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Remarks on Units of Skew Monoidal Categories

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Abstract This article shows that the axioms of a skew monoidal category are independent and that its unit is unique up to a unique isomorphism together with an analogue of this result for monoidal functors between skew monoidal categories. It is also noted that these results carry over to skew monoidales before some benefits of certain extra structure on the unit maps of a skew monoidal category are discussed.

Keywords Skew · Monoidal categories · Units

1 Introduction

Generalisations of the notion of monoidal category have been studied almost as long as the notion itself; one involves relaxing the invertibility of the maps expressing the associativity and unit conditions. Once invertibility is dropped then what is required is that the directions of these constraints be specified; one such choice leads to the notion of skew monoidal category. Mac Lane in [7] shows that a list of five axioms is sufficient for the coherence for monoidal categories. Kelly in [4] found that there were redundancies in that list and reduced it down to two. However this reduction relied on the invertibility of the associativity and unit maps. In the context of skew monoidal categories no such invertibility is assumed and so we require all of the five axioms of Mac Lane.

One of the first observations about a monoid is that its unit (if it exists) is unique, as shown by the equality $i = i.j = j$. In a monoidal category these equalities become isomorphisms $I \cong I \otimes J \cong J$; where now in this context there is also a uniqueness result. We show an analogous result for the units of a skew monoidal category. In this context we no

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longer have isomorphisms $I \cong I \otimes J$ or $I \otimes J \cong J$ but only the maps $I \longrightarrow I \otimes J$ and $I \otimes J \longrightarrow J$. Thus it might seem that uniqueness up to isomorphism is lost, but surprisingly, it turns out that the composite $I \longrightarrow I \otimes J \longrightarrow J$ *is* invertible, and we do still have a uniqueness result for this isomorphism.

Generalisations of monoidal categories have previously been considered with different choices for the directions of the non-invertible associativity and unit maps, or with fewer coherence data than those considered here. In [1], Altenkirch, Chapman and Uustalu, while studying relative monads, show a certain functor category is skew monoidal; they call it lax monoidal. Independently, and motivated by bialgebroids, Szlachányi in [9], first names and studies skew monoidal categories as such. In this text what we call a skew monoidal category is usually referred to as a left skew monoidal category, and what could have been referred to as a skew psuedomonoid we call a skew monoidale.

In Section 2 we establish that the units are isomorphic up to a unique isomorphism; this is the analogue for skew monoidal categories of Proposition 1.7 in [5]. This was shown for monoidal categories by Kock in [5], where earlier references are also given to these results by Saavedra Rivano in [8]. The coherence results for monoidal categories with units, by Mac Lane in [7], would imply that the isomorphisms between the units are unique. The proofs here follow the same methods employed in [5]. We then show the independence of the axioms for a skew monoidal category before concluding the section with a result on the unit conditions of a monoidal functor between skew monoidal categories. There is also a remark that the main results of Section 2 can be internalised to skew monoidales, that is, out of the cartesian monoidal 2-category **Cat** and lifted into a monoidal bicategory. In Section 3 we impose some extra structure on the unit maps of a skew monoidal category and remark on some consequences of this extra structure.

2 Skew Monoidal Categories

2.1 Skew Semimonoidal Categories

A *skew semimonoidal category* is a triple $(\mathcal{C}, \otimes, \alpha)$ where $\mathcal{C}$ is a category equipped with a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ (called *tensor product*), and a natural family of *lax constraints* α whose components have the directions

$$\alpha_{X,Y,Z} : (X \otimes Y) \otimes Z \longrightarrow X \otimes (Y \otimes Z)$$

subject to the condition that the following diagram commutes.

$$\begin{array}{ccccc} & & (W \otimes X) \otimes (Y \otimes Z) & & \\ & \nearrow^{\alpha_{W \otimes X, Y, Z}} & & \searrow^{\alpha_{W, X, Y \otimes Z}} & \\ ((W \otimes X) \otimes Y) \otimes Z & & & & W \otimes (X \otimes (Y \otimes Z)) \\ \downarrow{\scriptstyle \alpha_{W,X,Y} \otimes 1_Z} & & & & \uparrow{\scriptstyle 1_W \otimes \alpha_{X,Y,Z}} \\ (W \otimes (X \otimes Y)) \otimes Z & & \xrightarrow[\alpha_{W, X \otimes Y, Z}]{} & & W \otimes ((X \otimes Y) \otimes Z) \end{array} \quad (1)$$

2.2 The Category of Units

A skew monoidal category is a skew semimonoidal category equipped with a chosen unit, in a sense to be defined below. We shall see that if such a unit exists, it is unique up to isomorphism. Furthermore, this isomorphism is compatible, in the sense that it is a morphism in the *category of units*, which we now define.

Given a skew semimonoidal category $(\mathcal{C}, \otimes, \alpha)$, we form a category $\mathcal{U}(\mathcal{C})$ as follows. The *objects* are triples (I, λ, ρ) where I is an object of $\mathcal{C}$ and where λ and ρ are natural families of *lax constraints* whose components have directions

$$\lambda_X : I \otimes X \longrightarrow X$$

$$\rho_X : X \longrightarrow X \otimes I$$

subject to four conditions asserting that the following diagrams commute:

$$\begin{array}{ccc} (I \otimes X) \otimes Y & \xrightarrow{\alpha_{I,X,Y}} & I \otimes (X \otimes Y) \\ & {\scriptstyle \lambda_X \otimes 1_Y} \searrow \quad \swarrow {\scriptstyle \lambda_{X \otimes Y}} & \\ & X \otimes Y & \end{array} \tag{2}$$

$$\begin{array}{ccc} (X \otimes I) \otimes Y & \xrightarrow{\alpha_{X,I,Y}} & X \otimes (I \otimes Y) \\ {\scriptstyle \rho_X \otimes 1_Y} \uparrow & & \downarrow {\scriptstyle 1_X \otimes \lambda_Y} \\ X \otimes Y & \xrightarrow[1_{X \otimes Y}]{} & X \otimes Y \end{array} \tag{3}$$

$$\begin{array}{ccc} (X \otimes Y) \otimes I & \xrightarrow{\alpha_{X,Y,I}} & X \otimes (Y \otimes I) \\ & {\scriptstyle \rho_{X \otimes Y}} \nwarrow \quad \nearrow {\scriptstyle 1_X \otimes \rho_Y} & \\ & X \otimes Y & \end{array} \tag{4}$$

$$\begin{array}{ccc} I & \xrightarrow{1_I} & I \\ & {\scriptstyle \rho_I} \searrow \quad \nearrow {\scriptstyle \lambda_I} & \\ & I \otimes I & \end{array} \quad . \tag{5}$$

An *arrow* of $\mathcal{U}(\mathcal{C})$ from (I, λ, ρ) to (J, λ', ρ') is given by an arrow $\varphi : I \longrightarrow J$ in $\mathcal{C}$ such that the following two triangles commute.

$$\begin{array}{ccc} I \otimes X & \xrightarrow{\varphi \otimes 1_X} & J \otimes X \\ & {\scriptstyle \lambda_X} \searrow \quad \swarrow {\scriptstyle \lambda'_X} & \\ & X & \end{array} \qquad \begin{array}{ccc} X \otimes I & \xrightarrow{1_X \otimes \varphi} & X \otimes J \\ & {\scriptstyle \rho_X} \nwarrow \quad \nearrow {\scriptstyle \rho'_X} & \\ & X & \end{array} \tag{6}$$

The composition of arrows in $\mathcal{U}(\mathcal{C})$ is then given by the composition in $\mathcal{C}$.

Given two objects (I, λ, ρ) and (J, λ', ρ') of $\mathcal{U}(\mathcal{C})$ we *define* $\varphi_{I,J} : I \longrightarrow J$ to be the following composite

$$I \xrightarrow{\rho'_I} I \otimes J \xrightarrow{\lambda_J} J \ , \tag{7}$$

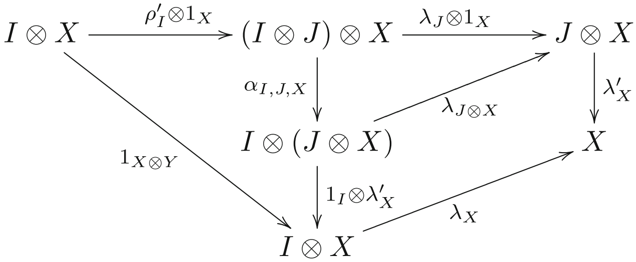
so with this notation $\varphi_{J,I} : J \longrightarrow I$ is the following composite

$$J \xrightarrow{\rho_J} J \otimes I \xrightarrow{\lambda'_I} I \ .$$

When no confusion arises we will call these maps just φ.

Lemma 2.1 *The map* $\varphi_{I,J}$ *defined by Eq. 7 is an arrow in* $\mathcal{U}(\mathcal{C})$ *from* (I, λ, ρ) *to* (J, λ', ρ').

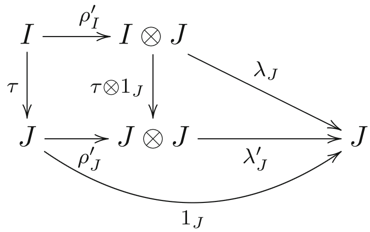
Proof We show that the first diagram of Eq. 6 commutes by considering the following diagram



in which the left-hand triangle commutes by Eq. 3 for (J, λ', ρ'), the right-hand triangle commutes by Eq. 2 for (I, λ, ρ), and the rectangle commutes by the naturality of λ. The right-hand side of Eq. 6 is analogous. □

Proposition 2.2 *There is exactly one morphism from* (I, λ, ρ) *to* (J, λ', ρ') *in* $\mathcal{U}(\mathcal{C})$.

Proof Suppose we have another morphism τ from I to J in $\mathcal{U}(\mathcal{C})$, and consider the following diagram.



The square commutes by the naturality of ρ', the triangle commutes by the assumption that τ satisfies the left-hand side of Eq. 6, and the semi-circle commutes by Eq. 5 for (J, λ', ρ'). This shows that $\tau = \varphi$. □

Corollary 2.3 *Any two objects* (I, λ, ρ) *and* (J, λ', ρ') *in* $\mathcal{U}(\mathcal{C})$ *are isomorphic.*

Proof Both $\varphi_{J,I} \circ \varphi_{I,J}$ and 1_I are arrows from (I, λ, ρ) to (I, λ, ρ) in $\mathcal{U}(\mathcal{C})$ so by uniqueness they are equal. That $\varphi_{I,J} \circ \varphi_{J,I} = 1_J$ is analogous. □

We may combine the previous two results into:

Theorem 2.4 *For a skew semimonoidal category* $\mathcal{C}$*, if* $\mathcal{U}(\mathcal{C})$ *is non-empty then it is equivalent to the terminal category.*

A skew semimonoidal category is a *skew monoidal category* if $\mathcal{U}(\mathcal{C})$ is non-empty. We denote a skew monoidal category by the 6-tuple $(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho)$. Proposition 2.2 and Corollary 2.3 then imply that;

Theorem 2.5 *The unit for a skew monoidal category* $(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho)$ *is unique up to a unique isomorphism.*

Next, we shall see that either λ or ρ determines the other.

Corollary 2.6 *If* (I, λ, ρ') *and* (I, λ, ρ) *are in* $\mathcal{U}(\mathcal{C})$ *then* $\rho' = \rho$.

Proof Consider the unique morphism $\varphi_{J,I} : (I, \lambda, \rho') \longrightarrow (I, \lambda, \rho)$ where $J = (I, \lambda, \rho')$. By Eq. 5, this must be 1_I; then by Eq. 6 we deduce that $\rho = \rho'$. □

Corollary 2.7 *If* (I, λ, ρ) *and* (I, λ', ρ) *are in* $\mathcal{U}(\mathcal{C})$ *then* $\lambda = \lambda'$.

Proof Dually, by reversing the tensor and the direction of arrows, we can instead repeat the above argument instead using $\varphi_{I,J}$. □

Remark 2.8 Equation 1, the pentagon equation, was not used in Proposition 2.2 or its Corollaries. This leads to the possibility of similar results about the units of skew versions of categories not satisfing (1) such as in [3].

Remark 2.9 The proof of Lemma 2.1 uses Eqs. 2, 3 and 4 but not 1 or 5, while the proof of Proposition 2.2 uses the Eqs. 5 and 6. Now suppose that λ and ρ satisfy only (2), (3) and (4). Then the composite $I \xrightarrow{\rho_I} I \otimes I \xrightarrow{\lambda_I} I$ satisfies (6) and so (5) becomes a special case of the uniqueness result in Proposition 2.2.

2.3 Independence of the Axioms

In this section we show that the five axioms for a skew monoidal category, given by Eqs. 1, 2, 3, 4 and 5, are independent. The underlying category we use is **Set** where the cartesian product between two sets is denoted by $\times$; we often identify the cartesian product of a one-point set with a set as the set itself and $X \times Y$ with $Y \times X$ in what follows.

For a set M, define a tensor product on **Set** by $X \otimes Y = M \times X \times Y$; this gives a functor $\mathbf{Set} \times \mathbf{Set} \xrightarrow{\otimes} \mathbf{Set}$. If M has a product $M \times M \xrightarrow{\cdot} M$ there is a natural transformation $\alpha : M \times M \times X \times Y \times Z \longrightarrow M \times X \times M \times Y \times Z$ given by sending (m, n, x, y, z) to $(m.n, x, m, y, z)$. Let I be a one-point set, and $1 \in M$. The map $\lambda : I \otimes X (= M \times X) \longrightarrow X$ defined by sending (m, x) to x and the map $\rho : X \longrightarrow X \otimes I (= M \times X)$ defined by sending x to $(x, 1)$ are both natural transformations. With these maps, Eq. 1 asks that the product on M is associative, Eq. 3 asks that $1 \in M$ is a right identity on M, and Eq. 4 asks that $1 \in M$ is a left identity on M. The remaining two equations are already satisfied under these maps and impose no extra structure on the set M. (These maps are based on the constructions in the first section of [6]).

We take for M the following three sets. The M defined by the table on the left has a left and right identity but is not associative, so Eq. 1 does not hold but the other four equations do.

.	1	a	b
1	1	a	b
a	a	1	b
b	b	a	1

.	1	a
1	1	a
a	1	a

.	1	a
1	1	1
a	a	a

The M defined by the table in the middle has no right identity, but has a left identity and is associative. In this case, Eq. 3 does not hold but the other four equations do. The M defined by the table on the right has no left identity, but has a right identity and is associative. In this case, the Eq. 4 does not hold but the other four equations do.

Thus each of the Eqs. 1, 3 and 4 is independent of the remaining four equations. By reversing the tensor, direction of arrows and the order of composition we notice that Eqs. 2 and 4 are dual, so statements such as independence holds for one if and only if it holds for the other. Thus independence of Eq. 2 follows from the independence of Eq. 4.

This leaves (5), for which we take the tensor product to be the cartesian product, so $X \otimes Y = X \times Y$. The map $\alpha\colon (X \times Y) \times Z \longrightarrow X \times (Y \times Z)$ is the usual associative isomorphism $(X\times Y)\times Z \cong X\times(Y\times Z)$ and I is given by $\{a, b\}$. The map $\lambda\colon I\times X \longrightarrow X$ defined by sending (i, x) to x and the map $\rho\colon X \longrightarrow X \times I$ defined by sending x to (x, a) are natural transformations. In this case, Eq. 5 asks for the elements of I to be identical which is not the case here, so Eq. 5 is not satisfied but it is easy to see that the other four equations hold.

With these four examples and duality we have shown that:

Proposition 2.10 *The five axioms for a skew monoidal category are independent.*

2.4 Monoidal Functors

Let $(\mathcal{C}, \otimes', I, \alpha', \lambda', \rho')$ and $(\mathcal{D}, \otimes, J, \alpha, \lambda, \rho)$ be skew monoidal categories. A *monoidal functor* from $\mathcal{C}$ to $\mathcal{D}$ is a triple (F, φ, F_0) where $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ is a functor of the underlying categories, F_0 is a morphism $J \longrightarrow F(I)$ in $\mathcal{D}$ and φ is a natural transformation with components $\varphi_{X,Y}\colon F(X) \otimes F(Y) \longrightarrow F(X \otimes' Y)$ such that the following diagrams commute.

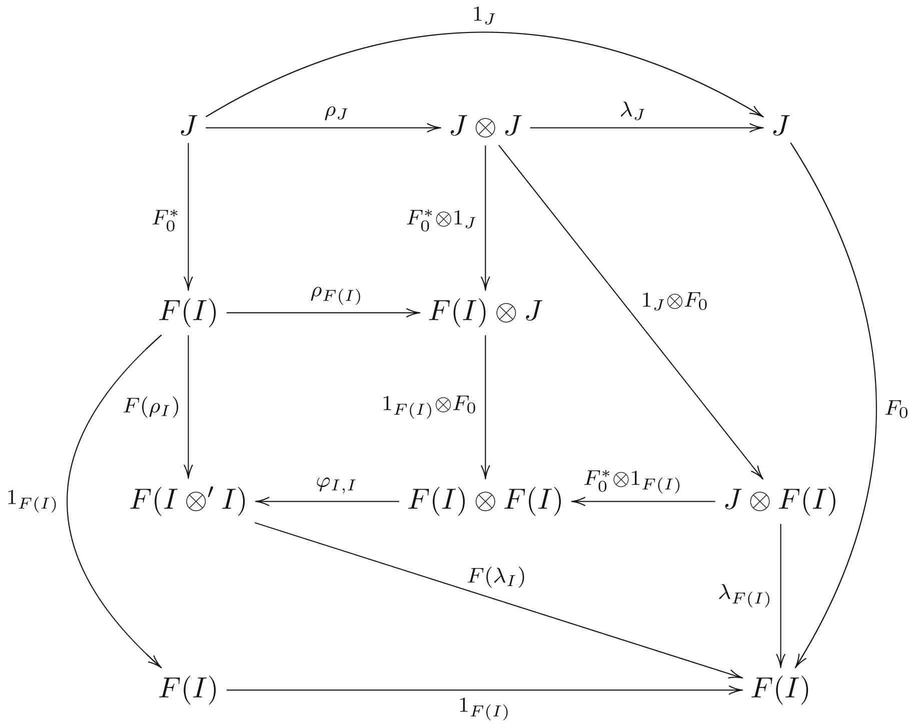
$$
\begin{array}{ccc}
(F(X)\otimes F(Y))\otimes F(Z) & \xrightarrow{\alpha_{F(X),F(Y),F(Z)}} & F(X)\otimes(F(Y)\otimes F(Z)) \\
\downarrow{\scriptstyle \varphi_{X,Y}\otimes 1_{F(Z)}} & & \downarrow{\scriptstyle 1_{F(X)}\otimes\varphi_{Y,Z}} \\
F(X\otimes' Y)\otimes F(Z) & & F(X)\otimes F(Y\otimes' Z) \\
\downarrow{\scriptstyle \varphi_{X\otimes' Y,Z}} & & \downarrow{\scriptstyle \varphi_{X,Y\otimes' Z}} \\
F((X\otimes' Y)\otimes' Z) & \xrightarrow[F(\alpha'_{X,Y,Z})]{} & F(X\otimes'(Y\otimes' Z))
\end{array}
\tag{8}
$$

$$
\begin{array}{ccc}
J\otimes F(X) & \xrightarrow{\lambda_{F(X)}} & F(X) \\
\downarrow{\scriptstyle F_0\otimes 1_{F(X)}} & & \uparrow{\scriptstyle F(\lambda'_X)} \\
F(I)\otimes F(X) & \xrightarrow[\varphi_{I,X}]{} & F(I\otimes' X)
\end{array}
\qquad
\begin{array}{ccc}
F(X) & \xrightarrow{\rho_{F(X)}} & F(X)\otimes J \\
\downarrow{\scriptstyle F(\rho'_X)} & & \downarrow{\scriptstyle 1_{F(X)}\otimes F_0} \\
F(X\otimes' I) & \xleftarrow[\varphi_{X,I}]{} & F(X)\otimes F(I)
\end{array}
\tag{9}
$$

A monoidal functor between skew monoidal categories is *normal* if F_0 is an isomorphism, and is *strong* if both φ and F_0 are isomorphisms. If the skew monoidal categories were monoidal then these are the usual notions of lax, normal, and strong monoidal functors.

Proposition 2.11 *Let* $(\mathcal{C}, \otimes', I, \alpha', \lambda', \rho')$ *and* $(\mathcal{D}, \otimes, J, \alpha, \lambda, \rho)$ *be skew monoidal categories and let* F *be a functor and* φ *a natural transformation such that* (8) *holds. Then there is at most one* F_0 *such that* (9) *holds.*

Proof Let F_0^* be another such morphism in $\mathcal{D}$, so in particular $F_0^* \colon J \longrightarrow F(I)$ satisfies the equations in Eq. 9. Consider the following diagram.

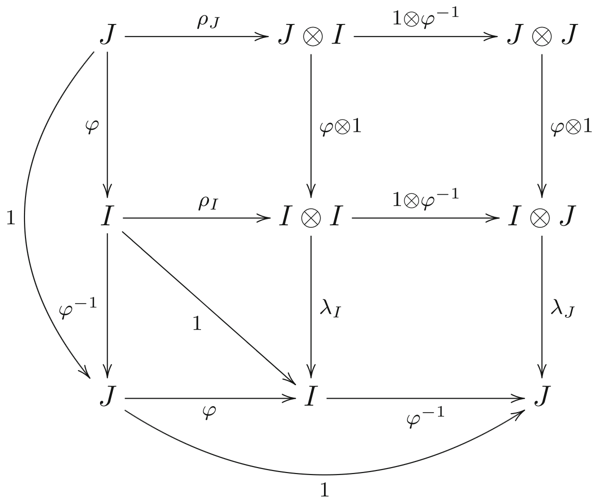


The part involving the semicircles on the top and left-hand side commute by Eq. 5. The top square commutes by the naturality of ρ, and the square below it commutes by the right-hand equation in (9). The triangle next to the squares commutes by the interchange law. The bottom triangle commutes by the left-hand equation in (9) and the remaining part of the diagram (on the right) commutes by the naturality of λ. The commutativity of the exterior gives the required uniqueness. □

Proposition 2.12 *Let* $(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho)$ *be a skew monoidal category . If there exists an object* J *and an isomorphism* $\varphi \colon J \longrightarrow I$ *in* $\mathcal{C}$ *then* (J, λ', ρ') *is also a unit of* $\mathcal{C}$*, where* $\lambda'_X \colon J \otimes X \longrightarrow X$ *and* $\rho'_X \colon X \longrightarrow X \otimes J$ *are given by the following composites.*

$$J \otimes X \xrightarrow{\varphi \otimes 1_X} I \otimes X \xrightarrow{\lambda_X} X \qquad\qquad X \xrightarrow{\rho_X} X \otimes I \xrightarrow{1_X \otimes \varphi^{-1}} X \otimes J$$

Proof These composites satisfy the four conditions in the definition, as can be shown by a straight forward transport of structure argument. We only supply the verification of the last condition. Consider the following diagram.



The top right square commutes by the interchange law, the bottom right square commutes by the naturality of λ, the top left square commutes by the naturality of ρ and below this, the upper triangle commutes by Eq. 5. □

Remark 2.13 If $F_0 : J \longrightarrow F(I)$ is an isomorphism in $\mathcal{D}$ then by Proposition 2.12, $F(I)$ is also a unit in $\mathcal{D}$. Now, by Proposition 2.2, there is a unique morphism between these units, namely $\varphi_{J,F(I)}$ and using the naturality of λ it can be shown that $\varphi_{J,F(I)} = F_0$.

Remark 2.14 Proposition 2.11 generalises the uniqueness result of Proposition 2.2, which we may recover on taking the two skew monoidal categories to be the same, F to be the identity functor, and φ the identity natural transformation. It also implies the uniqueness of units for monoids in a skew monoidal category by taking $\mathcal{C} = 1$.

Remark 2.15 We denote by **SkMon** the category with *objects* skew monoidal categories and *1-cells* monoidal functors and **SkSemiMon** the category with *objects* skew semimonoidal categories and *1-cells* semimonoidal functors (drop the F_0 conditions for the unit).

We denote the obvious forgetful functor where we drop all reference to units and any associated conditions by $V : \mathbf{SkMon} \longrightarrow \mathbf{SkSemiMon}$. For an object $\mathcal{C}$ of **SkSemiMon**, that is, $\mathcal{C}$ is a skew semimonoidal category, the fibre of V at $\mathcal{C}$ is the category $\mathcal{U}(\mathcal{C})$ of $\mathcal{C}$.

The uniqueness of F_0 in Proposition 2.11 implies that the forgetful functor V is faithful. Moreover, the uniqueness and existence results from Section 2.2 imply that V is also full on isomorphisms in **SkSemiMon**, and by Proposition 2.12 V is also an isofibration.

Remark 2.16 Skew monoidales in a monoidal bicategory were first defined in [6] as an internal version of a skew monoidal category. It should not be a surprise that the elementary nature of the proofs in Sections 2.2 and 2.4 allow for the results there to be generalised to skew monoidales. Following the scheme set out in these sections the reader can reconstruct the corresponding details for a skew monoidale and for monoidal morphisms between skew monoidales. For example, in [2], it is shown that

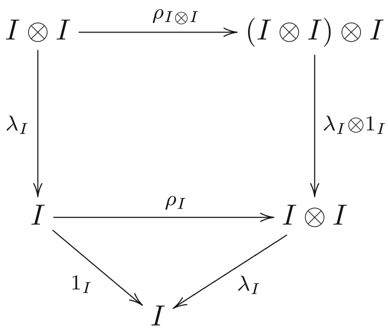
Proposition 2.17 *The unit of a skew monoidale is unique up to a unique isomorphism.*

3 Weakly Normal Skew Monoidal Categories

In this section we impose some extra structure on the unit maps of a skew monoidal category.

Proposition 3.1 *If* $(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho)$ *is a skew monoidal category then* $(I \otimes I, \lambda_X \circ (\lambda_I \otimes 1_X), (1_X \otimes \rho_I) \circ \rho_X)$ *is a unit of* $\mathcal{C}$ *if and only if* $\lambda_I : I \otimes I \longrightarrow I$ *is invertible (or equivalently,* ρ_I *is invertible).*

Proof Assuming that $(I \otimes I, \lambda_X \circ (\lambda_I \otimes 1_X), (1_X \otimes \rho_I) \circ \rho_X)$ is a unit we can use Lemma 2.1 and Proposition 2.2 and just show that $\varphi_{I \otimes I, I} = \lambda_I$ by considering the following diagram.



The square commutes by the naturality of ρ and the triangle commutes by Eq. 5.

Conversely, if $\lambda_I : I \otimes I \longrightarrow I$ is invertible then by Eq. 5 $\rho_I \circ \lambda_I = 1_{I \otimes I}$, so then $\lambda_I^{-1} = \rho_I$ and we can apply Proposition 2.12. □

A skew monoidal category $(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho)$ is *weakly normal* if it also satifies the condition that $\rho_I \circ \lambda_I = 1_{I \otimes I}$; equivalently, if λ_I or ρ_I (and so both) is invertible.

Proposition 3.2 *If* $(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho)$ *is a weakly normal skew monoidal category then the monoid* $\mathrm{End}(I)$ *of endomorphisms of the unit object* I *is commutative.*

Proof As $\lambda_I : I \otimes I \longrightarrow I$ is invertible, it induces an isomorphism $\psi : \mathrm{End}(I \otimes I) \longrightarrow \mathrm{End}(I)$ defined by $\psi(\gamma) = \lambda_I \circ \gamma \circ \lambda_I^{-1}$. For $f \in \mathrm{End}(I)$ we deduce, by the naturality of λ, that

$$\begin{aligned} f &= f \circ \lambda_I \circ \lambda_I^{-1} \\ &= \lambda_I \circ (1_I \otimes f) \circ \lambda_I^{-1} \\ &= \psi(1_I \otimes f) \,. \end{aligned}$$

Similarly, using the naturality of λ^{-1} we get $f = \psi(f \otimes 1_I)$.

So for $f, g \in \mathrm{End}(I)$ we have, by the interchange law, that

$$\begin{aligned} f \circ g &= \psi(f \otimes 1) \circ \psi(1 \otimes g) \\ &= \psi((f \otimes 1) \circ (1 \otimes g)) \\ &= \psi((1 \otimes g) \circ (f \otimes 1)) \\ &= \psi(1 \otimes g) \circ \psi(f \otimes 1) \\ &= g \circ f \ . \end{aligned}$$

□

Remark 3.3 Let R-**Mod** denote the category of left R-modules over some ring R. Regarding R as a left module over itself using its product, and noticing that $\mathrm{End}(R)$ is the monoid R if we regard R as a monoid under multiplication, we can use Lemma 3.2 to conclude that if R is a non-commutative ring then R is not the unit object for a weakly normal skew monoidal structure on R-**Mod**.

A skew monoidal category is *left normal* if λ is invertible. This implies that tensoring on the left by I is an equivalence. Using the naturality of λ and the invertibilty of λ_X we deduce that

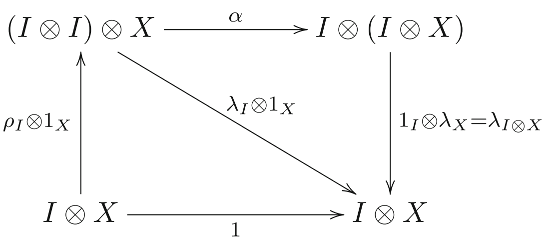
$$\lambda_{I \otimes X} = 1_I \otimes \lambda_X \ .$$

A skew monoidal category is *right normal* if ρ is invertible and *normal* if both λ and ρ are invertible.

Remark 3.4 If $\mathcal{C}$ is a left normal skew monoidal category then for any units I and J in $\mathcal{C}$ we have $I \otimes J \cong J$ and so $I \otimes J$ is also a unit by Proposition 3.1. Thus, if $\mathcal{C}$ is a left normal skew monoidal category then the $\otimes$ from $\mathcal{C}$ applied to $\mathcal{U}(\mathcal{C})$ gives $\mathcal{U}(\mathcal{C})$ the structure of a skew semimonoidal category.

Lemma 3.5 *If* $(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho)$ *only satisfies* (2) *and* (3) *with both* λ *and* ρ *being invertible then* (5) *holds.*

Proof Consider the following diagram.



The outside commutes by Eq. 3, the upper triangle commutes by Eq. 2 where we used the assumption of λ being invertible and the resulting identity that $1_I \otimes \lambda_X = \lambda_{I \otimes X}$, so then the lower triangle commutes. Now taking $X = I$ and using the assumption that ρ is a natural isomorphism we get (5). □

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