right triangulated categories can be thought of as triangulated categories whose shift functor is not an equivalence. We give intrinsic characterisations of when such categories are appearing as the (co-)aisle of a (co-)t-structure in an associated triangulated category. Along the way, we also give an interpretation of these structures in the language of extriangulated categories.

Keywords Aisle · Co-aisle · Right triangulated · Extriangulated · t-Structure · Co-t-structure

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1 Introduction

The notion of a t-structure in a triangulated category was introduced by [9] in the study of perverse sheaves. They are central to the study of stability conditions, in the sense of Bridgeland [16] and tilting theory [27] and are an area of interest in their own right [4, 25, 54]. Co-t-structures, introduced by Paukzstello [56] and Bondarko [15] (under the name of ‘weight structures’) are a more recent development and arise from dualising one of the axioms of a t-structure. They are, however, not quite a dual notion, for example, the *heart* of a t-structure is an abelian subcategory, whereas the co-heart of a co-t-structure is rarely so. Co-t-structures have been studied in the context of Auslander-Buchweitz approximation theory [30] and in (co-)silting theory [37, 42, 50], where t-structures also play a key role.

In more detail, a (co-)t-structure on a triangulated category \mathcal{T} consists of a pair of additive subcategories $(\mathcal{U}, \mathcal{V})$ called the ‘aisle’ and ‘co-aisle’ respectively satisfying three axioms. The first two of these axioms ask for $(\mathcal{U}, \mathcal{V})$ to be a *torsion pair* on \mathcal{T} and the third axiom requires that \mathcal{U} be closed under the (inverse) shift functor of \mathcal{T} (see Definition 4.1). We remark that there is not a consistent terminology in the literature here, for instance in some works, the term ‘torsion pair’ refers to a t-structure. There are some pleasant articles concerning the general torsion pairs on triangulated categories [19, 35, 69] though they have seemed attracted somewhat less interest than (co-)t-structures.

The corresponding notion for abelian categories is that of torsion pairs, which are extremely important objects [2, 17, 18]. Rump [59] and Bondal & Van den Bergh [3] discovered that every torsion and torsionfree class (the aisles and co-aisles of the abelian world) is a *quasi-abelian* category. Moreover, each quasi-abelian category appears as a tilting torsion(free) class in an associated abelian category. In this article, we give the triangulated analogues of these results for (co-)t-structures. Namely, we answer the question: What are precisely the categorical structures that appear as aisles and co-aisles of (co-)t-structures in triangulated categories?

We answer this question in detail for aisles of t-structures and co-aisles of co-t-structures, whence other co-aisles and aisles have a dual structure. As we mentioned above, the structures that we are interested in are closed under the shift of the ambient triangulated category, it is not hard to see that these subcategories have the structure of a *right triangulated category (with right semi-equivalence)* which we discuss next.

The data of a right triangulated category (or suspended category), first introduced in [41] consists of an additive category \mathcal{R} , an endofunctor $\Sigma : \mathcal{R} \rightarrow \mathcal{R}$ called ‘the shift of \mathcal{R} ’ and a class of right triangles of the form $A \rightarrow B \rightarrow C \rightarrow \Sigma A$ subject to essentially the same axioms as the triangles of a triangulated category (see Definition 2.16). Informally, a right triangulated category is a triangulated category whose shift functor is not necessarily an equivalence. Such categories (or their left-handed analogues) have been the subject of study in many articles [7, 10, 40, 43, 45, 47]. In [6], the class of ‘right triangulated categories with right semi-equivalence’ were introduced, these are the right triangulated categories whose shift functor is fully faithful and with image that is closed under extensions.

To every right triangulated category \mathcal{R} there is an associated triangulated category: the stabilisation $\mathcal{S}(\mathcal{R})$ (see Sect. 2.3.1 for details and construction). In the case where \mathcal{R} has a right semi-equivalence, $\mathcal{S}(\mathcal{R})$ can be thought of as the smallest triangulated category containing \mathcal{R} as a subcategory. Furthermore, we show in Lemma 4.4 that if \mathcal{R} is a (co-)aisle in a triangulated category then it is also a (co-)aisle in $\mathcal{S}(\mathcal{R})$. Thus we look to characterise when \mathcal{R} is a (co-)aisle in $\mathcal{S}(\mathcal{R})$.

We give an intrinsic characterisations of when a right triangulated category with right semi-equivalence, \mathcal{R} , appears as the aisle of a t-structure in terms of internal torsion pairs of \mathcal{R} and homological properties.

Theorem A (Theorem 4.11.) *Let \mathcal{R} be a right triangulated category with right semi-equivalence. Then the following are equivalent*

- (a) \mathcal{R} is the aisle of a t-structure $(\mathcal{R}, \mathcal{V})$ in $\mathcal{S}(\mathcal{R})$;
- (b) There is a torsion pair $(\Sigma\mathcal{R}, \mathcal{F})$ in \mathcal{R} ;
- (c) $\mathcal{S}(\mathcal{R})$ is equivalent to the co-stabilisation of \mathcal{R} . That is, $\mathcal{S}(\mathcal{R})$ has the following universal property: Every right triangle functor $G : \mathcal{T} \rightarrow \mathcal{R}$ with \mathcal{T} being a triangulated category factors uniquely through $\mathcal{S}(\mathcal{R})$ (see Sect. 2.3.1).

We note that there are related works [39, 44, 64] that give various characterisations of aisles and that our approach differs in the sense that we look to give characterisations intrinsic to the right triangulated category, that is, the aisle, rather than properties of the aisle related to the ambient triangulated category. We also present our characterisations of co-aisles of co-t-structures.

Theorem B (Theorem 4.6.) *Let \mathcal{R} be a right triangulated category with right semi-equivalence. Then the following are equivalent*

- (a) \mathcal{R} is the co-aisle of a co-t-structure $(\mathcal{U}, \mathcal{R})$ in $\mathcal{S}(\mathcal{R})$;
- (b) \mathcal{R} has enough projectives;
- (c) There is a torsion pair $(\text{Proj}\mathcal{R}, \Sigma\mathcal{R})$ in \mathcal{R} .

Moreover, suppose that for all $A, B \in \mathcal{R}$ we have that $\mathcal{R}(A, \Sigma^i B) = 0$ for $i \gg 0$, that is, \mathcal{R} is bounded. Then the above conditions are also equivalent to

- (d) $\text{Proj}\mathcal{R}$ is a silting subcategory of $\mathcal{S}(\mathcal{R})$.

Part (d) of the above result adds to the interpretations of silting subcategories in a triangulated category, which are surveyed in [46]. We also note that this result can be thought of as a generalisation of [7, Theorem 4.2] and that the work of [30] is useful in the proof. We also obtain the following as a direct consequence.

Corollary A (Corollary 4.7.) *There is a correspondence between silting subcategories of triangulated categories and bounded right triangulated categories with right semi-equivalence that have enough projectives.*

An immediate application is that the torsion pairs in Theorems A and B allow us to describe intermediate (co-)t-structures (Proposition 4.12).

In order to formally discuss the projective objects, like in Theorem B, and other homological properties of a right triangulated category, we use the language of extriangulated categories. These categories were introduced by Nakaoka and Palu in [52] as a simultaneous generalisation of exact categories and triangulated categories. The framework of extriangulated categories allows one to axiomatise properties of categories that have structural similarities to exact and/or triangulated categories but fall into neither class, for example extension closed subcategories of triangulated categories. The formalism of extriangulated categories then allows homological algebra to be applied to such categories which has been done successfully by many authors, for example [21, 53, 68]. For us, we are interested in the fact that right triangulated categories with right semi-equivalence enjoy homological properties close to those of triangulated categories. We formalise this similarity by showing the following.

Theorem C (Corollary 3.13.) *A right triangulated category has the natural structure of an extriangulated category if and only if the shift functor is a right semi-equivalence.*

Moreover, we are able to characterise which extriangulated categories have a natural right triangulated structure.

Theorem D (Theorem 3.12.) *The extriangles of an extriangulated category $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ induce a right triangulated structure with right semi-equivalence on \mathcal{C} if and only if the morphism $X \rightarrow 0$ is an \mathbb{E} -inflation for all objects $X \in \mathcal{C}$. In other words, the zero object of \mathcal{C} is the only \mathbb{E} -injective object and there are enough \mathbb{E} -injectives.*

To prove this, we use an extriangulated generalisation of the constructions of a right triangulated quotient category from a contravariantly finite subcategory (of an additive category) due to [14] and [6] (see Proposition 3.6). We also use ideas from relative homological algebra to characterise which extriangulated structures give rise to right triangulated quotient categories (with right semi-equivalence) (see Proposition 3.7).

The construction of these right triangulated structures can be thought of as a ‘one-sided’ analogue of the triangulated structure of the stable category of a Frobenius exact (or even extriangulated) category [26, 28]. See [13, 36] for similar constructions. Unsurprisingly, given the terminology, this one-sidedness also appears in Theorem D as triangulated categories are precisely the extriangulated categories satisfying both this condition and its dual [52, Proposition 3.22].

We also show that in the case of Frobenius extriangulated categories, aisles of t-structures in the triangulated stable category may be constructed as (shifts of) right triangulated quotients.

Theorem E (Theorem 5.1.) *Let $\mathcal{C} = (\mathcal{C}, \mathbb{E}, \mathfrak{s})$ be a Frobenius extriangulated category and $(\mathcal{U}, \mathcal{V})$ be a t-structure in the triangulated stable category $\underline{\mathcal{C}}$ and set $\mathcal{D} := \Sigma^{-1}\mathcal{V}$. Then there is an equivalence of right triangulated categories $\mathcal{U} \cong \Sigma_{\mathcal{D}}\underline{\mathcal{C}}_{\mathcal{D}}$, where $\underline{\mathcal{C}}_{\mathcal{D}}$ denotes the stable category of \mathcal{C} by the ideal of morphisms factoring through objects of \mathcal{D} and $\Sigma_{\mathcal{D}}$ is its shift functor.*

In [61, Proposition 3.9], it was shown that t-structures in an algebraic triangulated category correspond bijectively to certain complete cotorsion pairs in the associated Frobenius exact category. Additionally, it was observed in [51, Proposition 2.6] that t-structures in a triangulated category are precisely cotorsion pairs satisfying a shift closure property. Thus these results and Theorem E complement each other since, via these bijections, a t-structure $(\mathcal{U}, \mathcal{V})$ corresponds to a cotorsion pair $(\mathcal{U}, \Sigma^{-1}\mathcal{V})$ (Remark 5.2).

The article is organised as follows. In Sect. 2 we recall the necessary background material on relative homological algebra, extriangulated categories and right triangulated categories. In Sect. 3 we show how one may use relative homological algebra to construct new extriangulated structures and characterise the projectives and injectives of these new structures. We then investigate how such extriangulated structures induce a right triangulated structure on a quotient category and use this to prove Theorems C and D. We begin Sect. 4 with a discussion of torsion pairs in a right triangulated category with right semi-equivalence. We then go on to prove the characterisations of right triangulated categories as (co-)aisles of (co-)t-structures of Theorems A and B and use these to describe related classes of (co-)t-structures. We end in Sect. 5 by proving Theorem E using the right triangulated quotient categories from Sect. 3.

Although the theory of extriangulated categories and relative homological algebra is not necessary for Theorem A and Theorem B(a,c), we have chosen this order as it allows us to discuss the role of extriangulated categories throughout.

We note that dual results for left triangulated categories (with left semi-equivalence) hold but remain unstated. Categories are assumed to be additive and idempotent complete. When we say an additive subcategory, we mean a subcategory that is closed under isomorphisms, direct sums and direct summands.

2 Background

2.1 Relative Homological Algebra

We briefly recall some basic definitions from relative homological algebra introduced in [32] and state some useful properties. Let \mathcal{C} be an additive category and let $\text{Mor}_{\mathcal{C}}$ denote the category of morphisms in \mathcal{C} .

Definition 2.1 Let $\mathcal{D} \subset \text{Ob}(\mathcal{C})$ be a class of objects. A morphism $f : A \rightarrow B$ in \mathcal{C} is \mathcal{D} -*monic* if all morphisms $A \rightarrow D$ with $D \in \mathcal{D}$ factor through f

$$\begin{array}{ccc}
 A & \xrightarrow{f} & B \\
 \forall \downarrow & \swarrow \text{dashed} & \exists \\
 \mathcal{D} \ni D & &
 \end{array}$$

or, equivalently, $\mathcal{C}(f, -)|_{\mathcal{D}} : \mathcal{C}(B, -)|_{\mathcal{D}} \rightarrow \mathcal{C}(A, -)|_{\mathcal{D}}$ is an epimorphism. By $\text{Mon}(\mathcal{D})$ we denote the class of all \mathcal{D} -monic morphisms in \mathcal{C} . We define the notion of a \mathcal{D} -*epic* morphism and the class $\text{Epi}(\mathcal{D})$ dually.

Similarly, for a class of morphisms $\omega \subset \text{Ob}(\text{Mor}_{\mathcal{C}})$, an object $J \in \mathcal{C}$ is ω -*injective* if for every morphism $f : A \rightarrow B$ in ω , all morphisms $A \rightarrow J$ factor through f

$$\begin{array}{ccc}
 A & \xrightarrow{\forall f \in \omega} & B \\
 \forall \downarrow & \swarrow \text{dashed} & \exists \\
 J & &
 \end{array}$$

or, equivalently, $\mathcal{C}(f, J)$ is an epimorphism for all $f \in \omega$. By $\text{Inj}(\omega)$ we denote the class of all ω -injective objects in \mathcal{C} . We define the notion of an ω -*projective* object and the class $\text{Proj}(\omega)$ dually.

Example 2.2 (a) \mathcal{C} -monics are precisely sections.

(b) The injective objects of \mathcal{C} , $\text{Inj}\mathcal{C}$, are defined to be those objects that are injective relative to the class of monomorphisms of \mathcal{C} , \mathfrak{M} . That is $\text{Inj}\mathcal{C} = \text{Inj}(\mathfrak{M})$. \mathcal{C} is said to have *enough injective objects* if every object A in \mathcal{C} admits a monomorphism $g : A \hookrightarrow I$ where $I \in \text{Inj}\mathcal{C}$ called the *injective hull* of A . Suppose that \mathcal{C} has enough injective objects, then the class of $\text{Inj}\mathcal{C}$ -monic morphisms coincides with \mathfrak{M} , the class of monomorphisms in \mathcal{C} . Indeed, clearly $\mathfrak{M} \subseteq \text{Mon}(\text{Inj}\mathcal{C})$. For the converse, let $f : A \rightarrow B$ be $\text{Inj}\mathcal{C}$ -monic. Then the injective hull $g : A \hookrightarrow I$ of A factors as $g'f$ for some $g' : B \rightarrow I$. It follows that f is also a monomorphism.

(c) A left \mathcal{D} -approximation [8] of an object A is just a \mathcal{D} -monic morphism $A \rightarrow B$ such that $B \in \mathcal{D}$.

(d) In an extriangulated category (defined in the next section) $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ the \mathbb{E} -injectives are precisely $\text{Inj}(\{\mathbb{E}\text{-inflatons}\})$.

Definition 2.3 A commutative square

$$\begin{array}{ccc} A & \xrightarrow{x} & B \\ f \downarrow & & \downarrow f' \\ A' & \xrightarrow{x'} & B' \end{array}$$

in \mathcal{C} is a *weak pushout* if for all pairs of morphisms $g : A' \rightarrow C, y : B \rightarrow C$ such that $gf = yx$, there exists a (not necessarily unique) morphism $h : B' \rightarrow C$ such that $hf' = y$ and $hx' = g$:

$$\begin{array}{ccc} A & \xrightarrow{x} & B \\ f \downarrow & & \downarrow f' \\ A' & \xrightarrow{x'} & B' \end{array} \begin{array}{l} \xrightarrow{y} \\ \searrow \exists h \\ \xrightarrow{g} \end{array} C.$$

We collect some useful properties.

Lemma 2.4 *Let \mathcal{D} be a class of objects in \mathcal{C} . Then the following hold.*

- (a) $\text{Mon}(\mathcal{D})$ is closed under composition and retracts;
- (b) $\text{Mon}(\mathcal{D})$ is left divisive, that is, $gf \in \text{Mon}(\mathcal{D})$ implies that $f \in \text{Mon}(\mathcal{D})$;
- (c) $\text{Mon}(\mathcal{D})$ is closed under weak pushouts.

Proof (a) and (b) are easily verified. Let us show (c). Let

$$\begin{array}{ccc} A & \xrightarrow{x} & B \\ f \downarrow & & \downarrow f' \\ A' & \xrightarrow{x'} & B' \end{array}$$

be a weak pushout square in \mathcal{C} with $x \in \text{Mon}\mathcal{D}$. We must show that $x' \in \text{Mon}\mathcal{D}$. To this end, let $g : A' \rightarrow D$ be a morphism with $D \in \mathcal{D}$. Then, as x is \mathcal{D} -monic, there exists a morphism $y : B \rightarrow D$ such that $yx = gf$. Now, by the weak pushout property there exists a morphism $h : B' \rightarrow D$ such that $hx' = g$ as required. \square

The interested reader may look at [49] and [60] for more properties of \mathcal{D} -monics and ω -injectives. For other applications of these notions, look, for instance in [11, 20, 33].

2.2 Extriangulated Categories

In this section we put the background on extriangulated categories following [52, Section 2] where such categories were introduced. Let \mathcal{C} be an additive category and $\mathbb{E} : \mathcal{C}^{op} \times \mathcal{C} \rightarrow \text{Ab}$ be an additive bifunctor.

Definition 2.5 For any $A, C \in \mathcal{C}$, two pairs of composable morphisms in \mathcal{C}

$$A \xrightarrow{x} B \xrightarrow{y} C \text{ and } A \xrightarrow{x'} B' \xrightarrow{y'} C$$

are *equivalent* if there exists an isomorphism $b : B \rightarrow B'$ such that

$$\begin{array}{ccccc} A & \xrightarrow{x} & B & \xrightarrow{y} & C \\ \parallel & & \cong \downarrow b & & \parallel \\ A & \xrightarrow{x'} & B' & \xrightarrow{y'} & C \end{array}$$

commutes. We denote the equivalence class of $A \xrightarrow{x} B \xrightarrow{y} C$ by $[A \xrightarrow{x} B \xrightarrow{y} C]$, and by $\mathcal{S}(C, A)$ we denote the class of all such equivalence classes.

Notation 2.6 For $a : A \rightarrow A'$ write $a_* = \mathbb{E}(-, a) : \mathbb{E}(-, A) \rightarrow \mathbb{E}(-, A')$. Similarly, we write $c^* = \mathbb{E}(c, -) : \mathbb{E}(C', -) \rightarrow \mathbb{E}(C, -)$ for $c : C \rightarrow C'$.

Definition 2.7 [52, Definitions 2.9, 2.10] An assignment $\mathfrak{s}_{C,A} : \mathbb{E}(C, A) \rightarrow \mathcal{S}(C, A)$ for all $C, A \in \mathcal{C}$ is an *additive realisation* of \mathbb{E} if it satisfies the following axioms.

- (S1) $\mathfrak{s}(0) = [A \rightarrow A \oplus C \rightarrow C]$.
- (S2) $\mathfrak{s}(\delta) \oplus \mathfrak{s}(\delta') = \mathfrak{s}(\delta \oplus \delta') \in \mathcal{S}(C \oplus C', A \oplus A')$ for $\delta \in \mathbb{E}(C, A), \delta' \in \mathbb{E}(C', A')$.
- (S3) For all $\mathfrak{s}(\delta) = [A \xrightarrow{x} B \xrightarrow{y} C]$ and $\mathfrak{s}(\delta') = [A' \xrightarrow{x'} B' \xrightarrow{y'} C']$ such that $a_*\delta = c^*\delta'$ for $a : A \rightarrow A'$ and $c : C \rightarrow C'$ (that is, $(a, c) : \delta \rightarrow \delta'$ is a *morphism of \mathbb{E} -extensions*), there is a commutative diagram in \mathcal{C}

$$\begin{array}{ccccc} A & \xrightarrow{x} & B & \xrightarrow{y} & C \\ \downarrow a & & \downarrow \exists b & & \downarrow c \\ A' & \xrightarrow{x'} & B' & \xrightarrow{y'} & C' \end{array}$$

and in this case we say that the triple (a, b, c) *realises the morphism of \mathbb{E} -extensions* $(a, c) : \delta \rightarrow \delta'$.

Notation 2.8 If $\mathfrak{s}(\delta) = [A \xrightarrow{x} B \xrightarrow{y} C]$ we may also write

$$A \xrightarrow{x} B \xrightarrow{y} C \dashrightarrow^{\delta}$$

and call this an *extriangle*. In this situation we also call x an *\mathbb{E} -inflation* and y an *\mathbb{E} -deflation*. We may also say that $y : B \rightarrow C$ is the *cone of x* and $x : A \rightarrow B$ is the *co-cone of y* .

Definition 2.9 [52, Definition 2.12] A triple $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ is an *extriangulated category* if it satisfies the following conditions. In this case we call the pair $(\mathbb{E}, \mathfrak{s})$ an *external triangulation of \mathcal{C}* and \mathfrak{s} an *\mathbb{E} -triangulation of \mathcal{C}* .

- (ET1) $\mathbb{E} : \mathcal{C}^{op} \times \mathcal{C} \rightarrow \mathbf{Ab}$ is an additive bifunctor.
- (ET2) \mathfrak{s} is an additive realisation of \mathbb{E} .
- (ET3) For all commutative diagrams with extriangles as rows

$$\begin{array}{ccccccc} A & \longrightarrow & B & \longrightarrow & C & \dashrightarrow^{\delta} & \\ \downarrow a & & \downarrow b & & \downarrow \exists c & & \\ A' & \longrightarrow & B' & \longrightarrow & C' & \dashrightarrow^{\delta'} & \end{array}$$

there exists $c : C \rightarrow C'$ making the diagram commute and such that $(a, c) : \delta \rightarrow \delta'$ is a morphism of extriangles.

(ET3)^{op} Dual to (ET3).

(ET4) For any pair of extriangles of the form

$$A \xrightarrow{f} B \xrightarrow{f'} D \xrightarrow{-\delta} \quad , \quad B \xrightarrow{g} C \xrightarrow{g'} F \xrightarrow{-\delta'}$$

there exists a commutative diagram

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{f'} & D \xrightarrow{-\delta} \\ \parallel & & \downarrow g & & \downarrow h \\ A & \xrightarrow{gf} & C & \xrightarrow{e} & E \xrightarrow{-\delta''} \\ & & \downarrow g' & & \downarrow h' \\ & & F & = & F \\ & & \downarrow \delta' & & \downarrow (f')_* \delta' \end{array}$$

with columns and rows being extriangles such that $h^* \delta'' = \delta$ and $f_* \delta'' = (h')^* \delta'$ (that is, $(f, h') : \delta'' \rightarrow \delta'$ is a morphism of extriangles).

(ET4)^{op} Dual to (ET4).

For the rest of this section, let $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ be an extriangulated category.

Remark 2.10 [52, Remark 2.18] Let $\mathcal{X} \subseteq \mathcal{C}$ be a subcategory of \mathcal{C} that is closed under \mathbb{E} -extensions, that is, for all extriangles

$$A \longrightarrow B \longrightarrow C \dashrightarrow$$

the implication $A, C \in \mathcal{X} \Rightarrow B \in \mathcal{X}$ holds. Then $(\mathcal{X}, \mathbb{E}|_{\mathcal{X}}, \mathfrak{s}|_{\mathcal{X}})$ is an extriangulated category.

Lemma 2.11 [52, Corollary 3.12] Let $A \xrightarrow{x} B \xrightarrow{y} C \dashrightarrow$ be an extriangle. Then there are long exact sequences

$$\begin{aligned} \mathcal{C}(-, A) &\xrightarrow{\mathcal{C}(-, x)} \mathcal{C}(-, B) \xrightarrow{\mathcal{C}(-, y)} \mathcal{C}(-, C) \xrightarrow{\delta_{\#}} \mathbb{E}(-, A) \xrightarrow{\mathbb{E}(-, x)} \mathbb{E}(-, B) \xrightarrow{\mathbb{E}(-, y)} \mathbb{E}(-, C) \\ \mathcal{C}(C, -) &\xrightarrow{\mathcal{C}(y, -)} \mathcal{C}(B, -) \xrightarrow{\mathcal{C}(x, -)} \mathcal{C}(A, -) \xrightarrow{\delta^{\#}} \mathbb{E}(C, -) \xrightarrow{\mathbb{E}(y, -)} \mathbb{E}(B, -) \xrightarrow{\mathbb{E}(x, -)} \mathbb{E}(A, -) \end{aligned}$$

of functors and natural transformation in the functor categories $[\mathcal{C}^{op}, \mathbf{Ab}]$ and $[\mathcal{C}, \mathbf{Ab}]$ respectively. The maps $\delta_{\#}$ and $\delta^{\#}$ are given at $X \in \mathcal{C}$ by

$$\begin{aligned} (\delta_{\#})_X : \mathcal{C}(X, C) &\rightarrow \mathbb{E}(X, A), & (\delta^{\#})_X : \mathcal{C}(A, X) &\rightarrow \mathbb{E}(C, X) \\ f &\mapsto f^* \delta & g &\mapsto g_* \delta. \end{aligned}$$

Lemma 2.12 [48, Proposition 1.20] Let $\delta \in \mathbb{E}(C, A)$ be an extriangle and $a \in \mathcal{C}(A, A')$ a morphism. Then there exists a morphism b such that $(a, b, 1)$ realises the morphism of extriangles $(a, 1) : \delta \rightarrow a_* \delta$

$$\begin{array}{ccccc} A & \xrightarrow{x} & B & \xrightarrow{y} & C \dashrightarrow \\ \downarrow a & & \downarrow b & & \parallel \\ A' & \xrightarrow{x'} & B' & \xrightarrow{y'} & C \dashrightarrow \end{array} \quad \begin{array}{c} \delta \\ a_* \delta \end{array} \quad (1)$$

and such that the sequence $A \xrightarrow{\begin{bmatrix} -a \\ x \end{bmatrix}} A' \oplus B \xrightarrow{\begin{bmatrix} y & b \end{bmatrix}} C \dashrightarrow$ is an extriangle.

Corollary 2.13 The left hand square in Diagram 1 is a weak pushout and weak pullback.

Proof By Lemma 2.11, every \mathbb{E} -deflation of an extriangle is a weak cokernel of the \mathbb{E} -inflation and dually every \mathbb{E} -inflation is a weak kernel. The claim now follows from Lemma 2.12, since the weak pushout property is equivalent to $[y \ b] : A' \oplus B \rightarrow C$ being a weak cokernel of $\begin{bmatrix} -a \\ x \end{bmatrix} : A \rightarrow A' \oplus B$. □

Lemma 2.14 [52, Corollary 3.5] *Let*

$$\begin{array}{ccccc} A & \xrightarrow{x} & B & \xrightarrow{y} & C & \xrightarrow{-\delta} & \rightarrow \\ \downarrow a & & \downarrow b & & \downarrow c & & \\ A' & \xrightarrow{x'} & B' & \xrightarrow{y'} & C' & \xrightarrow{-\delta'} & \rightarrow \end{array}$$

be a morphism of extriangles. Then the following are equivalent

- (a) *a factors through x;*
- (b) $a_*\delta = c^*\delta' = 0$;
- (c) *c factors through y'.*

Definition 2.15 Let $\mathcal{C} = (\mathcal{C}, \mathbb{E}, s)$ be an extriangulated category. An object $P \in \mathcal{C}$ is \mathbb{E} -projective if $\mathbb{E}(P, C) = 0$ for all $C \in \mathcal{C}$. By $\text{Proj}_{\mathbb{E}}\mathcal{C}$ we denote the subcategory of \mathbb{E} -projective objects. We say that \mathcal{C} has enough \mathbb{E} -projectives if for all $C \in \mathcal{C}$ there exists an extriangle

$$A \longrightarrow P \longrightarrow C \dashrightarrow$$

with $P \in \text{Proj}_{\mathbb{E}}\mathcal{C}$. The notions of \mathbb{E} -injectives and having enough \mathbb{E} -injectives are defined dually. The subcategory of \mathbb{E} -injective objects is denoted by $\text{Inj}_{\mathbb{E}}\mathcal{C}$.

2.3 Right Triangulated Categories

We begin by recalling in full the definition of a right triangulated (or suspended) category first introduced in [41].

Definition 2.16 Let \mathcal{R} be an additive category and $\Sigma : \mathcal{R} \rightarrow \mathcal{R}$ an endofunctor. A *right triangulation* of the pair (\mathcal{R}, Σ) is a collection Δ of sequences of the form

$$A \longrightarrow B \longrightarrow C \longrightarrow \Sigma A$$

in \mathcal{R} that satisfy the following axioms.

- (R1) (i) Δ is closed under isomorphisms. That is, for every commutative diagram

$$\begin{array}{ccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & \Sigma A \\ \downarrow f & & \downarrow & & \downarrow & & \downarrow \Sigma f \\ A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & \Sigma(A') \end{array}$$

in \mathcal{R} whose vertical arrows are isomorphisms, one row belongs to Δ if and only if the other row also belongs to Δ .

- (ii) For every $A \in \mathcal{R}$, the sequence

$$0 \longrightarrow A \xrightarrow{1_A} A \longrightarrow 0$$

belongs to Δ .

(iii) Every morphism $x : A \rightarrow B$ in \mathcal{R} can be embedded into a sequence

$$A \xrightarrow{x} B \longrightarrow C_x \longrightarrow \Sigma A$$

in Δ .

(R2) If the sequence

$$A \xrightarrow{x} B \xrightarrow{y} C \xrightarrow{z} \Sigma A$$

is in Δ then so is the sequence

$$B \xrightarrow{y} C \xrightarrow{z} \Sigma A \xrightarrow{-\Sigma z} \Sigma B.$$

(R3) Every commutative diagram

$$\begin{array}{ccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & \Sigma A \\ \downarrow f & & \downarrow & & & & \downarrow \Sigma f \\ A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & \Sigma(A') \end{array}$$

in \mathcal{R} whose rows belong to Δ can be extended to a commutative diagram

$$\begin{array}{ccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & \Sigma A \\ \downarrow f & & \downarrow & & \downarrow & & \downarrow \Sigma f \\ A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & \Sigma(A'). \end{array}$$

(R4) Let

$$A \xrightarrow{x} B \xrightarrow{x'} X \xrightarrow{x''} \Sigma A, \quad B \xrightarrow{y} C \xrightarrow{y'} Y \xrightarrow{y''} \Sigma B,$$

and

$$A \xrightarrow{yx} C \xrightarrow{z'} Z \xrightarrow{z''} \Sigma A$$

be sequences in Δ . Then there is a commutative diagram

$$\begin{array}{ccccccc} A & \xrightarrow{x} & B & \xrightarrow{x'} & X & \xrightarrow{x''} & \Sigma A \\ \parallel & & \downarrow y & & \downarrow \alpha & & \parallel \\ A & \xrightarrow{yx} & C & \xrightarrow{z'} & Z & \xrightarrow{z''} & \Sigma A \\ \downarrow x & & \parallel & & \downarrow \beta & & \downarrow \Sigma x \\ B & \xrightarrow{y} & C & \xrightarrow{y'} & Y & \xrightarrow{y''} & \Sigma B \\ & & & & \downarrow \gamma & \swarrow \Sigma(x') & \\ & & & & \Sigma X & & \end{array}$$

in \mathcal{R} such that the dotted column belongs to Δ .

If Δ is a right triangulation of (\mathcal{R}, Σ) , then the triple $(\mathcal{R}, \Sigma, \Delta)$ is called a *right triangulated category* and the sequences in Δ are called *right triangles*. Following [6], if the functor Σ is fully faithful and its image, $\Sigma\mathcal{R}$, is closed under extensions we call Σ a *right semi-equivalence* and \mathcal{R} a *right triangulated category with right semi-equivalence*.

Definition 2.17 Let $(\mathcal{R}, \Sigma, \Delta)$ and $(\mathcal{R}', \Sigma', \Delta')$ be right triangulated categories. An additive functor $F : \mathcal{R} \rightarrow \mathcal{R}'$ is a *right triangle functor* if

- (a) There is a natural isomorphism $\zeta : F\Sigma \xrightarrow{\cong} \Sigma'F$;
- (b) For all right triangles $A \xrightarrow{x} B \xrightarrow{y} C \xrightarrow{z} \Sigma A$ in Δ , the sequence

$$FA \xrightarrow{Fx} FB \xrightarrow{Fy} FC \xrightarrow{\zeta_A \circ Fz} \Sigma'FA$$

is in Δ' .

Remark 2.18 Let $\mathcal{R} = (\mathcal{R}, \Sigma, \Delta)$ be a right triangulated category such that $\Sigma\mathcal{R}$ is closed under extensions. Then the subcategory $\Sigma\mathcal{R}$ naturally inherits the structure of a right triangulated category. Clearly, if Σ is a right semi-equivalence then $\Sigma\mathcal{R}$ is also a right triangulated category with right semi-equivalence. We will see in Sect. 5 examples where $\Sigma\mathcal{R}$ is a right triangulated category with right semi-equivalence but \mathcal{R} is not.

For homological properties of right triangulated categories we point the reader to [6].

We finish this subsection by noting that there are no finite right triangulated categories with right semi-equivalence that are not triangulated. This is one of the many ways to see that any additively finite (that is, the set of isomorphism classes of indecomposables in \mathcal{R} is finite) idempotent complete hom finite triangulated category admits no non-trivial (co-)t-structures. We refer to [5] for a discussion of such categories in the k -linear setting.

Lemma 2.19 *Let $\mathcal{R} = (\mathcal{R}, \Sigma, \Delta)$ be a hom finite right triangulated category with right semi-equivalence. Suppose that \mathcal{R} is additively finite, then Σ is an autoequivalence. In particular, $\mathcal{R} = (\mathcal{R}, \Sigma, \Delta)$ is a triangulated category.*

Proof Let $S = \{X_1, \dots, X_n\}$ be a set of isomorphism classes of indecomposable objects in \mathcal{R} . Since Σ is additive, fully faithful and \mathcal{R} is idempotent complete, Σ acts as a permutation on the set S and we deduce that $\Sigma^m = 1_{\mathcal{R}}$ for some $m \in \mathbb{N}$. □

2.3.1 (Co-)stabilisation of a Right Triangulated Category

We recall the definitions of two triangulated categories associated to a right triangulated category: the stabilisation and costabilisation.

The *stabilisation of a right triangulated category*, $\mathcal{R} = (\mathcal{R}, \Sigma, \Delta)$, consists of a pair $(\mathcal{S}(\mathcal{R}), s)$ where $\mathcal{S}(\mathcal{R})$ is a triangulated category and $s : \mathcal{R} \rightarrow \mathcal{S}(\mathcal{R})$ is a right triangle functor satisfying a universal property: For all right triangle functors $F : \mathcal{R} \rightarrow \mathcal{T}$ with \mathcal{T} being a triangulated category, there exists a unique triangle functor $F' : \mathcal{S}(\mathcal{R}) \rightarrow \mathcal{T}$ such that $F's = F$

$$\begin{array}{ccc} \mathcal{R} & \xrightarrow{s} & \mathcal{S}(\mathcal{R}) \\ \forall F \downarrow & \swarrow & \exists! F' \\ \mathcal{T} & & \end{array}$$

Dually, the *co-stabilisation of \mathcal{R}* consists of a pair $(\mathcal{C}(\mathcal{R}), c)$ where $\mathcal{C}(\mathcal{R})$ is a triangulated category and $c : \mathcal{C}(\mathcal{R}) \rightarrow \mathcal{R}$ is a right triangle functor satisfying a universal property: For all right triangle functors $G : \mathcal{T} \rightarrow \mathcal{R}$ with \mathcal{T} being a triangulated category, there exists a unique triangle functor $G' : \mathcal{T} \rightarrow \mathcal{C}(\mathcal{R})$ such that $G = cG'$.

The stabilisation and co-stabilisation of a right triangulated category always exist. Since we will use it explicitly, we recall the construction of the stabilisation from [12, Section 3.1] and [29]; see also [22]. For more information on the co-stabilisation, which may be constructed as the category of spectra, we refer to [24, Section 4.5-4.7]; see also [12, 38].

Let $\mathcal{R} = (\mathcal{R}, \Sigma, \Delta)$ be a right triangulated category.

Definition 2.20 We define the additive category $\mathcal{S}(\mathcal{R})$ as follows. The objects of $\mathcal{S}(\mathcal{R})$ are pairs (A, n) with $A \in \mathcal{R}$ and $n \in \mathbb{Z}$. The spaces of morphisms are given by

$$\mathcal{S}(\mathcal{R})[(A, n), (B, m)] = \operatorname{colim}_{\rightarrow k \in J} (\mathcal{R}(\Sigma^{n-k} A, \Sigma^{m-k} B)), \quad J = \{k \in \mathbb{Z} \mid k \leq \min\{n, m\}\}.$$

There is an autoequivalence of $\mathcal{S}(\mathcal{R})$ which is given on objects by $\Sigma(A, n) = (A, n + 1)$ and induced on morphisms by the natural map

$$\mathcal{R}(\Sigma^{n-k} A, \Sigma^{m-k} B) \rightarrow \mathcal{R}(\Sigma^{n+1-k} A, \Sigma^{m+1-k} B)$$

for all $k \leq \min\{m, n\}$. By abuse of notation, we denote this autoequivalence also by Σ .

The functor $s : \mathcal{R} \rightarrow \mathcal{S}(\mathcal{R})$ is given on objects by $s(A) = (A, 0)$ and for a morphism $f : A \rightarrow B$, $s(f : A \rightarrow B)$ is the zero-representative of $\mathcal{S}(\mathcal{R})((A, 0), (B, 0))$.

Triangles in $\mathcal{S}(\mathcal{R})$ are given by sequences isomorphic to sequences of the form

$$(A, n) \xrightarrow{x} (B, m) \xrightarrow{y} (C, l) \xrightarrow{z} (A, n + 1)$$

such that there exists $k \leq \min\{n, m, l\}$ such that

$$\Sigma^{n-k} A \xrightarrow{(-1)^k \Sigma^{-k} x} \Sigma^{m-k} B \xrightarrow{(-1)^k \Sigma^{-k} y} \Sigma^{l-k} C \xrightarrow{(-1)^k \Sigma^{-k} z} \Sigma^{n+1-k} A \tag{2}$$

is a right triangle in \mathcal{R} .

One may verify that $s \Sigma \cong \Sigma s$ and that s is a right triangle functor.

Remark 2.21 We make some observations.

- (a) If the endofunctor $\Sigma : \mathcal{R} \rightarrow \mathcal{R}$ is fully faithful, then the morphism spaces in $\mathcal{S}(\mathcal{R})$ are neater

$$\mathcal{S}(\mathcal{R})[(A, n), (B, m)] \cong \mathcal{R}(\Sigma^{n-k} A, \Sigma^{m-k} B), \quad \forall k \leq \min\{n, m\}.$$

Further, in this case, $s : \mathcal{R} \rightarrow \mathcal{S}(\mathcal{R})$ is fully faithful and $\mathcal{S}(\mathcal{R})$ is the smallest triangulated category that contains \mathcal{R} as a full right triangulated subcategory.

- (b) If $\Sigma : \mathcal{R} \rightarrow \mathcal{R}$ is a right semi-equivalence then the ‘there exists’ preceding Equation (2) can be replaced by ‘for all’.
- (c) If Σ is not dense, then every non-zero indecomposable object in $\mathcal{S}(\mathcal{R}) \setminus \mathcal{R}$ is isomorphic to an object of the form (A, n) with $A \in (\mathcal{R} \setminus \Sigma \mathcal{R})$ and $n < 0$. Indeed, let $X \in \mathcal{S}(\mathcal{R})$ be indecomposable, then by definition, $X \cong (B, m)$ for some indecomposable $B \in \mathcal{R}$ and $m \in \mathbb{Z}$. If $m \geq 0$ then $(B, m) \cong (\Sigma^m B, 0)$ and $X \in \mathcal{R}$, thus we may assume that $m \leq 0$. Now, if $B \in \mathcal{R} \setminus \Sigma \mathcal{R}$ then we are done. Else, $B = \Sigma B'$ for some $B' \in \mathcal{R}$ and $X \cong (B', m + 1)$. The statement now follows by an inductive argument since this process must terminate in fewer than m steps by the above. We note that given a presentation $X \cong (A, n)$ with $A \in (\mathcal{R} \setminus \Sigma \mathcal{R})$ indecomposable and $n < 0$, then n is maximal, that is, there exists no $B \in \mathcal{R}$ and $m > n$ such that $X \cong (B, m)$. Indeed, if this were the case, then we would have that $A \cong \Sigma^{m-n} B$, which is a contradiction.

Many categorical properties of $\mathcal{S}(\mathcal{R})$ are inherited from \mathcal{R} , for instance if \mathcal{R} has (co-)products then so does $\mathcal{S}(\mathcal{R})$. The following properties will be useful for our work.

Lemma 2.22 *Suppose that the endofunctor $\Sigma : \mathcal{R} \rightarrow \mathcal{R}$ is fully faithful and that for all $A, B \in \mathcal{R}$ we have that $\mathcal{R}(A, \Sigma^i B) = 0$ for $i \gg 0$. Then, for all $X, Y \in \mathcal{S}(\mathcal{R})$ we have that $\mathcal{S}(\mathcal{R})(X, \Sigma^i Y) = 0$ for all $i \gg 0$.*

Proof Let $X, Y \in \mathcal{S}(\mathcal{R})$ then there exists $A, B \in \mathcal{R}$ and $k > 0$ such that $\Sigma^k X \cong A$ and $\Sigma^k Y \cong B$. Therefore, for $i \gg 0$, we have that

$$\mathcal{S}(\mathcal{R})(X, \Sigma^i Y) \cong \mathcal{S}(\mathcal{R})(A, \Sigma^i B) \cong \mathcal{R}(A, \Sigma^i B) = 0.$$

□

We call a right triangulated category satisfying the condition of the above Lemma 2.22 *bounded*.

Lemma 2.23 Let $\mathcal{R} = (\mathcal{R}, \Sigma, \Delta)$ be a right triangulated category with right semi-equivalence. Then \mathcal{R} is extension closed as a subcategory of $\mathcal{S}(\mathcal{R})$.

Proof Let

$$(A, n) \xrightarrow{x} (B, m) \xrightarrow{y} (C, l) \xrightarrow{z} (A, n + 1)$$

be a triangle in $\mathcal{S}(\mathcal{R})$ with $(A, n), (C, l)$ in \mathcal{R} . We must show that $(B, m) \in \mathcal{R}$. We may assume that $n = l = 0$ and it suffices to show that $(B, m) \cong (B', m')$ for some $B' \in \mathcal{R}$ and $m' \geq 0$.

For that purpose, suppose $m < 0$, then $\min\{m, n, l\} = m$ and, by definition, there is a right triangle in \mathcal{R}

$$\Sigma^{-m} A \xrightarrow{(-1)^m \Sigma^{-m} x} B \xrightarrow{(-1)^m \Sigma^{-m} y} \Sigma^{-m} C \xrightarrow{(-1)^m \Sigma^{-m} z} \Sigma^{1-m} A.$$

As $\Sigma : \mathcal{R} \rightarrow \mathcal{R}$ is a right semi-equivalence, we deduce that $B \in \Sigma \mathcal{R}$ so that $B = \Sigma B_1$ for some $B_1 \in \mathcal{R}$. Then $(B, m) = (\Sigma B_1, m) \cong (B_1, m + 1)$. By repeating this argument m times, we find a B' so that $(B, m) \cong (B', 0) \in \mathcal{R}$ as required. □

Example 2.24 Let \mathcal{T} be a triangulated category and $\mathcal{R} \subset \mathcal{T}$ a subcategory that is closed under positive shifts and extensions. Then \mathcal{R} is a right triangulated category with right semi-equivalence and $\mathcal{S}(\mathcal{R}) \cong \text{cosusp}_{\mathcal{T}}(\mathcal{R})$, the smallest subcategory of \mathcal{T} containing \mathcal{R} that is closed under negative shifts and extensions.

For more examples of the stabilisation of right triangulated categories, see [12, Section 3.1] and [41].

3 Right Triangulated Categories as Extriangulated Categories

The aim of this section is show that right triangulated categories have a natural extriangulated structure precisely when the shift functor is a right semi-equivalence. We also describe which extriangulated categories have a right triangulated structure. To do this, we begin by using relative homological algebra to define new extriangulated structures from existing ones (Sect. 3.1). We then show when these new extriangulations induce right triangulated structures on a quotient category (Sect. 3.2). With these tools in hand we complete the above aims in Sect. 3.3.

3.1 Extriangulated Structures Using Relative Homological Algebra

In this section we show how one can use relative homological algebra to construct new extriangulated structures and characterise when these exact structures have enough injectives/

projectives. The existence of these extriangulated structures also follows from [31, Proposition 3.17] for $n = 1$ but we give an alternative proof using relative notions. We begin with an easy lemma.

Lemma 3.1 *Let $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ be an extriangulated category, \mathcal{D} be a class of objects in \mathcal{C} and δ be an extriangle. Then the following are equivalent*

- (a) *There exists representative of $\mathfrak{s}(\delta)$, $A \xrightarrow{x} B \xrightarrow{y} C$, such that x is \mathcal{D} -monic;*
- (b) *The \mathbb{E} -inflation of every representative of $\mathfrak{s}(\delta)$ is \mathcal{D} -monic.*

Proof (a) \Rightarrow (b) follows from Lemma 2.4(c). (b) \Rightarrow (a) is obvious. □

Proposition 3.2 *Let $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ be an extriangulated category and \mathcal{D} a class of objects in \mathcal{C} . Consider the classes of extriangles for $A, C \in \mathcal{C}$*

$$\mathbb{I}_{\mathcal{D}}(\mathcal{C}, A) = \{ A \xrightarrow{x} B \xrightarrow{y} C \xrightarrow{\delta} \in \mathbb{E}(\mathcal{C}, A) \mid x \in \text{Mon}(\mathcal{D}) \},$$

$$\mathbb{P}_{\mathcal{D}}(\mathcal{C}, A) = \{ A \xrightarrow{x} B \xrightarrow{y} C \xrightarrow{\delta} \in \mathbb{E}(\mathcal{C}, A) \mid y \in \text{Epi}(\mathcal{D}) \}, \text{ and}$$

$$\mathbb{D}_{\mathcal{D}}(\mathcal{C}, A) = \{ A \xrightarrow{x} B \xrightarrow{y} C \xrightarrow{\delta} \in \mathbb{E}(\mathcal{C}, A) \mid x \in \text{Mon}(\mathcal{D}), y \in \text{Epi}(\mathcal{D}) \} = \mathbb{I}_{\mathcal{D}} \cap \mathbb{P}_{\mathcal{D}}.$$

Then $(\mathbb{I}_{\mathcal{D}}, \mathfrak{s}|_{\mathbb{I}_{\mathcal{D}}})$, $(\mathbb{P}_{\mathcal{D}}, \mathfrak{s}|_{\mathbb{P}_{\mathcal{D}}})$ and $(\mathbb{D}_{\mathcal{D}}, \mathfrak{s}|_{\mathbb{D}_{\mathcal{D}}})$ all define external triangulations of \mathcal{C} . Moreover,

$$\mathcal{D} \subseteq \text{Inj}_{\mathbb{I}_{\mathcal{D}}}\mathcal{C} \subseteq \text{Inj}_{\mathbb{D}_{\mathcal{D}}}\mathcal{C} \text{ and}$$

$$\mathcal{D} \subseteq \text{Proj}_{\mathbb{P}_{\mathcal{D}}}\mathcal{C} \subseteq \text{Proj}_{\mathbb{D}_{\mathcal{D}}}\mathcal{C}.$$

Additionally, suppose that $\mathcal{D} = \text{Add}(\mathcal{D})$ then the following hold.

- (a) *If \mathcal{D} is covariantly finite and left \mathcal{D} -approximations are \mathbb{E} -inflations then $\mathcal{D} = \text{Inj}_{\mathbb{I}_{\mathcal{D}}}\mathcal{C}$ and \mathcal{C} has enough $\mathbb{I}_{\mathcal{D}}$ -injectives.*
- (b) *If \mathcal{D} is contravariantly finite and right \mathcal{D} -approximations are \mathbb{E} -deflations then $\mathcal{D} = \text{Proj}_{\mathbb{P}_{\mathcal{D}}}\mathcal{C}$ and \mathcal{C} has enough $\mathbb{P}_{\mathcal{D}}$ -projectives.*
- (c) *If \mathcal{D} is functorially finite, left (resp. right) \mathcal{D} -approximations are \mathbb{E} -inflations (resp. \mathbb{E} -deflations) and cones (resp. co-cones) of left (resp. right) \mathcal{D} -approximations are \mathcal{D} -epic (resp. \mathcal{D} -monic) then $\mathcal{D} = \text{Inj}_{\mathbb{D}_{\mathcal{D}}}\mathcal{C} = \text{Proj}_{\mathbb{D}_{\mathcal{D}}}\mathcal{C}$ and \mathcal{C} has enough $\mathbb{D}_{\mathcal{D}}$ -injectives and $\mathbb{D}_{\mathcal{D}}$ -projectives.*

Proof We prove the statements for $\mathbb{I}_{\mathcal{D}}$ whence the remaining claims follow from dual and combined arguments. Let $\delta \in \mathbb{I}_{\mathcal{D}}(\mathcal{C}', A)$, $a \in \mathcal{C}(A, A')$ and $c \in \mathcal{C}(C, C')$. By [31, Proposition 3.14] it is enough to show that $a_*\delta \in \mathbb{I}_{\mathcal{D}}(\mathcal{C}', A')$, $c^*\delta \in \mathbb{I}_{\mathcal{D}}(\mathcal{C}, A)$ and $\mathbb{I}_{\mathcal{D}}$ -inflations are closed under composition. The first follows from Corollary 2.13 and Lemma 2.4(c) and the third from Lemma 2.4(a). To show the second claim consider the morphism of extriangles $(1, c) : c^*\delta \rightarrow \delta$

$$\begin{array}{ccccc} A & \xrightarrow{x} & B & \xrightarrow{y} & C & \xrightarrow{c^*\delta} \\ \parallel & & \downarrow b & & \downarrow c & \\ A & \xrightarrow{x'} & B' & \xrightarrow{y'} & C' & \xrightarrow{\delta} \end{array} .$$

We see that $x' = bx \in \text{Mon}(\mathcal{D})$ and hence x is \mathcal{D} -monic by Lemma 2.4(b).

We verify the additional claims. By construction we have that the class of $\mathbb{I}_{\mathcal{D}}$ -inflations, ω , is contained in the class of \mathcal{D} -monic morphisms. Thus

$$\mathcal{D} \subseteq \text{Inj}(\text{Mon}(\mathcal{D})) \subseteq \text{Inj}(\omega) = \text{Inj}_{\mathbb{I}_{\mathcal{D}}}\mathcal{C}.$$

Now additionally suppose that $\mathcal{D} = \text{Add}(\mathcal{D})$, \mathcal{D} is covariantly finite and left \mathcal{D} -approximations are \mathbb{E} -inflations. Then for every object $I \in \text{Inj}_{\mathbb{E}\mathcal{D}}\mathcal{C}$ there is an $\mathbb{E}\mathcal{D}$ -inflation $i : I \rightarrow D$ with $D \in \mathcal{D}$. We deduce that I is a direct summand of \mathcal{D} and we are done. \square

Notation 3.3 Let $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ be an extriangulated category. For brevity, in the sequel we will say a subcategory \mathcal{D} of \mathcal{C} has property $(*)$ if $\mathcal{D} = \text{Add}(\mathcal{D})$, \mathcal{D} is covariantly finite and all left \mathcal{D} -approximations are \mathbb{E} -inflations.

We note that one may view the Frobenius property [26, 28] of external triangulations of an idempotent complete category through the lens of relative homological algebra.

Corollary 3.4 *An extriangulated category $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ is Frobenius if and only if there exists a functorially finite additive subcategory \mathcal{P} such that*

- (a) *All left \mathcal{P} -approximations are \mathbb{E} -inflations and all \mathbb{E} -inflations are \mathcal{P} -monic;*
- (b) *All right \mathcal{P} -approximations are \mathbb{E} -deflations and all \mathbb{E} -deflations are \mathcal{P} -epic.*

In other words, $\mathcal{P} = \text{Proj}_{\mathbb{D}\mathcal{P}}\mathcal{C} = \text{Inj}_{\mathbb{D}\mathcal{P}}\mathcal{C}$ and $\mathbb{D}\mathcal{P} = \mathbb{E}$.

3.2 Right Triangulated Stable Categories

We show how extriangulations can induce right triangulated structures on a quotient category. The construction is reminiscent of the triangulated structure of the stable category of a Frobenius exact category [26, 28].

Set-up: For the rest of this section, let $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ be an extriangulated category and $\mathcal{D} \subseteq \mathcal{C}$ be a subcategory satisfying property $(*)$. For each $A \in \mathcal{C}$ we make a choice of extriangle

$$A \xrightarrow{i_A} \mathcal{D}(A) \xrightarrow{p_A} \Sigma_{\mathcal{D}}A \xrightarrow{\delta_A} \rightarrow \quad (3)$$

where i_A is a left \mathcal{D} -approximation of A . We also define $\Sigma_{\mathcal{D}}$ on morphisms: Let $f : A \rightarrow B$. Then, as i_A is \mathcal{D} -monic, there exists $\mathcal{D}(f) : \mathcal{D}(A) \rightarrow \mathcal{D}(B)$ such that $\mathcal{D}(f)i_A = i_B f$ and thus, by (ET3) there exists a morphism $\Sigma_{\mathcal{D}}f : \Sigma_{\mathcal{D}}A \rightarrow \Sigma_{\mathcal{D}}B$ such that $(f, \Sigma_{\mathcal{D}}f) : \delta_A \rightarrow \delta_B$ is a morphism of extriangles

$$\begin{array}{ccccc} A & \xrightarrow{i_A} & \mathcal{D}(A) & \xrightarrow{p_A} & \Sigma_{\mathcal{D}}A & \xrightarrow{\delta_A} & \rightarrow \\ \downarrow f & & \exists \downarrow \mathcal{D}(f) & & \exists \downarrow \Sigma_{\mathcal{D}}f & & \\ B & \xrightarrow{i_B} & \mathcal{D}(B) & \xrightarrow{p_B} & \Sigma_{\mathcal{D}}B & \xrightarrow{\delta_B} & \rightarrow \end{array} \quad (4)$$

By $\mathcal{C}_{\mathcal{D}}$ we denote the stable category of \mathcal{C} by the ideal of morphisms factoring through objects of \mathcal{D} .

The following lemma is similar to [52, Claim6.1].

Lemma 3.5 *The construction above of $\Sigma_{\mathcal{D}}$ defines an additive endofunctor $\mathcal{C}_{\mathcal{D}} \rightarrow \mathcal{C}_{\mathcal{D}}$. Further, any other choice of extriangles δ_A yields a naturally isomorphic endofunctor.*

Proof We keep the notation of the above paragraph. First we show that $\Sigma_{\mathcal{D}}$ is well-defined on morphisms in $\mathcal{C}_{\mathcal{D}}$, that is, the construction of $\Sigma_{\mathcal{D}}f$ is independent of the choices made of $\mathcal{D}(f)$ and in (ET3). Indeed, let $g : \Sigma_{\mathcal{D}}A \rightarrow \Sigma_{\mathcal{D}}B$ be any morphism such that $(f, g) : \delta_A \rightarrow \delta_B$ is a morphism of extriangles. Then $g^*\delta_B = f_*\delta_A = (\Sigma_{\mathcal{D}}f)^*\delta_B$ and therefore $(g - \Sigma_{\mathcal{D}}f)^*\delta_B = 0$. By Lemma 2.14, $g - \Sigma_{\mathcal{D}}f$ factors through p_B and hence $\underline{g - \Sigma_{\mathcal{D}}f} = 0$ in $\mathcal{C}_{\mathcal{D}}$.

Now suppose for all $A \in \mathcal{C}$ we have another choice of extriangle

$$A \xrightarrow{i'_A} \mathcal{D}'(A) \xrightarrow{p'_A} \Sigma'_{\mathcal{D}}A \xrightarrow{\delta'_A}$$

which then results in another endofunctor $\Sigma'_{\mathcal{D}} : \underline{\mathcal{C}}_{\mathcal{D}} \rightarrow \underline{\mathcal{C}}_{\mathcal{D}}$. Then, since i_A is \mathcal{D} -monic, there exists $s : \mathcal{D}(A) \rightarrow \mathcal{D}'(A)$ such that $si_A = i'_A$. Then, by (ET3) there exists $t_A = t : \Sigma_{\mathcal{D}}A \rightarrow \Sigma'_{\mathcal{D}}A$ such that $(1, t) : \delta_A \rightarrow \delta'_A$ is a morphism of extriangles. Similarly, we obtain a morphism $t'_A = t' : \Sigma'_{\mathcal{D}}A \rightarrow \Sigma_{\mathcal{D}}A$ such that $(1, t') : \delta'_A \rightarrow \delta_A$ is a morphism of extriangles

$$\begin{array}{ccccccc} A & \xrightarrow{i_A} & \mathcal{D}(A) & \xrightarrow{p_A} & \Sigma_{\mathcal{D}}A & \xrightarrow{\delta_A} & \\ \parallel & & \exists \downarrow s & & \exists \downarrow t & & \\ A & \xrightarrow{i'_A} & \mathcal{D}'(A) & \xrightarrow{p'_A} & \Sigma'_{\mathcal{D}}A & \xrightarrow{\delta'_A} & \\ \parallel & & \exists \downarrow t' & & \exists \downarrow t' & & \\ A & \xrightarrow{i_A} & \mathcal{D}(A) & \xrightarrow{p_A} & \Sigma_{\mathcal{D}}A & \xrightarrow{\delta_A} & . \end{array}$$

We claim that $\{t_A\}_{A \in \underline{\mathcal{C}}_{\mathcal{D}}}$ is an isomorphism of functors $\Sigma_{\mathcal{D}} \rightarrow \Sigma'_{\mathcal{D}}$ with inverse given by $\{t'_A\}_{A \in \underline{\mathcal{C}}_{\mathcal{D}}}$. The fact that the t_A are isomorphisms in $\underline{\mathcal{C}}_{\mathcal{D}}$ follows from the observation that $(1, t') : \delta_A \rightarrow \delta'_A$ is a morphism of extriangles and hence, by using a similar argument to the above, we see that $t't = 1_{\Sigma_{\mathcal{D}}A}$ in $\underline{\mathcal{C}}_{\mathcal{D}}$. Dually, $tt' = 1_{\Sigma'_{\mathcal{D}}A}$.

It remains to verify that $\{t_A\}_{A \in \underline{\mathcal{C}}_{\mathcal{D}}}$ is a natural transformation. Let $f : A \rightarrow B$ we must show that $t_B \Sigma_{\mathcal{D}}f = \Sigma'_{\mathcal{D}}ft_A$. This follows from the observation that the pairs $(f, (t_B \Sigma_{\mathcal{D}}f))$ and $(f, (\Sigma'_{\mathcal{D}}ft_A))$ both define morphisms of extriangles $\delta_A \rightarrow \delta'_B$. □

Proposition 3.6 *The stable category $\underline{\mathcal{C}}_{\mathcal{D}}$ with the endofunctor $\Sigma_{\mathcal{D}}$ admits a right triangulation given by the collection of all sequences isomorphic to sequences of the form*

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma x$$

that fit into a commutative diagram in \mathcal{C}

$$\begin{array}{ccccccc} A & \xrightarrow{i_A} & \mathcal{D}(A) & \xrightarrow{p_A} & \Sigma_{\mathcal{D}}A & \xrightarrow{\delta_A} & \\ f \downarrow & & \downarrow & & \parallel & & \\ B & \xrightarrow{g} & C & \xrightarrow{h} & \Sigma_{\mathcal{D}}A & \xrightarrow{f_*\delta_A} & . \end{array}$$

Furthermore, $\Sigma_{\mathcal{D}}\underline{\mathcal{C}}_{\mathcal{D}}$ is always closed under extensions and $\Sigma_{\mathcal{D}}$ is fully faithful if and only if $\mathcal{D} \subset \text{Proj}_{\mathbb{D}}\mathcal{C}$.

Proof The arguments of [6, Theorem 3.3] and [14, Theorem 3.1] may be recycled to the extriangulated setting. □

We can precisely describe when $\Sigma_{\mathcal{D}}$ is a right semi-equivalence.

Proposition 3.7 *Suppose additionally that cones of left \mathcal{D} -approximations are \mathcal{D} -epic. Then the following hold.*

- (a) $\mathbb{I}_{\mathcal{D}} = \mathbb{D}_{\mathcal{D}}$.
- (b) $\underline{\mathcal{C}}_{\mathcal{D}}$ is a right triangulated category with right semi-equivalence.

(c) For all $A, C \in \mathcal{C}$ there is a functorial isomorphism of abelian groups

$$F = F_{C,A} : \underline{\mathcal{C}}_{\mathcal{D}}(C, \Sigma_{\mathcal{D}}A) \xrightarrow{\cong} \mathbb{D}_{\mathcal{D}}(C, A)$$

$$\underline{f} \mapsto f^* \delta_A.$$

Proof Claims (a) and (b) follow directly from Propositions 3.2 and 3.6. It remains to show (c):

F is well-defined: We must check that if $\underline{f} = \underline{g} \in \underline{\mathcal{C}}_{\mathcal{D}}(C, \Sigma_{\mathcal{D}}A)$ then $f^* \delta_A = g^* \delta_A$. Indeed, in this case, $\underline{f} - \underline{g} = 0$ and, since p_A is \mathcal{D} -epic by part (a), $f - g$ factors through p_A . Now, by Lemma 2.14, $(f - g)^* \delta_A = 0$ and we are done. Note that this also shows the injectivity of F , since $f^* \delta_A = 0$ implies that f factors through p_A and thus $\underline{f} = 0$.

F is bijective: It remains to show that F is surjective. Let $\gamma \in \mathbb{D}_{\mathcal{D}}(C, \overline{A})$ be realised by $A \xrightarrow{x} B \xrightarrow{y} C$. Then, since x is \mathcal{D} -monic there exists $g : B \rightarrow \mathcal{D}(A)$ such that $gx = i_A$. By (ET3), there then exists $f : C \rightarrow \Sigma_{\mathcal{D}}A$ such that $(1, f) : \gamma \rightarrow \delta_A$ is a morphism of extriangles. In other words, $\gamma = f^* \delta_A =: F(\underline{f})$.

$$\begin{array}{ccccc} A & \xrightarrow{x} & B & \xrightarrow{y} & C & \xrightarrow{f^* \delta_A} \\ \parallel & & \downarrow \exists g & & \downarrow \exists f & \\ A & \xrightarrow{i_A} & \mathcal{D}(A) & \xrightarrow{p_A} & \Sigma_{\mathcal{D}}A & \xrightarrow{\delta_A} \end{array}$$

F is a homomorphism of abelian groups: This is straightforward:

$$F(\underline{f} + \underline{f}') = (f + f')^* \delta_A = f^* \delta_A + f'^* \delta_A = F(\underline{f}) + F(\underline{f}').$$

Functoriality in the first argument: Let $\underline{c} \in \mathcal{C}(C, C')$. Then for all $\underline{f} \in \underline{\mathcal{C}}_{\mathcal{D}}(C', \Sigma_A)$

$$F_{C,A} \underline{\mathcal{C}}_{\mathcal{D}}(\underline{c}, \Sigma_{\mathcal{D}}A) : \underline{f} \mapsto \underline{f} \underline{c} \mapsto (f \underline{c})^* \delta_A$$

$$\mathbb{D}_{\mathcal{D}}(C, A) F_{C',A} : \underline{f} \mapsto f^* \delta_A \mapsto c^* f^* \delta_A = (f \underline{c})^* \delta_A$$

which proves the claim.

Functoriality in the second argument: Let $\underline{\alpha} \in \underline{\mathcal{C}}_{\mathcal{D}}(A, A')$ and set $\Sigma_{\mathcal{D}} \underline{\alpha} = \underline{a} \in \underline{\mathcal{C}}_{\mathcal{D}}(\Sigma_{\mathcal{D}}A, \Sigma_{\mathcal{D}}A')$. Then for all $\underline{f} \in \underline{\mathcal{C}}_{\mathcal{D}}(C, \Sigma_{\mathcal{D}}A)$

$$F_{C,A'} \underline{\mathcal{C}}_{\mathcal{D}}(C, \underline{a}) : \underline{f} \mapsto \underline{a} \underline{f} \mapsto (a \underline{f})^* \delta_A$$

$$\mathbb{D}_{\mathcal{D}}(C, A) F_{C,A} : \underline{f} \mapsto f^* \delta_A \mapsto \alpha_* f^* \delta_A = (f)^* \alpha_* \delta_A = (f)^* a^* \delta_A$$

where the last equality in the second line follows from the definition of $\Sigma_{\mathcal{D}}$. □

In light of Corollary 3.4, extriangulated categories $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ such that the injective stable category $\underline{\mathcal{C}}_{\text{Inj}} \mathbb{E}$ is a right triangulated category with right semi-equivalence have a ‘one-sided Frobenius’ property: There are enough injectives and each injective object is projective. This imbalance of projectives and injectives and also Lemma 2.19 indicate that we must look in extriangulated categories with infinitely many objects for examples of quotients that are right triangulated with right semi-equivalence.

Example 3.8 (a) Let Q be the infinite quiver

$$1 \leftarrow 2 \leftarrow 3 \leftarrow \dots$$

and consider the category $\mathcal{A} = \text{mod} K Q / \text{rad}^m$ for some $m > 1$. \mathcal{A} is an abelian category and with its maximal exact structure it is an extriangulated category. Observe that \mathcal{A}

has enough injectives and that each injective object is projective. Indeed, $I_r = P_{r+m-1}$. But not every projective object is injective, for instance $P_1 = S_1$ is not injective. Thus, by Theorem 3.6, the quotient category $\underline{\mathcal{A}}_{\text{Inj}\mathcal{A}}$ is a right triangulated category with semi-equivalence with the shift given by the co-syzygy functor.

- (b) Let $\mathcal{C} = (\mathcal{C}, \mathbb{E}, \mathfrak{s})$ be an extriangulated category and let $(\mathcal{U}, \mathcal{V})$ be a cotorsion pair in \mathcal{C} [52, Definition 4.1]. It is easily verified that \mathcal{V} satisfies property (*). Thus $\underline{\mathcal{C}}_{\mathcal{V}}$ is a right triangulated category. We will investigate examples of this flavour for the case of Frobenius extriangulated categories further in Sect. 5. Let us note that the class of subcategories satisfying property (*) is more general than the class of cotorsion pairs, since the subcategories giving cotorsion pairs must be closed under extensions.
- (c) Let \mathcal{T} be a compactly generated triangulated category. Recall that a subcategory \mathcal{X} of \mathcal{T} is *definable* if there is a class of morphisms ω between compact objects in \mathcal{T} such that $\mathcal{X} = \text{Inj}(\omega)$ [34, Section 4.1]; where it was also shown that every definable category admits left approximations. Thus, since in a triangulated category every morphism is an \mathbb{E} -inflation, it follows that $\underline{\mathcal{T}}_{\mathcal{X}}$ is a right triangulated category.

The author would be interested to know if is possible to classify when the examples in (b) and (c) that result in right triangulated categories with right semi-equivalence.

3.3 Right Triangulated Extriangulated Categories

We characterise right triangulated categories as extriangulated categories. We begin with some terminology and a useful lemma.

Definition 3.9 Let $(\mathcal{R}, \Sigma, \Delta)$ be a right triangulated category. We say that the right triangulation Δ *induces an extriangulated structure on \mathcal{R}* if there exists an external triangulation $(\mathbb{E}, \mathfrak{s})$ of \mathcal{R} such that for all right triangles $A \xrightarrow{x} B \xrightarrow{y} C \xrightarrow{z} \Sigma A$, there is an extriangle $A \xrightarrow{x} B \xrightarrow{y} C \xrightarrow{\delta} A$.

Lemma 3.10 [52, Proposition 3.30] *Let $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ be an extriangulated category and $\mathcal{D} \subseteq \text{Proj}_{\mathbb{E}}(\mathcal{C}) \cap \text{Inj}_{\mathbb{E}}(\mathcal{C})$ be a full, additive, replete subcategory. Then the stable category $\underline{\mathcal{C}}_{\mathcal{D}}$ inherits an external triangulation, $(\underline{\mathbb{E}}, \underline{\mathfrak{s}})$, given by*

- (a) $\underline{\mathbb{E}}(C, A) = \mathbb{E}(C, A)$ for all $A, C \in \mathcal{C}$;
- (b) $\underline{\mathbb{E}}(\underline{c}, \underline{a}) = \mathbb{E}(c, a)$ for all $a \in \mathcal{C}(A, A'), c \in \mathcal{C}(C', C)$;
- (c) $\underline{\mathfrak{s}}(\delta) = [A \xrightarrow{x} B \xrightarrow{y} C]$ where $\mathfrak{s}(\delta) = [A \xrightarrow{x} B \xrightarrow{y} C]$ for all extriangles δ .

Example 3.11 We give two important classes of examples.

- (a) Let \mathcal{T} be a triangulated category. Then the triangulation of \mathcal{T} is a right triangulation which induces an extriangulated structure on \mathcal{T} . See [52, Section 3.3]. This was a motivating example for the introduction of extriangulated categories.
- (b) Let $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ be an extriangulated category and $\mathcal{D} \subset \mathcal{C}$ be a subcategory satisfying property (*) and such that all cones of left \mathcal{D} -approximations are \mathcal{D} -epic. By Proposition 3.2 we have that $\mathcal{D} \subseteq \text{Proj}_{\mathbb{D}_{\mathcal{D}}}(\mathcal{C}) \cap \text{Inj}_{\mathbb{D}_{\mathcal{D}}}(\mathcal{C})$ and it follows from Proposition 3.7(c) that the right triangulated structure on $\underline{\mathcal{C}}_{\mathcal{D}}$ of Proposition 3.6 coincides with the extriangulated structure of Lemma 3.10 induced by the external triangulation $(\mathbb{D}_{\mathcal{D}}, \mathfrak{s}|_{\mathbb{D}_{\mathcal{D}}})$. In other words, the right triangulation of $\underline{\mathcal{C}}_{\mathcal{D}}$ (with right semi-equivalence) induces an extriangulation on $\underline{\mathcal{C}}_{\mathcal{D}}$. Let us note that this could also be deduced from a combination of Lemma 2.23, Remark 2.10 and the above Example.

We may now state and prove the main result of this section.

Theorem 3.12 *Let $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ be an extriangulated category. Then the following are equivalent*

- (a) *There exists a fully faithful additive endofunctor $\Sigma : \mathcal{C} \rightarrow \mathcal{C}$ such that $\mathbb{E}(-, ?) \cong \mathcal{C}(-, \Sigma ?)$ and that the image $\Sigma \mathcal{C}$ is closed under \mathbb{E} -extensions;*
- (b) *$\text{Inj}_{\mathbb{E}}(\mathcal{C}) = \{0\}$ and there are enough \mathbb{E} -injectives;*
- (c) *There is a right triangulation of \mathcal{C} that induces the extriangulated structure $(\mathbb{E}, \mathfrak{s})$.*

Proof (a) \Rightarrow (b): We claim that, for each $A \in \mathcal{C}$, the morphism $1_{\Sigma A} \in \mathcal{C}(\Sigma A, \Sigma A) \cong \mathbb{E}(A, \Sigma A)$ is realised by the sequence $[A \rightarrow 0 \rightarrow \Sigma A]$. Indeed, we may use a similar argument to that of [52, Lemma3.21]: Let $\mathfrak{s}(1_{\Sigma A}) = [A \xrightarrow{x} E \xrightarrow{y} \Sigma A]$, then by Lemma 2.11 there is a long exact sequence in $[\mathcal{C}^{\text{op}}, \text{Ab}]$

$$\mathcal{C}(-, A) \xrightarrow{\mathcal{C}(-, x)} \mathcal{C}(-, E) \xrightarrow{\mathcal{C}(-, y)} \mathcal{C}(-, \Sigma A) \xrightarrow{(1_{\Sigma A})_{\#} = \text{id}} \mathcal{C}(-, \Sigma A) \xrightarrow{\mathcal{C}(-, \Sigma x)} \mathcal{C}(-, \Sigma E).$$

It follows that $y = 0$ and $\Sigma x = 0$. Thus $x = 0$ since Σ is faithful. Now the exactness of $0 \rightarrow \mathcal{C}(-, E) \rightarrow 0$ implies that $E \cong 0$. Thus, if I is an \mathbb{E} -injective object then it is a direct summand of 0 and so $I = 0$.

(b) \Rightarrow (a),(c): In this case, the subcategory $\{0\}$ satisfies $\mathbb{E} = \mathbb{I}_{\{0\}} = \mathbb{D}_{\{0\}}$. Thus the stable category $\underline{\mathcal{C}}_{\{0\}} \cong \mathcal{C}$ has a right triangulated structure with right semi-equivalence by Propositions 3.6 and 3.7 with shift functor given by $\Sigma X = \Sigma_{\{0\}} X := \text{cone}(X \rightarrow 0)$. The claims follow from Example 3.11(b). In particular, by Proposition 3.7(c) there is a functorial isomorphism

$$\mathbb{E}(-, ?) = \mathbb{D}_{\{0\}}(-, ?) \cong \underline{\mathcal{C}}_{\{0\}}(-, \Sigma_{\{0\}} ?) = \mathcal{C}(-, \Sigma ?).$$

(c) \Rightarrow (b): By the axiom (R3)(iii), for all $A \in \mathcal{C}$, the morphism $A \rightarrow 0$ is the first morphism in a right triangle. Thus, by assumption, $A \rightarrow 0$ is an \mathbb{E} -inflation and the claim follows. \square

As a direct consequence, we see that, in general, right triangulated categories do not have a natural extriangulated structure.

Corollary 3.13 *Let $(\mathcal{R}, \Sigma, \Delta)$ be a right triangulated category. Then Δ induces an extriangulated structure on \mathcal{R} if and only if Σ is a right semi-equivalence.*

Remark 3.14 The concept of negative (first) extensions of an extriangulated category has been recently introduced and studied [1, 23]. For a triangulated category \mathcal{T} one may take $\mathbb{E}^{-1}(-, ?) = \mathcal{T}(-, \Sigma^{-1} ?)$ as a negative first extension. Since by Lemma 2.23 a right triangulated category with right semi-equivalence \mathcal{R} is an extension closed subcategory of the triangulated category $\mathcal{S}(\mathcal{R})$ there is a natural first negative extension structure on \mathcal{R} given by $\mathbb{E}^{-1}(C, A) := \mathcal{S}(\mathcal{R})(C, \Sigma^{-1} A) \cong \mathcal{R}(\Sigma C, A)$ for all $A, C \in \mathcal{R}$.

4 Aisles and Co-aisles

In this section, we show that the language of extriangulated categories allows us, under some assumptions, to give an intrinsic characterisation of which right triangulated categories with right semi-equivalence occur as (co)-aisles of (co)-t-structures in its stabilisation. Let $\mathcal{R} = (\mathcal{R}, \Sigma, \Delta)$ be a right triangulated category with right semi-equivalence that is not triangulated (i.e. Σ is not dense). We add this final assumption as many statements are trivial when \mathcal{R} is triangulated and this allows us to freely use Remark 2.21(c).

4.1 Torsion Pairs

We begin by recalling some important definitions.

Definition 4.1 Let $\mathcal{C} = (\mathcal{C}, \mathbb{E}, \mathfrak{s})$ be an extriangulated category. A pair of additive subcategories, $(\mathcal{U}, \mathcal{V})$, of \mathcal{C} is a *torsion pair in \mathcal{C}* if

- (a) $\mathcal{C}(\mathcal{U}, \mathcal{V}) = 0$;
- (b) For all $C \in \mathcal{C}$ there exists an extriangle

$$U \xrightarrow{u} C \xrightarrow{v} V \dashrightarrow \tag{5}$$

with $U \in \mathcal{U}$ and $V \in \mathcal{V}$.

If \mathcal{C} is a triangulated category (viewed naturally as an extriangulated category) with shift functor Σ , a torsion pair in \mathcal{C} , $(\mathcal{U}, \mathcal{V})$, is a *t-structure* [9] (resp. *co-t-structure* [15, 56]) if $\Sigma\mathcal{U} \subseteq \mathcal{U}$ (resp. $\Sigma^{-1}\mathcal{U} \subseteq \mathcal{U}$) with *heart* $\mathcal{H} = \mathcal{U} \cap \Sigma\mathcal{V}$ (resp. *co-heart* $\mathcal{M} = \mathcal{U} \cap \Sigma^{-1}\mathcal{V}$). We call the subcategory \mathcal{U} the *aisle* of the (co-)t-structure and \mathcal{V} the *co-aisle*. A (co-)t-structure is *bounded* if the equalities $\mathcal{C} = \bigcup_{n \in \mathbb{Z}} \Sigma^n \mathcal{U} = \bigcup_{m \in \mathbb{Z}} \Sigma^m \mathcal{V}$ hold.

The reader should be aware that the terminology and notation of torsion pairs and (co-)t-structures in a triangulated category varies, often by a shift, throughout the literature and that co-t-structures were introduced under the name ‘weight structures’ in [15].

Remark 4.2 Let $(\mathcal{U}, \mathcal{V})$ be a torsion pair in $\mathcal{C} = (\mathcal{C}, \mathbb{E}, \mathfrak{s})$. Then it follows quickly from the definition that the following properties hold.

- (a) \mathcal{U}, \mathcal{V} are closed under extensions.
- (b) $\mathcal{U} = \{A \in \mathcal{C} \mid \mathcal{C}(A, \mathcal{V}) = 0\}$ and $\mathcal{V} = \{A \in \mathcal{C} \mid \mathcal{C}(\mathcal{U}, A) = 0\}$.
- (c) The morphism u (resp. v) in the extriangle (5) is a right \mathcal{U} (resp. left \mathcal{V}) approximation of C .

Before we proceed, let us compare the above definition of a torsion pair in \mathcal{R} with other notions appearing in the literature:

- (a) A torsion pair $(\mathcal{U}, \mathcal{V})$ in \mathcal{R} is a *right torsion pair* [47] if Σ preserves \mathcal{V} -monics.
- (b) When \mathcal{R} is equipped with a negative first extension structure, \mathbb{E}^{-1} , (see Remark 3.14) a torsion pair $(\mathcal{U}, \mathcal{V})$ in \mathcal{R} is an *s-torsion pair* [1] if $\mathbb{E}^{-1}(\mathcal{U}, \mathcal{V}) = 0$. In this case the extriangle (5) is essentially unique and assignments $C \mapsto U$ and $C \mapsto V$ are functorial [1, Proposition 3.7].

The next lemma shows that right torsion pairs and s-torsion pairs in \mathcal{R} coincide (when we equip \mathcal{R} with the natural first negative extension structure $\mathbb{E}^{-1}(-, ?) = \mathcal{S}(\mathcal{R})(-, \Sigma^{-1}?)$) and that such torsion pairs remind us of t-structures; which are precisely the s-torsion pairs in a triangulated category. For a class of objects \mathcal{X} in \mathcal{R} , by $\Sigma^{-1}\mathcal{X}$ we denote the class of objects $\{X \in \mathcal{R} \mid \Sigma X \in \mathcal{X}\}$.

Lemma 4.3 *Let $(\mathcal{U}, \mathcal{V})$ be a torsion pair in \mathcal{R} . Then the following are equivalent*

- (a) $\Sigma\mathcal{U} \subseteq \mathcal{U}$;
- (b) $\Sigma^{-1}\mathcal{V} \subseteq \mathcal{V}$;
- (c) $\mathcal{R}(\Sigma\mathcal{U}, \mathcal{V}) = 0$;
- (d) Σ preserves \mathcal{V} -monics.

Proof (a)⇒(b): Suppose that $\Sigma\mathcal{U} \subseteq \mathcal{U}$ and let $Y \in \Sigma^{-1}\mathcal{V}$. Then there is a right triangle

$$U \xrightarrow{u} Y \longrightarrow V \longrightarrow \Sigma U$$

with $U \in \mathcal{U}$ and $V \in \mathcal{V}$. Consider the morphism $\Sigma u : \Sigma U \rightarrow \Sigma Y$. By assumption, $\Sigma U \in \mathcal{U}$ and $\Sigma Y \in \mathcal{V}$; thus, $\Sigma u = 0$. Since Σ is faithful, $u = 0$ and we deduce that Y is a direct summand of $V \in \mathcal{V}$.

(b)⇒(a): Let $X \in \mathcal{U}$, then there is a right triangle

$$U \longrightarrow \Sigma X \xrightarrow{v} V \longrightarrow \Sigma U$$

with $U \in \mathcal{U}$ and $V \in \mathcal{V}$. Then, as $\Sigma\mathcal{R}$ is closed under extensions, we have that $V \in \Sigma\mathcal{R}$. Thus there exists $V' \in \Sigma^{-1}\mathcal{V} \subseteq \mathcal{V}$ such that $\Sigma V' = V$. Moreover, there is a right triangle of the form

$$X \xrightarrow{v'} V' \longrightarrow U \longrightarrow \Sigma X$$

where $\Sigma v' = -v$. Since \mathcal{U} is closed under extensions, we deduce that $V' \in \mathcal{U}$. Together, we have that $V' \in \mathcal{U} \cap \mathcal{V} = \{0\}$. Thus, $V = 0, v = 0$ and ΣX is a direct summand of $U \in \mathcal{U}$.

(a)⇒(c): Follows from the fact that $\mathcal{R}(\mathcal{U}, \mathcal{V}) = 0$.

(c)⇒(d): Let $f : A \rightarrow B$ be a \mathcal{V} -monic morphism. There are right triangles

$$U \xrightarrow{u} A \xrightarrow{v} V \longrightarrow \Sigma U$$

and

$$U' \xrightarrow{u'} \Sigma A \xrightarrow{v'} V' \longrightarrow \Sigma U'$$

with $U, U' \in \mathcal{U}$ and $V, V' \in \mathcal{V}$. Then, as f is \mathcal{V} -monic, there exists $g : B \rightarrow V$ such that $gf = v$. Since $\mathcal{R}(\Sigma\mathcal{U}, \mathcal{V}) = 0$ there exists $a : \Sigma U \rightarrow U'$ such that $\Sigma u = u'a$. Thus, by the axiom (R3) = (ET3) there exists $c : \Sigma V \rightarrow V'$ such that $c(\Sigma v) = v'$

$$\begin{array}{ccccc} \Sigma U & \xrightarrow{\Sigma u} & \Sigma A & \xrightarrow{\Sigma v} & \Sigma V & \longrightarrow & \Sigma^2 U \\ \downarrow \exists a & & \parallel & & \downarrow \exists c & & \\ U' & \xrightarrow{u'} & \Sigma A & \xrightarrow{v'} & V' & \longrightarrow & \Sigma U'. \end{array}$$

Thus

$$v' = c(\Sigma v) = c\Sigma(gf) = c(\Sigma g)(\Sigma f).$$

This finishes the proof since v' is a left \mathcal{V} -approximation of ΣA and so all morphisms from ΣA to \mathcal{V} factor through v' .

(d)⇒(a). Let $U \in \mathcal{U}$. Then, since $\mathcal{R}(U, \mathcal{V}) = 0$ we have that $U \rightarrow 0$ is \mathcal{V} -monic. Thus, by assumption $\Sigma U \rightarrow 0$ is also \mathcal{V} -monic from which we deduce that $\Sigma U \in \mathcal{U}$. □

4.2 Co-aisles of Co-t-Structures

The next lemma justifies why we will look to describe \mathcal{R} as a (co-)aisle in $\mathcal{S}(\mathcal{R})$.

Lemma 4.4 *Let \mathcal{T} be a triangulated category, $(\mathcal{V}, \mathcal{W})$ (resp. $(\mathcal{U}, \mathcal{V})$) be a t-structure (resp. co-t-structure) in \mathcal{T} and $\mathcal{S} := \text{cosusp}_{\mathcal{T}}\mathcal{V} \cong \mathcal{S}(\mathcal{V})$. Then $(\mathcal{V}, \mathcal{W} \cap \mathcal{S})$ (resp. $(\mathcal{U} \cap \mathcal{S}, \mathcal{V})$) is a t-structure (resp. co-t-structure) in \mathcal{S} .*

Proof We prove the t-structure case, whence statement for co-t-structures will follow dually. Let $(\mathcal{V}, \mathcal{W})$ be a t-structure in \mathcal{T} and set $\mathcal{S} := \text{cosusp}_{\mathcal{T}}\mathcal{V} \cong \mathcal{S}(\mathcal{V})$. Clearly, $\mathcal{S}(\mathcal{V}, \mathcal{W} \cap \mathcal{S}) = 0$ and $\Sigma\mathcal{V} \subset \mathcal{V}$. Thus it remains to show that $\mathcal{S} = \mathcal{V} * (\mathcal{W} \cap \mathcal{S})$. Since $\mathcal{V}, (\mathcal{W} \cap \mathcal{S}) \subset \mathcal{S}$ and \mathcal{S} is closed under extensions, $\mathcal{V} * (\mathcal{W} \cap \mathcal{S}) \subset \mathcal{S}$. To show the converse, let $A \in \mathcal{S}$ then there exists a triangle in \mathcal{T}

$$V \longrightarrow A \longrightarrow W \longrightarrow \Sigma V$$

with $V \in \mathcal{V}$ and $W \in \mathcal{W}$. Thus $W \in \mathcal{S} * \Sigma\mathcal{V} \subset \mathcal{S} * \mathcal{S} \subset \mathcal{S}$. □

We also note that the boundedness of right triangulated categories (Lemma 2.22) relates to the boundedness of (co-)t-structures.

Lemma 4.5 *Suppose that \mathcal{R} is the co-aisle of a co-t-structure, $(\mathcal{U}, \mathcal{R})$ in $\mathcal{S}(\mathcal{R})$. If \mathcal{R} is bounded then the co-t-structure $(\mathcal{U}, \mathcal{R})$ is bounded. A dual statement holds for a t-structure $(\mathcal{R}, \mathcal{V})$ in $\mathcal{S}(\mathcal{R})$.*

Proof Let $(\mathcal{U}, \mathcal{R})$ be a co-t-structure in $\mathcal{S}(\mathcal{R})$. By construction of $\mathcal{S}(\mathcal{R})$, the equality $\mathcal{S}(\mathcal{R}) = \bigcup_{n \in \mathbb{Z}} \Sigma^n \mathcal{R}$ holds. It remains to verify that $\mathcal{S}(\mathcal{R}) = \bigcup_{m \in \mathbb{Z}} \Sigma^m \mathcal{U}$. Observe that since \mathcal{R} is bounded, in light of Lemma 2.22, for all $X \in \mathcal{S}(\mathcal{R})$ we have that $\mathcal{S}(\mathcal{R})(\mathcal{R}, \Sigma^i X) = 0$ for $i \gg 0$. Thus, by Remark 4.2, $\Sigma^i X \in \mathcal{V}$ for $i \gg 0$; whence the claim follows. □

Before we give our characterisations of \mathcal{R} as the co-aisle of a co-t-structure, we make some comments. Note, by Proposition 3.7, the \mathbb{E} -projectives of \mathcal{R} are precisely the objects $P \in \mathcal{R}$ satisfying $\mathcal{R}(P, \Sigma-) = 0$. Recall from [63] that a subcategory $\mathcal{X} = \text{add}\mathcal{X}$ of a triangulated category \mathcal{T} is *silting* if $\mathcal{T}(\mathcal{X}, \Sigma^{>0}\mathcal{X}) = 0$ and $\mathcal{T} = \text{thick}\mathcal{X}$, the smallest triangulated subcategory of \mathcal{T} containing \mathcal{X} that is closed under direct summands. We are now ready to give characterisations of \mathcal{R} as the co-aisle of a co-t-structure in $\mathcal{S}(\mathcal{R})$ in terms of torsion pairs, \mathbb{E} -projectives of \mathcal{R} and silting subcategories.

Theorem 4.6 *The following are equivalent*

- (a) \mathcal{R} is the co-aisle of a co-t-structure $(\mathcal{U}, \mathcal{R})$ in $\mathcal{S}(\mathcal{R})$;
- (b) \mathcal{R} has enough \mathbb{E} -projectives;
- (c) There is a torsion pair $(\text{Proj}_{\mathbb{E}}\mathcal{R}, \Sigma\mathcal{R})$ in \mathcal{R} .

Moreover, if \mathcal{R} is bounded then the above conditions are also equivalent to

- (d) $\text{Proj}_{\mathbb{E}}\mathcal{R}$ is a silting subcategory of $\mathcal{S}(\mathcal{R})$.

Proof (a) \Rightarrow (b): Suppose there is a co-t-structure $(\mathcal{U}, \mathcal{R})$ in $\mathcal{S}(\mathcal{R})$. We must show that for all $X \in \mathcal{R}$ there is a right triangle

$$R \rightarrow P \rightarrow X \rightarrow \Sigma R$$

in \mathcal{R} with $P \in \text{Proj}\mathcal{R}$. Since $(\mathcal{U}, \mathcal{R})$ is a co-t-structure in $\mathcal{S}(\mathcal{R})$ there is a triangle in $\mathcal{S}(\mathcal{R})$

$$U \rightarrow \Sigma^{-1}X \rightarrow R \rightarrow \Sigma U$$

with $U \in \mathcal{U}, R \in \mathcal{R}$. By rotating we obtain the triangle

$$R \rightarrow \Sigma U \rightarrow X \rightarrow \Sigma R.$$

We claim that $\Sigma U \in \text{Proj}_{\mathbb{E}}\mathcal{R}$. The fact that $\Sigma U \in \mathcal{R}$ follows from this triangle since \mathcal{R} is closed under extensions in $\mathcal{S}(\mathcal{R})$ by Lemma 2.23. It remains to verify that ΣU is \mathbb{E} -projective in \mathcal{R} , that is $\mathcal{R}(\Sigma U, \Sigma\mathcal{R}) = 0$:

$$0 = \mathcal{S}(\mathcal{R})(U, \mathcal{R}) \cong \mathcal{S}(\mathcal{R})(\Sigma U, \Sigma\mathcal{R}) \cong \mathcal{R}(\Sigma U, \Sigma\mathcal{R})$$

where we have the first equality from the properties of co-t-structures.

(b)⇒(c): Let $X \in \mathcal{R}$. By assumption there is a right triangle $R \rightarrow P \rightarrow X \rightarrow \Sigma R$ in \mathcal{R} with $P \in \text{Proj}_{\mathbb{E}}\mathcal{R}$. We rotate this to the right triangle

$$P \rightarrow X \rightarrow \Sigma R \rightarrow \Sigma P$$

whence the claim follows as $\mathcal{R}(P, \Sigma\mathcal{R}) = 0$.

(c)⇒(a): We use a similar argument to [30, Theorem 3.11]. Suppose that there is a torsion pair $(\text{Proj}_{\mathbb{E}}\mathcal{R}, \Sigma\mathcal{R})$ in \mathcal{R} and set $\mathcal{P} := \text{Proj}_{\mathbb{E}}\mathcal{R}$ and $\mathcal{U} := \{Y \in \mathcal{S}(\mathcal{R}) \mid \mathcal{S}(\mathcal{R})(Y, \mathcal{R}) = 0\}$. Let $0 \neq X \in \mathcal{S}(\mathcal{R})$, we must show that there is a triangle

$$U \rightarrow X \rightarrow R \rightarrow \Sigma U$$

in $\mathcal{S}(\mathcal{R})$ with $U \in \mathcal{U}$ and $R \in \mathcal{R}$. It is enough to verify this for X being indecomposable. If $X \in \mathcal{R}$ then we may simply take the triangle $0 \rightarrow X \rightarrow X \rightarrow 0$. If $X \in \mathcal{S}(\mathcal{R}) \setminus \mathcal{R}$ then by Remark 2.21(c), there exists $A \in \mathcal{R} \setminus \Sigma\mathcal{R}$ and $n < 0$ such that $X \cong (A, n) \cong \Sigma^n A$.

We proceed by an inductive argument on n . By assumption, there is a right triangle in \mathcal{R}

$$P \rightarrow A \rightarrow \Sigma R \rightarrow \Sigma P$$

with $P \in \mathcal{P}$ and $R \in \mathcal{R}$. This rotates to a triangle

$$\Sigma^n P \rightarrow \Sigma^n A \rightarrow \Sigma^{n+1} R \rightarrow \Sigma^{n+1} P \tag{6}$$

in $\mathcal{S}(\mathcal{R})$. Let us show that $\Sigma^n P \in \mathcal{U}$:

$$\mathcal{S}(\mathcal{R})(\Sigma^n P, \mathcal{R}) \cong \mathcal{S}(\mathcal{R})(P, \Sigma^{-n}\mathcal{R}) \cong \mathcal{R}(P, \Sigma^{-n}\mathcal{R}) = 0$$

since $\mathcal{R}(P, \Sigma\mathcal{R}) = 0$ and $n < 0$. For $n = -1$, the base case, we have that $\Sigma^{n+1} R = R \in \mathcal{R}$ the triangle (6) suffices as $\Sigma^{-1} P \in \mathcal{U}$. For $n < -1$, by the induction hypothesis, there is a triangle in $\mathcal{S}(\mathcal{R})$

$$U \rightarrow \Sigma^{n+1} R \rightarrow V \rightarrow \Sigma U$$

with $U \in \mathcal{U}$ and $V \in \mathcal{R}$. We apply (ET4)^{op} to this triangle and triangle (6) to obtain

$$\begin{array}{ccccccc}
 \Sigma^n P & \longrightarrow & E & \longrightarrow & U & \longrightarrow & \Sigma^{n+1} P \\
 \parallel & & \downarrow & & \downarrow & & \parallel \\
 \Sigma^n P & \longrightarrow & \Sigma^n A & \longrightarrow & \Sigma^{n+1} R & \longrightarrow & \Sigma^{n+1} P \\
 & & \downarrow & & \downarrow & & \\
 & & V & \xlongequal{\quad} & V & & \\
 & & \downarrow & & \downarrow & & \\
 & & \Sigma E & \longrightarrow & \Sigma U & &
 \end{array} \tag{7}$$

and claim that the triangle $E \rightarrow \Sigma^n A \rightarrow V \rightarrow \Sigma E$ satisfies the required conditions. Since $V \in \mathcal{R}$ by construction, we only have to show that $E \in \mathcal{U}$. We have already seen above that $\Sigma^n P \in \mathcal{U}$, thus the top row of the Diagram (7) shows that $E \in \mathcal{U}$ since \mathcal{U} is closed under extensions.

(d)⇔(a): Since \mathcal{R} is bounded, by Lemma 2.22 the co-t-structure $(\mathcal{U}, \mathcal{R})$ is bounded in $\mathcal{S}(\mathcal{R})$. Observe that $\text{Proj}_{\mathbb{E}}\mathcal{R}$ is the co-heart of this co-t-structure. The claim then follows from [30, Corollary 5.9] where it was shown that a subcategory of a triangulated category is silting precisely when it is the co-heart of a bounded co-t-structure. \square

As a consequence, we obtain the following.

Corollary 4.7 *There is a correspondence between silting subcategories of triangulated categories and bounded right triangulated categories with right semi-equivalence that have enough projectives.*

The above result is a triangulated analogue of a result due to Bondal & van den Bergh ([3, Proposition B.3]) and Rump ([59, Theorem 2] and [58, Theorem 1]), which states that there is a correspondence between tiltings in abelian categories and quasi-abelian categories. For more information on quasi-abelian categories and the construction of the right associated category, see the above references and [62, 65, 66]. When working with derived categories/module categories of nice finite dimensional algebras these correspondences have a natural connection which we explain in the next paragraph.

4.3 Tilting and Silting Connection

Let us first describe the tilting–quasi-abelian correspondence in the abelian setting in more detail: Let \mathcal{Q} be a quasi-abelian category, then \mathcal{Q} appears as a tilting torsion class in the *right associated* abelian category \mathcal{A} . In particular, $\text{Proj}\mathcal{Q}$ is a tilting subcategory of \mathcal{A} , that is, every object is a subobject of an object in $\text{Proj}\mathcal{Q}$.

There is a natural way to associate a right triangulated category (with right semi-equivalence) to a quasi-abelian category. For a quasi-abelian category \mathcal{Q} as above, we define the right triangulated category $\mathcal{R}_{\mathcal{Q}} := \text{susp}_{\text{D}^b(\mathcal{A})}\mathcal{Q}$, that is, the smallest subcategory of $\text{D}^b(\mathcal{A})$ that is closed under positive shifts and extensions that contains \mathcal{Q} .

Lemma 4.8 *In the notation as above, the following hold.*

- (a) $\mathcal{R}_{\mathcal{Q}} = \{X^\bullet \in \text{D}^b(\mathcal{A}) \mid H^0(X^\bullet) \in \mathcal{Q}, H^i(X^\bullet) = 0 \forall i < 0\}$;
- (b) $\mathcal{S}(\mathcal{R}_{\mathcal{Q}}) = \text{D}^b(\mathcal{A})$;
- (c) *If \mathcal{A} has finite global dimension, $\mathcal{R}_{\mathcal{Q}}$ is bounded;*
- (d) *If \mathcal{Q} has a projective generator then $\text{Proj}\mathcal{R}_{\mathcal{Q}} = \text{Proj}\mathcal{Q}$ and $\mathcal{R}_{\mathcal{Q}}$ has enough projectives.*

Proof The left-to-right inclusion of part (a) is obvious. For the converse, note that it is enough to show that $\Sigma\mathcal{A} = \{\Sigma X \mid X \in \mathcal{A}\} \subseteq \mathcal{R}_{\mathcal{Q}}$. To this end, let $X \in \mathcal{A}$, then, as $\text{Proj}\mathcal{Q}$ is a tilting subcategory of \mathcal{A} , there is a short exact sequence

$$0 \rightarrow X \rightarrow Q \rightarrow Q' \rightarrow 0$$

in \mathcal{A} with $Q \in \mathcal{Q}$. Since \mathcal{Q} is a torsion class in \mathcal{A} , it is closed under quotients in \mathcal{A} and thus $Q' \in \mathcal{Q}$. This short exact sequence gives rise to a triangle in $\text{D}^b(\mathcal{A})$

$$X \rightarrow Q \rightarrow Q' \rightarrow \Sigma X, \tag{8}$$

which we rotate to obtain the triangle

$$Q' \rightarrow \Sigma X \rightarrow \Sigma Q \rightarrow \Sigma Q'.$$

The claim now follows since $Q', \Sigma Q \in \mathcal{R}_{\mathcal{Q}}$ and $\mathcal{R}_{\mathcal{Q}}$ is closed under extensions. (b) and (c) follow from (a). For (d), let P be a projective generator of \mathcal{Q} . We will show that every object A in $\mathcal{R}_{\mathcal{Q}}$ admits a right triangle

$$B \rightarrow P^J \rightarrow A \rightarrow \Sigma B$$

in $\mathcal{R}_{\mathcal{Q}}$ for a set J . Note that it suffices to show this for indecomposable objects. Since for $X \in \Sigma\mathcal{R}_{\mathcal{Q}}$ we may make take the triangle

$$\Sigma^{-1}X \rightarrow 0 \rightarrow X \rightarrow X$$

we only have to show the claim for objects in $\mathcal{R}_Q \setminus \Sigma \mathcal{R}_Q \subset \mathcal{Q} \sqcup \Sigma \mathcal{A}$. For every object $Q' \in \mathcal{Q}$ there is a short exact sequence in \mathcal{Q} (and hence also in \mathcal{A}) $0 \rightarrow K \rightarrow P^J \rightarrow Q' \rightarrow 0$ which induces a triangle in $D^b(\mathcal{A})$

$$K \rightarrow P^J \rightarrow Q' \rightarrow \Sigma K$$

which is a right triangle in \mathcal{R}_Q since all terms are in \mathcal{R}_Q . For $X \in \mathcal{A}$ we apply (ET4)^{op} to the above triangle and a rotation of the triangle (8) to obtain the following commutative diagram of triangles in $D^b(\mathcal{A})$

$$\begin{array}{ccccccc} K & \longrightarrow & E & \longrightarrow & Q & \longrightarrow & \Sigma K \\ \parallel & & \downarrow & & \downarrow & & \\ K & \longrightarrow & P^J & \longrightarrow & Q' & \longrightarrow & \Sigma K \\ & & \downarrow & & \downarrow & & \\ & & \Sigma X & \xlongequal{\quad} & \Sigma X & & \\ & & \downarrow & & \downarrow & & \\ & & \Sigma E & & \Sigma Q & & \end{array}$$

Since $K, Q \in \mathcal{R}_Q$ we have that $E \in \mathcal{R}_Q$. Thus, the middle column of the above diagram, $E \rightarrow P^J \rightarrow \Sigma X \rightarrow \Sigma E$, gives us the desired right triangle in \mathcal{R}_Q . □

Corollary 4.9 *Let \mathcal{Q} be a quasi-abelian category with an injective cogenerator I and a projective generator P and let \mathcal{A} be the right associated abelian category of \mathcal{Q} . Suppose that, \mathcal{Q} is artinian with respect to subobjects and $\text{End}_{\mathcal{Q}}(I)$ is an artin algebra of finite global dimension. Then P is a tilting module in \mathcal{A} and $\text{add}P$ is a silting subcategory of $D^b(\mathcal{A})$.*

Proof The first claim is a combination of (the duals of) [65, Theorem III.4.9 and Corollary III.4.11]. The second claim follows from Theorem 4.6 and Lemma 4.8. □

Remark 4.10 Let Λ be a finite dimensional algebra over a field k and set $\mathcal{A} = \text{mod}\Lambda$. By [2, Theorems 2.7 & 3.2] and [37, Theorem 2.2] (see also [42]) there are bijections between the following

- (a) Functorially finite torsion classes in \mathcal{A} ;
- (b) Isomorphism classes of τ -tilting modules in \mathcal{A} ;
- (c) Two-term silting complexes in $K^b(\text{proj}\mathcal{A})$;
- (d) Intermediate co-t-structures of $K^b(\text{proj}\mathcal{A})$.

and that the first bijection restricts to a bijection between tilting torsion classes in \mathcal{A} and isomorphism classes of tilting modules in \mathcal{A} . Along these bijections, a functorially finite torsion class \mathcal{T} corresponds to the τ -tilting module M , where M is a projective generator of \mathcal{T} . M , in turn, corresponds to the 2-term silting complex $(P^1 \rightarrow P^0) \in K^b(\text{proj}\mathcal{A})$, where $P^1 \rightarrow P^0 \rightarrow M \rightarrow 0$ is a projective presentation of M in \mathcal{A} . The co-t-structure corresponding to $(P^1 \rightarrow P^0)$ has co-aisle $\text{susp}_{K^b(\text{proj}\mathcal{A})}(P^1 \rightarrow P^0)$.

When Λ is of finite global dimension we have that $K^b(\text{proj}\mathcal{A}) \cong D^b(\mathcal{A})$. Thus, by Lemma 4.8, following the correspondences above from (a) to (d), a tilting torsion class \mathcal{T} is associated to (the co-t-structure with co-aisle) $\mathcal{R}_{\mathcal{T}}$ as we defined at the start of this subsection. Of course, without the finite global dimension assumption on Λ , one could define $\mathcal{R}_{\mathcal{T}}$ as $\text{susp}_{K^b(\text{proj}\mathcal{A})}(P^1 \rightarrow P^0)$ but this is somewhat less natural.

4.4 Aisles of t-Structures

We now present our characterisations of aisles of t-structures.

Theorem 4.11 *The following are equivalent*

- (a) \mathcal{R} is the aisle of a t-structure $(\mathcal{R}, \mathcal{V})$ in $\mathcal{S}(\mathcal{R})$;
- (b) There is a torsion pair $(\Sigma\mathcal{R}, \mathcal{F})$ in \mathcal{R} ;
- (c) There is an equivalence of triangulated categories $\phi : \mathcal{S}(\mathcal{R}) \rightarrow \mathcal{C}(\mathcal{R})$

$$\begin{array}{ccc}
 & \mathcal{R} & \\
 s \swarrow & & \nwarrow c \\
 \mathcal{S}(\mathcal{R}) & \xrightarrow[\cong]{\phi} & \mathcal{C}(\mathcal{R})
 \end{array}$$

such that $c\phi s = 1_{\mathcal{R}}$.

Proof The equivalence of (a) and (b) follows from dualising the arguments used in the proof of Theorem 4.6.

(a) \Rightarrow (c): We show that in this case, $\mathcal{S}(\mathcal{R})$ satisfies the universal property of the co-stabilisation of \mathcal{R} . Let $r : \mathcal{S}(\mathcal{R}) \rightarrow \mathcal{R}$ denote the functor induced by the t-structure $(\mathcal{R}, \mathcal{V})$. Let \mathcal{T} be a triangulated category and $F : \mathcal{T} \rightarrow \mathcal{R}$ be a right triangle functor. Define the functor $F' : \mathcal{T} \rightarrow \mathcal{S}(\mathcal{R})$ by $F'X = (FX, 0)$. Clearly $rF' = F$ and F' is a triangle functor.

We quickly verify that F' is unique with this property. Suppose that $F'' : \mathcal{T} \rightarrow \mathcal{S}(\mathcal{R})$ is another triangle functor such that $rF'' = F$. Then for all $X \in \mathcal{T}$ there are triangles in $\mathcal{S}(\mathcal{R})$

$$\begin{aligned}
 rF'X \cong FX &\rightarrow F'X \rightarrow V' \rightarrow \Sigma FX \\
 rF''X \cong FX &\rightarrow F''X \rightarrow V'' \rightarrow \Sigma FX
 \end{aligned}$$

with $\mathcal{S}(\mathcal{R})(\mathcal{R}, V') = 0 = \mathcal{S}(\mathcal{R})(\mathcal{R}, V'')$. Observe that, for $n \gg 0$, $\Sigma^n F'X, \Sigma^n F''X \in \mathcal{R}$. Thus, $\Sigma^n F'X \cong \Sigma^n FX \cong \Sigma^n F''X$. So, since F' and F'' are triangle functors, $F''\Sigma^n X \cong F'\Sigma^n X$ and we are done.

(c) \Rightarrow (a): We will show that that $c\phi$ is right adjoint to the inclusion s whence we will be done by [39, Proposition 1.2]. It is enough to show that for all $X, Y \in \mathcal{R}$ and $m \in \mathbb{Z}$ that there is a natural isomorphism

$$\mathcal{R}(X, c\phi(Y, m)) \cong \mathcal{S}(\mathcal{R})(sX, (Y, m)).$$

If $m \geq 0$ then $(Y, m) = s(\Sigma^m Y)$ and thus, by assumption $c\phi(Y, m) = c\phi s(\Sigma^m Y) \cong \Sigma^m Y$. On the other hand

$$\mathcal{S}(\mathcal{R})(sX, (Y, m)) = \mathcal{S}(\mathcal{R})((X, 0), (Y, m)) \cong \mathcal{R}(X, \Sigma^m Y)$$

and we are done.

If $m < 0$, then $\mathcal{S}(\mathcal{R})(sX, (Y, m)) \cong \mathcal{R}(\Sigma^{-m} X, Y)$. On the other hand, $\mathcal{R}(X, c\phi(Y, m)) \cong \mathcal{R}(\Sigma^{-m} X, \Sigma^{-m} c\phi(Y, m))$ and

$$\Sigma^{-m} c\phi(Y, m) \cong c\phi \Sigma^{-m}(Y, m) = c\phi(Y, 0) \cong c\phi s Y \cong Y$$

and we are done since all isomorphisms used are natural. □

We end this section by noting that the language of the torsion pairs in \mathcal{R} also allows us to describe related classes of (co-)t-structures.

Proposition 4.12 $\mathcal{R} = (\mathcal{R}, \Sigma, \Delta)$ be a right triangulated category with right semi-equivalence.

(a) Suppose that there is a co- t -structure $(\mathcal{U}, \mathcal{R})$ in $\mathcal{S}(\mathcal{R})$. Then there is a bijection

$$\begin{aligned} &\{(\mathcal{X}, \mathcal{Y}) \text{ co-}t\text{-structure in } \mathcal{S}(\mathcal{R}) \mid \mathcal{Y} \subseteq \mathcal{R}\} \\ &\quad \longleftrightarrow \{(\mathcal{A}, \mathcal{B}) \text{ torsion pair in } \mathcal{R} \mid \Sigma\mathcal{B} \subseteq \mathcal{B} \subseteq \Sigma\mathcal{R}\} \\ &\quad (\mathcal{X}, \mathcal{Y}) \longmapsto (\mathcal{R} \cap \Sigma\mathcal{X}, \Sigma\mathcal{Y}) \\ &\quad (\Sigma^{-1}(\mathcal{U} * \mathcal{A}), \Sigma^{-1}\mathcal{B}) \longleftarrow (\mathcal{A}, \mathcal{B}) \end{aligned}$$

which preserves the inclusion of aisles and restricts to a bijection

$$\begin{aligned} &\{(\mathcal{X}, \mathcal{Y}) \text{ co-}t\text{-structure in } \mathcal{S}(\mathcal{R}) \mid \Sigma\mathcal{R} \subseteq \mathcal{Y} \subseteq \mathcal{R}\} \\ &\quad \longleftrightarrow \{(\mathcal{A}, \mathcal{B}) \text{ torsion pair in } \mathcal{R} \mid \Sigma^2\mathcal{R} \subseteq \mathcal{B} \subseteq \Sigma\mathcal{R}\}. \end{aligned}$$

(b) Suppose that there is a t -structure $(\mathcal{R}, \mathcal{V})$ in $\mathcal{S}(\mathcal{R})$. Then there is a bijection

$$\begin{aligned} &\{(\mathcal{X}, \mathcal{Y}) \text{ } t\text{-structure in } \mathcal{S}(\mathcal{R}) \mid \mathcal{X} \subseteq \mathcal{R}\} \\ &\quad \longleftrightarrow \{(\mathcal{A}, \mathcal{B}) \text{ torsion pair in } \mathcal{R} \mid \Sigma\mathcal{A} \subseteq \mathcal{A} \subseteq \Sigma\mathcal{R}\} \\ &\quad (\mathcal{X}, \mathcal{Y}) \longmapsto (\Sigma\mathcal{X}, \mathcal{R} \cap \Sigma\mathcal{Y}) \\ &\quad (\Sigma^{-1}\mathcal{A}, \Sigma^{-1}(\mathcal{V} * \mathcal{B})) \longleftarrow (\mathcal{A}, \mathcal{B}) \end{aligned}$$

which preserves the inclusion of aisles and restricts to a bijection

$$\begin{aligned} &\{(\mathcal{X}, \mathcal{Y}) \text{ } t\text{-structure in } \mathcal{S}(\mathcal{R}) \mid \Sigma\mathcal{R} \subseteq \mathcal{X} \subseteq \mathcal{R}\} \\ &\quad \longleftrightarrow \{(\mathcal{A}, \mathcal{B}) \text{ torsion pair in } \mathcal{R} \mid \Sigma^2\mathcal{R} \subseteq \mathcal{A} \subseteq \Sigma\mathcal{R}\}. \end{aligned}$$

Proof We do not write the proof of (a) since it is a straightforward generalisation of the proof of [66, Theorem 4.4]. Part (b) can be proved in a similar way. Alternatively, since all torsion pairs in (b) are \mathfrak{s} -torsion pairs, the bijections of (b) are, after shifting, special cases of [1, Theorem 3.9]. □

The (co-) t -structures in the second correspondences in parts (a) and (b) are called *intermediate* with respect to the (co-) t -structure of \mathcal{R} . Let us mention related work on this property. Intermediate t -structures also correspond to torsion pairs in the heart [11, 27] and have applications in, for example, stability conditions [67] and algebraic geometry [57]. In the case of co- t -structures, it has been shown that intermediate co- t -structures correspond to certain two term silting subcategories [37] and to cotorsion pairs in the ‘extended coheart’ [55].

5 Aisles Through Quotients

We fix some conventions. Let $\mathcal{C} = (\mathcal{C}, \mathbb{E}, \mathfrak{s})$ be a Frobenius extriangulated category, with projectives-injectives \mathcal{I} and $\underline{\mathcal{C}} = \underline{\mathcal{C}}_{\mathcal{I}}$ be the stable category, which has the natural structure of triangulated category [52, Corollary 7.4] with shift functor Σ . By $\mathcal{I}(X)$ we denote the object such that $X \rightarrow \mathcal{I}(X)$ is a minimal left \mathcal{I} -approximation.

In this section, we show the following.

Theorem 5.1 *Let $(\mathcal{U}, \mathcal{V})$ be a t -structure in $\underline{\mathcal{C}}$ and set $\mathcal{D} = \Sigma^{-1}\mathcal{V}$. Then there is an equivalence of right triangulated categories $\Sigma_{\mathcal{D}}\underline{\mathcal{C}}_{\mathcal{D}} \cong \mathcal{U}$.*

Remark 5.2 In [61, Proposition 3.9], it was shown that t-structures in an algebraic triangulated category correspond bijectively to certain complete cotorsion pairs in the associated Frobenius exact category. Along that bijection a t-structure $(\mathcal{U}, \mathcal{V})$ in $\underline{\mathcal{C}}$ corresponds to the cotorsion pair $(\mathcal{U}, \Sigma^{-1}\mathcal{V})$ in \mathcal{C} . This bijection generalises immediately to the extriangulated setting. Further, when applied to the case of a triangulated category \mathcal{T} (which is Frobenius extriangulated category [52, Proposition 3.22]), this specialises to the observation of [51, Proposition 2.6] that t-structures in \mathcal{T} are in bijection with cotorsion pairs $(\mathcal{U}, \mathcal{V})$ in \mathcal{T} such that $\Sigma\mathcal{U} \subseteq \mathcal{U}$. Thus, these results and Theorem 5.1 complement each other.

We note that a class of objects $\mathcal{V} \subset \text{Obj}(\mathcal{C}) = \text{Obj}(\underline{\mathcal{C}})$ defines subcategories of both \mathcal{C} and $\underline{\mathcal{C}}$.

Lemma 5.3 *Let $\mathcal{D} = \text{Add}\mathcal{D}$ be a covariantly finite additive subcategory of $\underline{\mathcal{C}}$. Then \mathcal{D} is also a covariantly finite additive subcategory of \mathcal{C} . Moreover, left \mathcal{D} -approximations are \mathbb{E} -inflations in \mathcal{C} .*

Proof For any $X \in \mathcal{C}$, let $\underline{\alpha} : X \rightarrow D$ and $\beta : X \rightarrow \mathcal{I}(X) = I$ be a left \mathcal{D} -approximation of X in $\underline{\mathcal{C}}$ and a left \mathcal{I} -approximation of X in \mathcal{C} respectively. We claim that $\begin{bmatrix} \alpha \\ \beta \end{bmatrix} : X \rightarrow D \oplus I$ is a left \mathcal{D} -approximation of X in \mathcal{C} .

Indeed, since \mathcal{D} is additive, $0 \cong I \subset \mathcal{D}$, so $D \oplus I \in \mathcal{D}$. Let $f : X \rightarrow W$ be a morphism in \mathcal{C} with $W \in \mathcal{D}$. Then there exists $(g : D \rightarrow W) \in \underline{\mathcal{C}}(D, W)$ such that $\underline{f} = g\underline{\alpha}$. Now, $\underline{f} - g\underline{\alpha} = 0$ so $f - g\underline{\alpha}$ factors through \overline{I} and so must factor through the left \mathcal{I} -approximation of X . Thus there exists $g' : I \rightarrow W$ such that $f - g\underline{\alpha} = g'\beta$. Rearranging we have

$$\begin{array}{ccc} X & \xrightarrow{\begin{bmatrix} \alpha \\ \beta \end{bmatrix}} & D \oplus I \\ f \downarrow & \swarrow & \left[\begin{array}{c} g \\ g' \end{array} \right] \\ W & & \end{array}$$

The remaining claim follows from Lemma 2.12. □

Let $(\mathcal{U}, \mathcal{V})$ be a t-structure on $\underline{\mathcal{C}}$, so that for all objects X there exists a unique (up to isomorphism) triangle in $\underline{\mathcal{C}}$

$$\mathcal{U}(X) \xrightarrow{u_X} X \xrightarrow{v_X} \mathcal{V}(X) \xrightarrow{w_X} \Sigma\mathcal{U}(X)$$

therefore there is a unique (up to isomorphism) triangle in $\underline{\mathcal{C}}$

$$\Sigma^{-1}\mathcal{U}(\Sigma X) \xrightarrow{u'_X} X \xrightarrow{v'_X} \Sigma^{-1}\mathcal{V}(\Sigma X) \xrightarrow{w'_X} \mathcal{U}(\Sigma X). \tag{9}$$

Note that this triangle is also the canonical triangle of X with respect to the t-structure $(\Sigma^{-1}\mathcal{U}, \Sigma^{-1}\mathcal{V})$. Set $\mathcal{D} := \Sigma^{-1}\mathcal{V} \subset \mathcal{C}$, which by Lemma 5.3 satisfies property $(*)$ and thus the category $\underline{\mathcal{C}}_{\mathcal{D}}$ is a right triangulated category by Proposition 3.6. We also set $\mathcal{V}'(X) := \Sigma^{-1}\mathcal{V}(\Sigma X)$ and $J(X) := \mathcal{V}'(X) \oplus \mathcal{I}(X)$.

Lemma 5.4 *For all $X \in \mathcal{C}$, $\Sigma_{\mathcal{D}}X \cong \mathcal{U}(\Sigma X)$ in $\underline{\mathcal{C}}$.*

Proof By the definition of the functor $\Sigma_{\mathcal{D}}$ and Lemmas 2.12 and 5.3, $\Sigma_{\mathcal{D}}X$ fits into a commutative diagram of extriangles in \mathcal{C}

$$\begin{array}{ccccccc} X & \longrightarrow & \mathcal{I}(X) & \longrightarrow & \Sigma X & \dashrightarrow & \\ v'_X \downarrow & & \downarrow & & \parallel & & \\ \mathcal{V}'(X) & \longrightarrow & \Sigma_{\mathcal{D}}X & \longrightarrow & \Sigma X & \dashrightarrow & \end{array}$$

where $v'_X : X \rightarrow \mathcal{V}(X)$ fits into the triangle (9). Thus $\Sigma_{\mathcal{D}}X = \text{cone}_{\underline{\mathcal{C}}}(v'_X) \cong \mathcal{U}(\Sigma X)$ in $\underline{\mathcal{C}}$. □

Consider the diagram

$$\begin{array}{ccc}
 & \mathcal{C} & \xrightarrow{\text{can}} \mathcal{C}_{\mathcal{D}} \\
 & \text{can} \downarrow & \nearrow \exists G \\
 \mathcal{U} & \xrightarrow[\text{can}]{i} \underline{\mathcal{C}} &
 \end{array}$$

where G is the unique additive functor rendering the diagram commutative which exists since $\mathcal{I} \subseteq \mathcal{D}$. The next result finishes the proof of Theorem 5.1.

Proposition 5.5 *The composition $\psi := Gi : \mathcal{U} \rightarrow \mathcal{C}_{\mathcal{D}}$ is a full and faithful right triangle functor and induces an equivalence of right triangulated categories $\mathcal{U} \cong \Sigma_{\mathcal{D}}\mathcal{C}_{\mathcal{D}}$.*

Proof ψ is a right triangulated functor: Note that for all $U \in \mathcal{U}, \mathcal{U}(U) \cong U$ and $\mathcal{V}(U) \cong 0$ in $\underline{\mathcal{C}}$. Thus, by Lemma 5.4, there is a natural isomorphism of functors $\Sigma|_{\mathcal{U}} \cong \Sigma_{\mathcal{D}}|_{\mathcal{U}}$ and $\mathcal{I}(U) \cong J(U)$ so that (right) triangles in \mathcal{U} are also right triangles in $\mathcal{C}_{\mathcal{D}}$.

ψ is full: Follows from the fact that i and G are both full.

ψ is faithful: Let $f : X \rightarrow Y$ be a morphism in \mathcal{U} . Suppose that $\psi(f) = 0$. Then f must factor through $\mathcal{D} = \Sigma^{-1}\mathcal{V}$ in $\underline{\mathcal{C}}$. But since $\underline{\mathcal{C}}(\mathcal{U}, \mathcal{V}) = 0$ and $\Sigma^{-1}\mathcal{V} \subset \mathcal{V}$ this is only possible if $f = 0$.

$\text{Im } \psi \cong \Sigma_{\mathcal{D}}\mathcal{C}_{\mathcal{D}}$: Let $\Sigma_{\mathcal{D}}X \in \Sigma_{\mathcal{D}}\mathcal{C}_{\mathcal{D}}$, by Lemma 5.4, $\Sigma_{\mathcal{D}}X \cong \mathcal{U}(\Sigma X)$ in $\underline{\mathcal{C}}$ and hence also in $\mathcal{C}_{\mathcal{D}}$. Thus $\psi(\mathcal{U}(\Sigma X)) \cong \Sigma_{\mathcal{D}}X$. Conversely, let $Y \in \mathcal{U}$. Then $Y \cong \Sigma_{\mathcal{D}}(\Sigma^{-1}Y) \in \mathcal{C}_{\mathcal{D}}$ by Lemma 5.4. □

Corollary 5.6 $\Sigma_{\mathcal{D}}G\Sigma^{-1} : \underline{\mathcal{C}} \rightarrow \Sigma_{\mathcal{D}}\mathcal{C}_{\mathcal{D}} \cong \mathcal{U}$ is right adjoint to the inclusion $\mathcal{U} \hookrightarrow \underline{\mathcal{C}}$. In particular, $\Sigma_{\mathcal{D}}G\Sigma^{-1} \cong \mathcal{U}(-)$.

Remark 5.7 Theorem 5.1 cannot be directly dualised to give co-aisles of co-t-structures since the aisle of a co-t-structure is contravariantly finite. The author would be interested to know if there is a way to work around this problem.

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