Applied Categorical Structures (2019) 27:111–123
https://doi.org/10.1007/s10485-018-9545-z

Universal Central Extensions of Lie Crossed Modules Over a Fixed Lie Algebra

Behrouz Edalatzadeh[1]

Received: 21 May 2016 / Accepted: 22 October 2018 / Published online: 29 October 2018

Abstract
Let L be a Lie algebra over a field of arbitrary characteristic. In this paper, we give a necessary and sufficient condition for the existence of universal central extensions in the category of crossed modules of Lie algebras over L. Also, we determine the structure of the universal central extension of a crossed L-module and show that the kernel of this extension is related to the first non-abelian homology of L.

Keywords Universal central extension · Crossed L-module · Non-abelian homology

Mathematics Subject Classification 17B60 · 18G60 · 17B99

1 Introduction

Originally the concept of the universal central extension appeared in the contexts of group theory to characterize perfect groups. Analogously, the structure of perfect Lie algebras and their universal central extensions and the relation between the second homology group and the universal central extension of a perfect Lie algebra were investigated in [10,14,26]. Similar results in the category of Leibniz algebras were obtained in [7].

It was shown in [5,6] that there is a common approach to universal central extensions in the category of (all) crossed modules of Lie algebras. Many authors discussed similar concepts in the category of (all) crossed and pre-crossed modules of groups, one can see [1,2,16] for instance. Casas and Van der Linden in [8] established the general theory of universal central extensions in a semi-abelian category with respect to a Birkhoff subcategory, based on the general approach of central extensions due to Janelidze and Kelly [20]. Everaert and Gran in [12] extended some results on the low-dimensional homology and central series in quasi-pointed, exact and protomodular categories, that in particular are applied to the category of internal crossed modules over a fixed base object.

Communicated by George Janelidze.

✉ Behrouz Edalatzadeh
edalatzadeh@gmail.com

1 Department of Mathematics, Faculty of Science, Razi University, Kermanshah, Iran

In the present work, we study universal central extensions in the category of crossed L-modules over a fixed Lie algebra L, where extensions are central with respect to the subcategory of crossed L-modules with aspherical commutator submodule. It should be noted that the category of crossed modules of Lie algebras over L is not semi-abelian, simply since it is not pointed, hence the main results of this paper about universal central extensions can not be straightly obtained from [8]. However, the basic set-up of the theory fits into the framework developed in [12], where the general notion of central extension defined in [20] is studied in the context of quasi-pointed exact homological categories.

In Sect. 4, we give a construction of the universal central extension based on the non-abelian tensor product of Lie algebras. This is used in Sect. 5, where the Schur multiplier of a pair of Lie algebras is interpreted in terms of a Hopf-type formula involving a free crossed module.

2 Preliminaries

We first remind the reader of the notion of an action between Lie algebras. Let L and M be two Lie algebras over a field Λ. An action of L on M is a Λ-bilinear map $L \times M \to M$, $(l, m) \mapsto {}^{l}m$ satisfying

$$^{[l,l']}m = {}^{l}({}^{l'}m) - {}^{l'}({}^{l}m), \qquad {}^{l}[m, m'] = [{}^{l}m, m'] + [m, {}^{l}m'],$$

for all $l, l' \in L$ and $m, m' \in M$. We follow the notations and terminology of [25] for the *L–center* and *L–commutator* of M which are defined as follows:

$$Z(L, M) = \langle m \in M |\, {}^{l}m = 0, \;\text{ for all }\; l \in L\rangle,$$
$$[L, M] = \langle {}^{l}m \mid l \in L, m \in M\rangle.$$

Definition 2.1 Let L and M be Lie algebras such that L acts on M. A *crossed L-module* (M, ∂) is a Lie homomorphism $\partial : M \to L$ satisfying the following conditions:

(i) $\partial({}^{l}m) = [l, \partial(m)]$, for all $l \in L, m \in M$,
(ii) ${}^{\partial(m)}(m') = [m, m']$, for all $m, m' \in M$.

It is readily checked that the kernel of ∂ is an L-invariant ideal contained in the center of M and the image of ∂ is an ideal of L.

Example 2.2 (i) Let N be an ideal of the Lie algebra L. Then (N, i) is a crossed L-module, where $i : N \hookrightarrow L$ is the inclusion map and the action of L on N is induced by the Lie bracket of L. In particular, $\mathrm{Id} : L \to L$ and $0 : 0 \hookrightarrow L$ are crossed modules and so we can consider any Lie algebra as a crossed module in any of these ways. A subterminal object in the category of crossed L-modules is a subobject of the terminal object (L, Id); these are precisely the crossed L-modules of the form (N, i), where N is an ideal of L.
(ii) Let $\partial : M \to L$ be a surjective Lie homomorphism whose kernel lies in the center of M. If we consider the action of L on M as ${}^{l}m = [\bar{l}, m]$, where $\bar{l}$ is any element in the pre-image of l, then (M, ∂) is a crossed L-module.
(iii) Let M be an L-module and $0 : M \to L$ the zero homomorphism. Then $(M, 0)$ is a crossed L-module.

Crossed modules of Lie algebras are algebraic objects which can be viewed as a simultaneous generalization of the concepts of ideals and modules over Lie algebras. They were introduced in 1982 by Kassel and Loday [21]. Janelidze in [19] characterized internal crossed modules in a semi-abelian category in terms of internal object actions, in such a way that a category equivalence between internal crossed modules and internal categories is obtained. Of course, crossed modules of Lie algebras are an instance of this general concept.

Definition 2.3 Let (M_i, ∂_i), $i = 1, 2$ be crossed L-modules. A *morphism* $f : (M_1, \partial_1) \to (M_2, \partial_2)$ of crossed L-modules is a commutative diagram of Lie homomorphisms

$$\begin{array}{ccc} M_1 & \xrightarrow{f} & M_2 \\ & {\scriptstyle \partial_1} \searrow \quad \swarrow {\scriptstyle \partial_2} & \\ & L & \end{array}$$

such that f is an L-equivariant Lie homomorphism, i.e. $f({}^{l}m) = {}^{l}f(m)$ for all $m \in M_1$ and $l \in L$. It is worth noting that $\ker(f) \subseteq \ker(\partial_1)$ and so $(\ker(f), 0)$ and $(M_1/\ker(f), \bar{\partial}_1)$ are also crossed L-modules, where $\bar{\partial}_1$ is the morphism induced by ∂_1. It should be mentioned that it is possible that there are no morphisms between two crossed L-modules, for instance in the case that ∂ is not zero there are no morphisms from (M, ∂) to $(0, 0)$.

For a Lie algebra L, we denote the category of crossed modules of Lie algebras over L by $\mathbf{CM}(L)$. It can be seen that $\mathbf{CM}(L)$ is a subcategory of the category of all Lie crossed modules which is denoted by $\mathbf{CM}$. In fact, $\mathbf{CM}(L)$ is the fibre category over L under the functor $U : \mathbf{CM} \to \mathbf{Lie}$, which takes a crossed module to its target. Nevertheless, there are some basic differences between these categories. For instance, $\mathbf{CM}(L)$ is not a semi-abelian category, because it does not have a zero object, while $0 : 0 \to 0$ is the zero object of $\mathbf{CM}$. However, $(0, 0)$ and (L, Id) are the initial and final objects of $\mathbf{CM}(L)$ respectively and therefore $\mathbf{CM}(L)$ is a quasi-pointed category.
According to the notions of quasi-pointed categories, we say that the sequence

$$(M_1, \partial_1) \overset{f}{\rightarrowtail} (M_2, \partial_2) \overset{g}{\twoheadrightarrow} (M_3, \partial_3),$$

of crossed L-modules is *exact* if the following sequence of Lie algebras is exact

$$0 \to M_1 \overset{f}{\to} M_2 \overset{g}{\to} M_3 \to 0.$$

This means that f is the kernel of g and g is the cokernel of f. Note that, in this condition $\partial_1 = \partial_3 \circ (g \circ f) = 0$, hence for any $m_1, m_1' \in M_1$ we have $[m_1, m_1'] = {}^{\partial_1(m_1)}m_1' = 0$ and so M_1 must be an abelian Lie algebra.

Let $\mathbf{AC\text{-}CM}(L)$ be the full subcategory of $\mathbf{CM}(L)$ consisting of crossed L-modules of Lie algebras (M, ∂) such that the commutator submodule $([L, M], \partial_{|[L,M]})$ is aspherical, i.e. $\ker(\partial_{|[L,M]}) = 0$. The inclusion of $\mathbf{AC\text{-}CM}(L)$ in $\mathbf{CM}(L)$ has a left adjoint $F : \mathbf{CM}(L) \to \mathbf{AC\text{-}CM}(L)$ which assigns to a crossed L-module (M, ∂) the quotient $(M/(\ker(\partial) \cap [L, M]), \bar{\partial})$, where $\bar{\partial} : M/(\ker(\partial) \cap [L, M]) \to L$ and the action of L on $M/(\ker(\partial) \cap [L, M])$ are induced by (M, ∂). Note that F preserves the initial object $(0, 0)$.

Recall from [13] that a functor F between protomodular (pointed) categories is called *protoadditive* if it preserves split short exact sequences. Although $\mathbf{CM}(L)$ is not pointed, we may employ this notion for our further considerations.

Proposition 2.4 *The reflector F is protoadditive.*

Proof Suppose

$$(K, 0) \longrightarrow (M_1, \partial_1) \underset{s}{\overset{f}{\rightleftarrows}} (M, \partial)$$

is a split short exact sequence of crossed L-modules. For any $m_1 \in M_1$ there exists $k \in K$ and $m \in M$ such that $m_1 = k + s(m)$, hence for all $l \in L$

$$f({}^l m_1) = {}^l f(k) + {}^l f(s(m)) = {}^l m.$$

This implies that ${}^l m = 0$ if and only if ${}^l m_1 \in \ker(f)$. Also, we have

$${}^l m_1 = {}^l k + s({}^l m),$$

which deduces ${}^l m_1 \in \ker(f)$ if and only if ${}^l m_1 \in [L, K]$. This means that by applying the commutator $[-, L]$, we get the split exact sequence of Lie algebras

$$[L, K] \longrightarrow [L, M_1] \overset{[1_L, f]}{\rightleftarrows} [L, M].$$

This also implies that

$$[L, K] \cap K \longrightarrow [L, M_1] \cap \ker(\partial_1) \overset{[1_L, f]}{\rightleftarrows} [L, M] \cap \ker(\partial),$$

is a short exact sequence, which (via the 3×3 Lemma) proves that

$$F(K, 0) \longrightarrow F(M_1, \partial_1) \underset{Fs}{\overset{Ff}{\rightleftarrows}} F(M, \partial),$$

is a split short exact sequence. □

Now, we define the concept of a central extension in $\mathbf{CM}(L)$ that is determined by the adjunction

$$\mathbf{CM}(L) \overset{F}{\rightleftarrows} \mathbf{AC\text{-}CM}(L),$$

Our definition of central extension fits into Janelidze and Kelly's theory of central extensions [20].

For a given extension of crossed L-modules

$$(K, 0) \rightarrowtail (M_1, \partial_1) \overset{f}{\twoheadrightarrow} (M, \partial), \qquad ((*))$$

suppose $\pi_1, \pi_2 : R[f] \to (M_1, \partial_1)$ are the kernel pair projections and $s : R[f] \to (M_1, \partial_1)$ the subdiagonal morphism. The extension $(*)$ is central if π_1 is trivial, i.e. in the following diagram the right-hand side square is pullback

$$\begin{array}{ccccc} (K, 0) & \longrightarrow & R[f] & \underset{s}{\overset{\pi_1}{\rightleftarrows}} & (M_1, \partial_1) \\ \downarrow{\scriptstyle \eta_{(K,0)}} & & \downarrow & & \downarrow \\ F(K, 0) & \longrightarrow & FR[f] & \underset{Fs}{\overset{F\pi_1}{\rightleftarrows}} & F(M_1, \partial_1) \end{array}$$

where η is the unit of the adjunction. Since F is protoadditive by [4, Lemma 1, Proposition 7] $\eta_{(K,0)}$ is an isomorphism if and only if the right-hand side square is pullback. This means that $(K, 0)$ is in $\mathbf{AC\text{-}CM}(L)$ if and only if $(*)$ is a central extension. Now, we are ready to summarize the above discussions in the following proposition.

Proposition 2.5 *A surjective morphism of crossed L-modules $f : (M_1, \partial_1) \twoheadrightarrow (M, \partial)$ is a central extension of (M, ∂) if and only if* $\ker(f) \subseteq Z(L, M_1)$. *This is equivalent to* $[\ker(f), M_1] = 0$.

Let $\mathbf{Set}(L)$ be the category of set mappings with target L and its morphisms are all maps of sets such that the corresponding triangles are commutative. For any object $f : X \to L$ of the category $\mathbf{Set}(L)$, let $A(X)$ be the free Λ-module on the set X, UL the universal enveloping algebra of L and $\mathcal{L}(UL \otimes_\Lambda A(X))$ the free Lie algebra generated by $UL \otimes_\Lambda A(X)$. The action of L on $UL \otimes_\Lambda A(X)$ induces an L-module structure on $\mathcal{L}(UL \otimes_\Lambda A(X))$. Using the fact that L is a UL-module, the L-homomorphism

$$\bar{f} : UL \otimes_\Lambda A(X) \to L$$
$$a \otimes x \mapsto a \cdot f(x)$$

induces an L-equivariant homomorphism $\bar{\partial} : \mathcal{L}(UL \otimes_\Lambda A(X)) \to L$. Let I be tha ideal of $\mathcal{L}(UL \otimes_\Lambda A(X))$ generated by the elements $[x, y] - {}^{\bar{\partial}(x)}y$. Since I is L-invariant, if we set $\mathcal{C}(X) = \mathcal{L}(UL \otimes_\Lambda A(X))/I$ then the induced homomorphism $\partial : \mathcal{C}(X) \to L$ is a crossed L-module. The forgetful functor $\mathcal{U} : \mathbf{CM}(L) \to \mathbf{Set}(L)$ has has a left adjoint functor $\mathcal{F}$, where $\mathcal{F}(X \xrightarrow{f} L) = (\mathcal{C}(X), \partial)$, see [10] for more details.

Projective objects with respect to the regular epimorphisms (= surjective morphisms) in $\mathbf{CM}(L)$ are called *projective crossed L-modules*. The construction of the functor $\mathcal{F}$ ensures that $\mathbf{CM}(L)$ has enough projective objects, i.e. for each crossed L-module (M, ∂) there exists a surjection $f : (P, \delta) \twoheadrightarrow (M, \partial)$ such that (P, δ) is a projective crossed L-module. By a *projective presentation* of (M, ∂), we mean an exact sequence of crossed L-modules

$$(\ker(f), 0) \rightarrowtail (P, \delta) \overset{f}{\twoheadrightarrow} (M, \partial),$$

such that (P, δ) is a projective crossed L-module.

3 Universal Central Extensions

A Lie algebra is called perfect if it coincides with its derived subalgebra. In [14], it was shown that any perfect Lie algebra L admits a universal central extension, i.e. the (unique) central extension of Lie algebras

$$0 \to M \to L^* \xrightarrow{\pi} L \to 0,$$

such that for any central extension

$$0 \to M_1 \to K_1 \xrightarrow{\pi_1} L \to 0,$$

there exists a unique Lie homomorphism $f : L^* \to K_1$ such that $\pi_1 \circ f = \pi$. In this section, we generalize these concepts to the central extensions of crossed L-modules. Note that, by considering L as a crossed L-module in the usual form of (L, Id), the results of this section can generalize the main results of [26].

Definition 3.1 Let $E : (M_1, \partial_1) \overset{f_1}{\twoheadrightarrow} (M, \partial)$ and $E' : (M_2, \partial_2) \overset{f_2}{\twoheadrightarrow} (M, \partial)$ be two central extensions of the crossed L-module (M, ∂). Then we say that the extension E *covers* (uniquely covers) E' if there exists a morphism (or a unique morphism, respectively) $\phi : (M_1, \partial_1) \to (M_2, \partial_2)$ such that the following diagram commutes:

$$\begin{array}{ccc} E:(M_1,\partial_1) & \xrightarrow{f_1} & (M,\partial) \\ \Big\downarrow \phi & & \Big\downarrow \mathrm{Id} \\ E':(M_2,\partial_2) & \xrightarrow{f_2} & (M,\partial) \end{array}$$

A central extension E of the crossed L-module (M, ∂) is called *universal* if it uniquely covers any central extension of (M, ∂). On the other hand, the universal central extension of (M, ∂) is the initial object amongst all central extensions of (M, ∂).

The next proposition follows immediately from the fact that an initial object is unique up to isomorphism.

Proposition 3.2 *If E and E' are universal central extensions of (M, ∂) then there is an isomorphism $(M_1, \partial_1) \to (M_2, \partial_2)$. In other words, when it exists, the universal central extension of (M, ∂) is unique up to isomorphism of crossed L-modules.*

As we shall see, a crossed L-module (M, ∂) admits a universal central extension if and only if $[L, M] = M$. Just as in the characterisation of central extensions, this condition only depends on the action of L on M: this action must be perfect with respect to the coinvariants reflector. Reformulated in terms of the reflector $F : \mathbf{CM}(L) \to \mathbf{AC}\text{-}\mathbf{CM}(L)$, this means that $F(M, \partial)$ is a subterminal object (N, i) such that $[L, N] = N$. Let indeed (N, i) denote the support of (M, ∂), i.e., $i : N \to L$ is the monomorphism (and $p : M \to N$ the regular epimorphism) in the regular epi-mono factorisation $\partial = i \circ p$ of $\partial : M \to L$. Then in the diagram with short exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K\cap[L,M] & \longrightarrow & [L,M] & \longrightarrow & [L,N] & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & K & \longrightarrow & M & \xrightarrow{p} & N & \longrightarrow & 0 \end{array}$$

where $K = \ker(\partial) = \ker(p)$, the middle vertical arrow is an isomorphism if and only if so are the outer arrows. For the one on the left this means that $K \cong K \cap [L, M]$, which is equivalent to the condition that $F(M, \partial)$ is the support (N, i) of (M, ∂). In other words, $F(M, \partial)$ is subterminal. The isomorphism $[L, N] \cong N$ on the right says that the conjugation action of L on N is perfect with respect to taking coinvariants.

Lemma 3.3 *Let (M, ∂) be a crossed L-module and $E : (M_1, \partial_1) \overset{f_1}{\twoheadrightarrow} (M, \partial)$, $E' : (M_2, \partial_2) \overset{f_2}{\twoheadrightarrow} (M, \partial)$ be central extensions of (M, ∂). The following statements hold:*

(i) If E is a universal central extension then (M, ∂) and (M_1, ∂_1) are L-perfect.
(ii) If (M_1, ∂_1) is L-perfect then E covers E' if and only if E uniquely covers E'.
(iii) If E is a universal central extension then any central extension of (M_1, ∂_1) splits.
(iv) If (M_1, ∂_1) is L-perfect and any central extension of (M_1, ∂_1) splits then E is a universal central extension.

Proof (i) Let $\delta : M_1 \oplus \frac{M_1}{[L,M_1]} \to L$ be the Lie homomorphism defined by $\delta(m, m' + [L, M_1]) = \partial_1(m)$ for all $m, m' \in M_1$. If we define the action of L on $M_1 \oplus \frac{M_1}{[L,M_1]}$ as

$${}^{l}(m, m' + [L, M_1]) = ({}^{l}m, 0),$$

then $(M_1 \oplus \frac{M_1}{[L,M_1]}, \delta)$ is a crossed L-module. Now, consider the following central extension

$$E'' : \left(M_1 \oplus \frac{M_1}{[L, M_1]}, \delta\right) \overset{\psi}{\twoheadrightarrow} (M, \partial),$$

where $\psi(m, m' + [L, M_1]) = f(m)$. Suppose for $i = 1, 2$,

$$\phi_i : (M_1, \partial_1) \to \left(M_1 \oplus \frac{M_1}{[L, M_1]}, \delta\right)$$

are morphisms defined by $\phi_1(m) = (m, 0 + [L, M_1])$ and $\phi_2(m) = (m, m + [L, M_1])$ for all $m \in M_1$. Hence $\psi \circ \phi_1 = \psi \circ \phi_2 = f_1$, so by the universal property of E, we conclude that $\phi_1 = \phi_2$. It follows that $M_1 = [L, M_1]$. Also, f_1 is a surjection and so we have

$$M = f_1(M_1) = f_1([L, M_1]) = [L, f_1(M_1)] = [L, M].$$

(ii) Suppose $\phi_i : M_1 \to M_2$ $(i = 1, 2)$ are morphisms such that $f_2 \circ \phi_1 = f_1 = f_2 \circ \phi_2$. Let $\phi : M_1 \to M_2$ be the linear map defined by $\phi(m) = \phi_1(m) - \phi_2(m)$ for all $m \in M_1$. Hence $f_2(\phi(M_1)) = 0$ so $\phi(M_1) \subseteq Z(L, M_2)$ and

$$\phi(M_1) = \phi([L, M_1]) \subseteq [L, \phi(M_1)] = 0,$$

which implies $\phi_1 = \phi_2$.
(iii) Let $E'' : (M_3, \partial_3) \overset{f_3}{\twoheadrightarrow} (M_1, \partial_1)$ be any central extension. Then it can be easily checked that $f_1 \circ f_3 : (M_3, \partial_3) \twoheadrightarrow (M, \partial)$ is a central extension of (M, ∂). Since E is universal, there exists a morphism $g : (M_1, \partial_1) \to (M_3, \partial_3)$ such that $f_3 \circ g = \mathrm{Id}_{M_1}$, this yields the desired result.
(iv) By (ii), it is enough to prove E covers E'. Put

$$M_3 = \{(m_1, m_2) \in M_1 \oplus M_2 \mid f_1(m_1) = f_2(m_2)\},$$

and $\partial_3(m_1, m_2) = \partial_1(m_1)$ then (M_3, ∂_3) is a crossed L-module and the natural projection $(M_3, \partial_3) \overset{\pi_1}{\twoheadrightarrow} (M_1, \partial_1)$ is a central extension. By the hypothesis, there exists a morphism $g : (M_1, \partial_1) \to (M_3, \partial_3)$ and so there is a morphism $\pi_2 \circ g : (M_1, \partial_1) \to (M_2, \partial_2)$, hence E covers E'. □

The structure of (weakly) universal central extensions in a semi-abelian category with enough projectives was detected in [8], that is far from our case of $\mathbf{CM}(L)$ which is quasi-pointed. In the following theorem, we use a projective presentation to construct the universal central extension of (M, ∂).

Theorem 3.4 *Let (M, ∂) be an L-perfect crossed L-module. If*

$$(R, 0) \rightarrowtail (P, \delta) \overset{\beta}{\twoheadrightarrow} (M, \partial)$$

is a projective presentation of (M, ∂) then

$$E : \left(\frac{[L, P]}{[L, R]}, \bar{\delta}\right) \twoheadrightarrow (M, \partial),$$

is the universal central extension of (M, ∂).

Proof Let $(M_1, \partial_1) \overset{f_1}{\twoheadrightarrow} (M, \partial)$ be a central extension of the crossed L-module (M, ∂). As (P, δ) is projective, there is a morphism $\phi_1 : (P, \delta) \twoheadrightarrow (M_1, \partial_1)$ such that the following diagram commutes:

$$\begin{array}{ccccc} (R,0) & \rightarrowtail & (P,\delta) & \xrightarrow{\beta} & (M,\partial) \\ \downarrow & & \downarrow{\scriptstyle \phi_1} & & \downarrow{\scriptstyle \mathrm{Id}} \\ (\ker(f_1),0) & \rightarrowtail & (M_1,\partial_1) & \xrightarrow{f_1} & (M,\partial) \end{array}$$

$[L, R] \subseteq \ker(\delta) \cap \ker(\beta) \cap \ker(\phi_1)$, so β induces the central extension $\beta' : (\frac{P}{[L,R]}, \bar{\delta}) \twoheadrightarrow (M, \partial)$ and ϕ_1 induces the morphism $\phi_2 : (\frac{P}{[L,R]}, \bar{\delta}) \to (M_1, \partial_1)$ such that the following diagram commutes:

$$\begin{array}{ccc} (\frac{P}{[L,R]},\bar{\delta}) & \xrightarrow{\beta} & (M,\partial) \\ \downarrow{\scriptstyle \phi_2} & & \downarrow{\scriptstyle \mathrm{Id}} \\ (M_1,\partial_1) & \xrightarrow{f_1} & (M,\partial) \end{array}$$

Now, we can obtain a morphism $\phi : (\frac{[L,P]}{[L,R]}, \bar{\delta}) \to (M_1, \partial_1)$ by composing of the inclusion morphism $i : (\frac{[L,P]}{[L,R]}, \bar{\delta}) \to (\frac{P}{[L,R]}, \bar{\delta})$ with ϕ_2. To show that ϕ is unique, by Lemma 3.3 (ii), it is sufficient to verify $[L, [L, P]] + [L, R] = [L, P]$, i.e. that $[L, \frac{[L,P]}{[L,R]}] = \frac{[L,P]}{[L,R]}$. The inclusion $[L, [L, P]] + [L, R] \subseteq [L, P]$ is obvious. For the converse,

$$\frac{P}{R} = M = [L, M] \cong [L, \frac{P}{R}] = \frac{[L, P] + R}{R}$$

so any element $p \in P$ can be written as $p = {}^{l'}p' + r$ for some $p' \in P, l' \in L, r \in R$. Obviously, we have

$${}^{l}p = {}^{l}({}^{l'}p' + r) = {}^{l}({}^{l'}p') + {}^{l}r \in [L, [L, P]] + [L, R],$$

which implies the required assertion. □

Example 3.5 (i) Let L be an arbitrary Lie algebra. The inclusion crossed L-module $(0, i)$ admits the universal central extension

$$(0, i) \twoheadrightarrow (0, i).$$

(ii) Let L be a perfect Lie algebra and $0 \to M \to L^* \xrightarrow{\pi} L \to 0$ be the universal central extension of L. Using Example 2.2 (ii), we can see that

$$(L^*, \pi) \twoheadrightarrow (L, \mathrm{Id}),$$

is the universal central extension of the crossed L-module (L, Id).

A crossed module of Lie algebras $\partial : M \to L$ is said to be abelian (respectively, perfect) if M and L are abelian Lie algebras and L acts trivialy on M (respectively, L is a perfect Lie algebra and M is L-perfect). The category of abelian crossed modules, **ACM**, is a subcategory of **CM**. The inclusion of abelian crossed modules in **CM** admits a left adjoint ab : **CM** $\to$ **ACM** which assigns to a crossed module $\partial : M \to L$ the abelian crossed module $\bar{\partial} : M/[L, M] \to L/[L, L]$. Casas and Ladra in [6] considered the universal central extensions of **CM** with respect to this adjunction. They proved that a crossed module $\partial : M \to L$ has a universal central extension in **CM** if and only if $\partial : M \to L$ is a perfect crossed module. It should be mentioned that if L is not a perfect Lie algebra then $i : 0 \to L$ is not perfect in **CM** and so it does not admit any universal central extensions in the category of crossed modules, but as mentioned in Example 3.5, $(0, i)$ has a universal central extension in $\mathbf{CM}(L)$.

4 Relation to the Non-abelian Tensor Product

In this section, we construct the universal central extension of a crossed L-module in terms of non-abelian tensor product of Lie algebras. At first, let us recall from [10] the definition and basic properties of the non-abelian tensor product of Lie algebras. Suppose M_1 and M_2 are Lie algebras that act on each other. The *non-abelian tensor product* $M_1 \otimes M_2$ is the algebra generated by the elements $m_1 \otimes m_2$ with $(m_1, m_2) \in M_1 \times M_2$, subject to the relations

$$c(m_1 \otimes m_2) = cm_1 \otimes m_2 = m_1 \otimes cm_2, \qquad [m_1, m_1'] \otimes m_2 = m_1 \otimes ({}^{m_1'}m_2) - m_1' \otimes ({}^{m_1}m_2)$$
$$(m_1 + m_1') \otimes m_2 = m_1 \otimes m_2 + m_1' \otimes m_2, \qquad m_1 \otimes [m_2, m_2'] = ({}^{m_2'}m_1) \otimes m_1 - ({}^{m_2}m_1) \otimes m_2'$$
$$m_1 \otimes (m_2 + m_2') = m_1 \otimes m_2 + m_1 \otimes m_2', \qquad [(m_1 \otimes m_2), (m_1' \otimes m_2')] = -({}^{m_2}m_1) \otimes ({}^{m_1'}m_2').$$

for all $m_1, m_1' \in M_1, m_2, m_2' \in M_2$ and scalar $c \in \Lambda$.
For any Lie algebra T, a bilinear map $\bar{f} : M_1 \oplus M_2 \to T$ is called a Lie pairing if for any $m_1, m_1' \in M_1, m_2, m_2' \in M_2$

$$\bar{f}([m_1, m_1'], m_2) = \bar{f}(m_1, {}^{m_1'}m_2) - \bar{f}(m_1', {}^{m_1}m_2),$$
$$\bar{f}(m_1, [m_2, m_2']) = \bar{f}({}^{m_2'}m_1, m_1) - \bar{f}({}^{m_2}m_1, m_2'),$$
$$\bar{f}([{}^{m_2}m_1, {}^{m_1'}m_2']) = -[\bar{f}(m_1, m_2), \bar{f}(m_1', m_2')].$$

The non-abelian tensor product $M_1 \otimes M_2$ can be identified by its universal property; given a Lie algebra T and a Lie pairing $\bar{f} : M_1 \oplus M_2 \to T$, there is a unique Lie homomorphism $f : M_1 \otimes M_2 \to T$ such that $\bar{f}(m_1, m_2) = f(m_1 \otimes m_2)$. Note that, if M_1 and M_2 act on each other trivially then the non abelian tensor product $M_1 \otimes M_2$ is isomorphic to the usual tensor product of Λ-modules $M_1/[M_1, M_1] \otimes_\Lambda M_2/[M_2, M_2]$. See [10] for more details.

Let (M_1, ∂_1) and (M_2, ∂_2) be crossed L-modules. Then M_1 and M_2 act on each other by the action of L via ∂_1 and ∂_2, i.e. for any $m_1 \in M_1$ and $m_2 \in M_2$

$${}^{m_1}m_2 = {}^{\partial(m_1)}m_2 \quad , \qquad {}^{m_2}m_1 = {}^{\partial(m_2)}m_1.$$

In this case, L acts on $M_1 \otimes M_2$ by the rule

$${}^{l}(m_1 \otimes m_2) = {}^{l}m_1 \otimes m_2 + m_1 \otimes {}^{l}m_2,$$

for all $m_1 \in M_1, m_2 \in M_2$ and $l \in L$. Using the action of L on $M_1 \otimes M_2$, the following proposition constructs the crossed L-module $[\partial_1, \partial_2] : M_1 \otimes M_2 \to L$.

Proposition 4.1 *Let (M_1, ∂_1) and (M_2, ∂_2) be crossed L-modules. Then $(M_1 \otimes M_2, [\partial_1, \partial_2])$ is a crossed L-module, where $[\partial_1, \partial_2]$ is the commutator map which is defined on generators by $[\partial_1, \partial_2](m_1 \otimes m_2) = [\partial_1(m_1), \partial_2(m_2)]$ for all $m_1 \in M_1, m_2 \in M_2$.*

Let (M, ∂) be a crossed L-module. One can check that $\delta_M : L \otimes M \to M$ defined on generators by $\delta_M(l \otimes m) = {}^{l}m$ is an L-equivariant Lie homomorphism, where the action of L on itself is given by the adjoint map. In this case, the cokernel and the kernel of δ_M are called *the zeroth* and *the first non-abelian homology groups of the Lie algebra L with*

coefficients in the crossed L-module (M, ∂) and are denoted by $H_0(L, M)$ and $H_1(L, M)$, respectively. Guin in [17] introduced these low degree non-abelian homology modules. The higher degree non-abelian homology modules were introduced in [18] by Inassaridze et al., as the non-abelian left derived functors of the non-abelian tensor product.
The following theorem determines the kernel of a universal central extension in terms of the non-abelian homology of L.

Theorem 4.2 *Let* (M, ∂) *be an* L*-perfect crossed* L*-module. The sequence*

$$(H_1(L, M), 0) \rightarrowtail (L \otimes M, [\mathrm{Id}, \partial]) \overset{\delta_M}{\twoheadrightarrow} (M, \partial),$$

is the universal central extension of (M, ∂).

Proof Suppose $f : (M_1, \partial_1) \twoheadrightarrow (M, \partial)$ is a central extension of (M, ∂). For any $m \in M$ choose $\bar{m} \in M_1$ be such that $f(\bar{m}) = m$. Clearly, the mapping $(l, m) \mapsto {}^{l}\bar{m}$ is a well-defined Lie pairing and so by the universal property of the non-abelian tensor product there exists a Lie homomorphism $\phi : L \otimes M \to M_1$ such that $\phi(l \otimes m) = {}^{l}\bar{m}$. ϕ is an L-equivariant Lie homomorphism because for all $m \in M$ and $l_1, l_2 \in L$

$$\begin{aligned} \phi({}^{l_1}(l_2 \otimes m)) &= \phi([l_1, l_2] \otimes m + l_2 \otimes {}^{l_1}m) \\ &= {}^{[l_1,l_2]}\bar{m} + {}^{l_2}({}^{l_1}\bar{m}) \\ &= {}^{l_1}({}^{l_2}\bar{m}) \\ &= {}^{l_1}\phi(l_2 \otimes m). \end{aligned}$$

Clearly $f \circ \phi = \delta_M$ and $\partial_1 \circ \phi = [\mathrm{Id}, \partial]$, so ϕ is a morphism of crossed L-modules. To prove that ϕ is unique, by Lemma 3.3(ii), it is enough to show that $[L, L \otimes M] = L \otimes M$. But for any $l_1, l_2 \in L$ and $m \in M$, we have

$$l_1 \otimes ({}^{l_2}m) = {}^{l_1}(l_2 \otimes m),$$

hence the equality $[L, M] = M$ implies that $L \otimes M \subseteq [L, L \otimes M]$. The converse is trivial. □

Remark 4.3 A short exact sequence of L-modules $M_1 \rightarrowtail M_2 \twoheadrightarrow M$ is called an L*-central extension* of M whenever L acts on M_1 trivially. Analogously as [3,23] (in Lie algebras), one can deduce that M admits a universal central extension in the category of L-modules if and only if $[L, M] = M$. Note that, since the category of L-modules is a semi-abelian category, by considering central extensions and perfect objects with respect to the Birkhoff subcategory of L-module with trivial action, one can obtain this assertion directly by applying the results of [8]. In this case, $K \rightarrowtail IL \otimes_{UL} M \twoheadrightarrow M$ is the universal central extension of M, where K is isomorphic to the first (Chevalley-Eilenberg) homology of L with coefficients in M and UL, IL are the universal enveloping algebra and the augmentation ideal of L, respectively. As any L-module M can be considered as the crossed L-module $(M, 0)$, Theorem 4.2 implies that if $[L, M] = M$ then $IL \otimes_{UL} M \cong L \otimes M$. See also [5, Proposition 13] for a similar result.

5 An Application to a Pair of Lie Algebras

Let L be a Lie algebra over an arbitrary field Λ and $H_n(L)$ be the n-th homology group of L with coefficients in Λ. If N is an ideal of L then there exists a long exact sequence

$$\cdots \to H_n(L; N) \to H_n(L) \to H_n\left(\frac{L}{N}\right) \to H_{n-1}(L; N) \to \cdots$$
$$\to H_2(L; N) \to H_2(L) \to H_2\left(\frac{L}{N}\right) \to H_1(L; N) \to H_1(L) \to H_1\left(\frac{L}{N}\right) \to 0.$$

Further details about the construction of this sequence in general case for semi-abelian categories can be found in [11], [15, Theorem 2.6]. The abelian Lie algebra $H_2(L; N)$ is called the *Schur multiplier of the pair* (L, N) and is denoted by $M(L, N)$. A central extension of the inclusion crossed module (N, i) is called a *relative central extension* of the pair of Lie algebras (L, N). The theory of relative central extensions and their connections with the Schur multiplier of a pair of Lie algebras have been studied in several papers and some of the known notions in the contexts of central extensions have been generalized, see [9,24] for instance. A relative central extension of which the kernel is the abelian crossed L-module $(M(L, N), 0)$ is called a *relative cover* of (L, N). Using Theorem 4.2, we can obtain the following result which determines the relative cover of a special pair of Lie algebras.

Proposition 5.1 *Let* (L, N) *be a pair of Lie algebras such that* $[L, N] = N$ *then*

$$(M(L, N), 0) \rightarrowtail (L \otimes N, [\mathrm{Id}, i]) \overset{[\mathrm{Id}, i]}{\twoheadrightarrow} (N, i),$$

is the universal central extension of the inclusion crossed module (N, i).

Proof It is readily checked that the subspace $L\square N$ of $L \otimes N$ generated by the elements $n \otimes n$ for all $n \in N$ is a central ideal of $L \otimes N$. There is a natural exact sequence of Lie algebras

$$\Gamma\left(\frac{N}{[L, N]}\right) \to L \otimes N \to L \wedge N \to 0,$$

where $\Gamma(-)$ is the universal quadratic functor and $L \wedge N = L \otimes N / L\square N$, see [10] for more details. But $N = [L, N]$ implies that $L \otimes N \cong L \wedge N$. Now, by [10, Theorem 35], there is an isomorphism

$$M(L, N) \cong \ker(L \otimes N \overset{[\mathrm{Id}, i]}{\to} L).$$

This completes the proof. □

Example 5.2 Let L be a two dimensional Lie algebra with a basis $\{x, y\}$ and the multiplication given by $[x, y] = x$ and $N = \langle x \rangle$. It can be checked that $L \otimes N$ is the two dimensional abelian Lie algebra generated by $\{a = x \otimes x, b = y \otimes x\}$ and L acts on the generators of $L \otimes N$ by

$$^{x}a = 0 \quad , \quad ^{x}b = a$$
$$^{y}a = -2a \quad , \quad ^{y}b = -b.$$

Hence $M(L, N)$ is the one dimensional ideal generated by the set $\{x \otimes x\}$.

We recall from [9, Theorem 24] that if N is an ideal of L and (P, δ) is a projective crossed L-module with $\delta(P) = N$ then

$$H_2(N) \cong \ker(\delta) \cap [P, P],$$

that can be seen as a crossed version of the Hopf formula

$$H_2(N) \cong \frac{R \cap [F, F]}{[F, R]},$$

where $0 \to R \to F \to N \to 0$ is a free presentation of N. Now, we can give a similar result for the Schur of a pair of Lie algebras in terms of a projective presentation.

Theorem 5.3 *Let (L, N) be a pair of Lie algebras such that $[L, N] = N$ and (P, δ) any projective crossed L-module such that $\delta(P) = N$. Then $M(L, N) \cong \ker(\delta) \cap [L, P]$.*

Proof It is obvious that

$$(\ker(\delta), 0) \rightarrowtail (P, \delta) \twoheadrightarrow (N, i),$$

is a projective presentation of the inclusion crossed L-module (N, i). By Theorem 3.4

$$\left(\frac{\ker(\delta) \cap [L, P]}{[L, \ker(\delta)]}, 0\right) \rightarrowtail \left(\frac{P}{[L, \ker(\delta)]}, \bar{\delta}\right) \twoheadrightarrow (N, i),$$

is the universal central extension (N, i). But $\ker(\delta) \subseteq Z(L, P)$ and so we have

$$\frac{\ker(\delta) \cap [L, P]}{[L, \ker(\delta)]} = \ker(\delta) \cap [L, P].$$

Now, Proposition 5.1 gives the result. □

Remark 5.4 Let $\mu : L \to Q$ be a surjective Lie algebra morphism. A *relative central extension* of μ by an abelian Lie algebra A is an exact sequence

$$0 \to A \to M \xrightarrow{\partial} L \xrightarrow{\mu} Q \to 0,$$

such that (M, ∂) is a crossed L-module and $A \subseteq Z(L, M)$. A morphism between two relative central extensions $0 \to A_1 \to M_1 \xrightarrow{\partial_1} L \xrightarrow{\mu} Q \to 0$ and $0 \to A_2 \to M_2 \xrightarrow{\partial_2} L \xrightarrow{\mu} Q \to 0$, is a morphism $f : (M_1, \partial_1) \to (M_2, \partial_2)$ of crossed L-modules. A relative central extension $0 \to A \to M \xrightarrow{\partial} L \xrightarrow{\mu} Q \to 0$, is called *universal* if there exists a unique morphism from it to other relative central extensions of μ. A Lie version of results in [22] implies that μ admits a universal relative central extension if and only if $[L, \ker \mu] = \ker \mu$. Note that, we can obtain this result by applying Lemma 3.3 to the crossed L-module $(\ker \mu, i)$, where i is the inclusion map.

Acknowledgements The author is greatly indebted to the referee, whose valuable criticisms and suggestions led me to rearrange the paper.

References

1. Arias, D., Casas, J.M., Ladra, M.: On universal central extensions of precrossed and crossed modules. J. Pure Appl. Algebra **210**, 177–191 (2007)
2. Arias, D., Ladra, M., Grandjeán, A.R.: Universal central extensions of precrossed modules and Milnor's relative K_2. J. Pure Appl. Algebra **184**, 139–154 (2003)
3. Blanc, D., Peschke, G.: The plus construction, Postnikov towers and universal central module extensions. Israel J. Math. **132**, 109–123 (2002)
4. Bourn, D.: 3×3 Lemma and protomodularity. J. Algebra **236**, 778–795 (2001)
5. Casas, J.M.: Universal central extension and the second invariant of homology of crossed modules in Lie algebras. Commun. Algebra **27**, 3811–3821 (1999)

6. Casas, J.M., Ladra, M.: Perfect crossed modules in Lie algebras. Commun. Algebra **23**(5), 1625–1644 (1995)
7. Casas, J.M., Ladra, M.: Stem extensions and stem covers of Leibniz algebras. Georgian. Math. J. **9**(4), 659–669 (2002)
8. Casas, J.M., Van der Linden, T.: Universal central extensions in semi-abelian categories. Appl. Categ. Struct. **22**, 253–268 (2014)
9. Edalatzadeh, B.: Some notes on the Schur multiplier of a pair of Lie algebras. J. Lie Theory **23**, 483–492 (2013)
10. Ellis, G.: A non-abelian tensor product of Lie algebras. Glasgow Math. J. **39**, 101–120 (1991)
11. Everaert, T.: An approach to non-abelian homology based on Categorical Galois Theory. Ph.D. thesis, Vrije Universiteit Brussel (2007)
12. Everaert, T., Gran, M.: On low-dimensional homology in categories. Homol. Homot. Appl. **9**, 275–293 (2007)
13. Everaert, T., Gran, M.: Protoadditive functors, derived torsion theories and homology. J. Pure Appl. Algebra **219**, 36293676 (2015)
14. Garland, H.: The arithmetic theory of loop groups. Inst. Ht. Études Sci. Publ. Math. **52**, 5–136 (1980)
15. Goedecke, J., Van der Linden, T.: On satellites in semi-abelian categories: Homology without projectives. Math. Proc. Camb. Philos. Soc. **147**, 629–657 (2009)
16. Grandjeán, A.R., Ladra, M.: $H_2(T, G, \partial)$ and central extensions for crossed modules. Proc. Edinb. Math. Soc. **42**(1), 169–177 (1999)
17. Guin, D.: Cohomologie des algèbres de Lie croisées et K-théorie de Milnor additive. Ann. Inst. Fourier **45**, 93–118 (1995). Grenoble
18. Inassaridze, N., Khmaladze, E., Ladra, M.: Non-abelian (co)homology of Lie algebras. Glasgow Math. J. **46**, 417–429 (2004)
19. Janelidze, G.: Internal crossed modules. Georgian Math. J. **10**, 99–114 (2003)
20. Janelidze, G., Kelly, G.M.: Galois theory and a general notion of central extension. J. Pure Appl. Algebra **97**, 135–161 (1994)
21. Kassel, C., Loday, J.L.: Extensions centrales d'algèbres de Lie. Ann. Inst. Fourier **32**, 119–142 (1982). Grenoble
22. Loday, J.-L.: Cohomologie et groups de Steinberg relatifs. J. Algebra **54**, 178–202 (1978)
23. Meier, W., Strebel, R.: Homotopy groups of acyclic spaces. Q. J. Math. Oxford Ser. **32**(2), 81–95 (1981)
24. Moghadam, M.R.R., Parvaneh, F.: On the isoclinism of a pair of Lie algebras and factor sets. Asian Eur. J. Math. **2**(2), 213–225 (2009)
25. Saeedi, F., Salemkar, A.R., Edalatzadeh, B.: The commutator subalgebra and Schur multiplier of a pair of nilpotent Lie algebras. J. Lie Theory **21**, 491–498 (2011)
26. Salemkar, A.R., Edalatzadeh, B., Mohammadzadeh, H.: On covers of perfect Lie algebras. Algebra Colloq. **18**, 419–427 (2011)