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Semisimple and G-Equivariant Simple Algebras Over Operads

Pavel Etingof[1]

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Abstract Let G be a finite group. There is a standard theorem on the classification of G-equivariant finite dimensional simple commutative, associative, and Lie algebras (i.e., simple algebras of these types in the category of representations of G). Namely, such an algebra is of the form $A = \mathrm{Fun}_H(G, B)$, where H is a subgroup of G, and B is a simple algebra of the corresponding type with an H-action. We explain that such a result holds in the generality of algebras over a linear operad. This allows one to extend Theorem 5.5 of Sciarappa (arXiv:1506.07565) on the classification of simple commutative algebras in the Deligne category $\mathrm{Rep}(S_t)$ to algebras over any finitely generated linear operad.

Keywords Simple algebra · Semisimple algebra · Operad · Equivariant

1 Semisimple Algebras Over Operads

1.1 Algebras

Let C be a linear operad over a field F [1]. E.g., C can be the operad of commutative associative unital algebras, associative unital algebras, or Lie algebras (the latter if $1/2 \in F$).

Recall [1] that a C-*algebra* is a vector space A over F with a collection of linear maps $\alpha_n : C(n) \to \mathrm{Hom}_F(A^{\otimes n}, A)$ compatible with the operadic structure. Clearly, a direct product of finitely many C-algebras is a C-algebra.

Given a C-algebra A, we can define the space $E_A \subset \mathrm{End}_F(A)$ spanned over F by operators of the form $\alpha_n(c)(a_1, ..., a_{j-1}, ?, a_j, ..., a_{n-1})$ for various $n \geq 2$, $c \in C(n)$, and $a_i \in A$. By the definition of an operad, E_A is a (possibly non-unital) subalgebra of $\mathrm{End}_F(A)$.

✉ Pavel Etingof
etingof@math.mit.edu

1 Massachusetts Institute of Technology (MIT), Cambridge, MA, USA

We also denote by L_A the image of $C(1)$ in $\mathrm{End}_F(A)$. Clearly, L_A is a unital subalgebra and $L_A E_A = E_A L_A = E_A$. Thus $R_A := L_A + E_A$ is a unital subalgebra of $\mathrm{End}_F(A)$, and E_A is an ideal in R_A.

Lemma 1.1 *One has*[1] $E_{A_1\oplus...\oplus A_m} = E_{A_1} \oplus ... \oplus E_{A_m}$.

Proof It is clear that $E_{A_1\oplus...\oplus A_m} \subset E_{A_1} \oplus ... \oplus E_{A_m}$. Let $a_i \in A_r$, $c \in C(n)$, and $b = \alpha_n(c)(a_1, ..., a_{j-1}, ?, a_j, ..., a_{n-1}) \in E_{A_r}$. Let $b' := (0, ..., b, ..., 0)$ (where b is at the r-th place). Then we have $b' = \alpha_n(c)(a'_1, ..., a'_{j-1}, ?, a'_j, ..., a'_{n-1})$, where $a'_i = (0, ..., a_i, ..., 0)$. Hence $b' \in E_{A_1\oplus...\oplus A_m}$. Thus $E_{A_1\oplus...\oplus A_m} \supset E_{A_1} \oplus ... \oplus E_{A_m}$. □

1.2 Ideals

By an *ideal* in a C-algebra A we mean a subspace $I \subset A$ such that for any $n \geq 1$, $c \in C(n)$, $j \in [1, n]$, and $T \in A^{\otimes j-1} \otimes I \otimes A^{\otimes n-j}$ one has $\alpha_n(c)T \in I$.

Lemma 1.2 (i) *$I \subset A$ is an ideal if and only if it is an R_A-submodule of A.*
(ii) *$A = A_1 \oplus ... \oplus A_m$ as an R_A-module if and only if it is so as a C-algebra.*

Proof (i) This follows directly from the definition.
(ii) The "if" direction is clear. To prove the "only if" direction, note that by (i) A_i are ideals in A, hence $\alpha_n(..., x, ..., y, ...) = 0$ once $x \in A_i$ and $y \in A_j$ with $j \neq i$, which implies the statement.

□

It is clear that if $I \subset A$ is an ideal then A/I is a C-algebra, and $E_{A/I}, L_{A/I}, R_{A/I}$ are homomorphic images of E_A, L_A, R_A in $\mathrm{End}_F(A/I)$.

1.3 Simple and Semisimple Algebras

From now on we assume that A is a finite dimensional C-algebra. We say that A is *simple* if any ideal in A is either 0 or A (i.e., A is a simple R_A-module), and $E_A \neq 0$.[2]

Lemma 1.3 *If A is a simple C-algebra then $E_A = R_A$, and it is a central simple algebra (over some finite field extension of F).*

Proof Since A is a faithful simple R_A-module, R_A is central simple. Since $E_A \neq 0$ and E_A is an ideal in R_A, we have $E_A = R_A$. □

[1]Categorically, it is more natural to regard the direct sum $A_1 \oplus ... \oplus A_m$ as a direct product, but there is no difference since it is finite. So, we will call it a direct product, but use the sign $\oplus$ instead of $\times$ to emphasize that our constructions are linear over a field.

[2]Note that this recovers the standard definition for commutative, associative, and Lie algebras. Moreover, while in the commutative and associative case, the condition $E_A \neq 0$ is automatic for $A \neq 0$ because of the unit axiom, in the Lie case it is needed (as an abelian Lie algebra is not simple). Note also that if $C(n) = 0$ for $n \neq 1$ (i.e., when C is an ordinary algebra), then $E_A = 0$ automatically, so there are no simple C-algebras, even though there may exist simple C-modules.

We say that A is *semisimple* if A is a direct product of a finite (possibly empty) collection of simple C-algebras: $A = A_1 \oplus ... \oplus A_m$.

Lemma 1.4 *Let $A = A_1 \oplus ... \oplus A_m$ be a semisimple C-algebra with simple constituents A_i. Then the only ideals in A are $\oplus_{i \in S} A_i \subset A$, where $S \subset [1, m]$.*

Proof Clearly, the subspaces in the lemma are ideals. Conversely, let $I \subset A$ be an ideal. Let $a = (a_1, ..., a_m) \in I$. By Lemmas 1.1 and 1.3, the projection operator $P_i : A \to A$ to A_i along $\oplus_{j \neq i} A_j$ is contained in E_A. Thus, $P_i a = (0, ..., a_i, ..., 0) \in I$. This implies the statement. □

1.4 The Radical

Let A' be the maximal semisimple quotient of A as an R_A-module (it exists by the standard theory of finite dimensional algebras). Let $\overline{A}$ be the quotient of A' by the kernel of the action of E_A (which is an R_A-submodule of A). Define the *radical* $\mathrm{Rad}(A)$ of A to be the kernel of the projection of A onto $\overline{A}$. So the radical of $A/\mathrm{Rad}(A) = \overline{A}$ is zero. In particular, if A is a semisimple C-algebra, then $\mathrm{Rad}(A) = 0$.

Theorem 1.5 (i) *$\overline{A}$ is a semisimple C-algebra. In particular, $Rad(A) = 0$ if and only if A is semisimple.*
(ii) *If $I \subset A$ is an ideal, then A/I is a semisimple C-algebra if and only if I contains $Rad(A)$.*

Proof (i) By the definition, $\overline{A}$ is a semisimple R_A-module, such that E_A acts by nonzero on all its simple summands. Hence by Lemma 1.2(ii), $\overline{A}$ is a semisimple C-algebra.
(ii) The "if" direction holds by (i) and Lemma 1.4. To prove the "only if" direction, let $I \subset A$ be an ideal such that A/I is a semisimple C-algebra: $A/I = A_1 \oplus ... \oplus A_m$. Then by Lemma 1.2(ii) A/I is a semisimple $R_{A/I}$-module and hence R_A-module, with simple constituents A_i, and the action of E_A on A_i is nonzero. Thus $I \supset \mathrm{Rad}(A)$. □

2 *G*-Equivariant Simple Algebras Over Operads

Now let G be a finite group, and A be a C-algebra with an action of G. Let us say that A is a *simple G-equivariant C-algebra* if the only G-invariant ideals in A are 0 and A, and $E_A \neq 0$.

Lemma 2.1 (i) *If B is a simple C-algebra then we have $Aut(B^{\oplus n}) = S_n \ltimes Aut(B)^n$.*
(ii) *If A is a simple G-equivariant C-algebra then A is semisimple as a usual C-algebra. Moreover, G acts transitively on the simple constituents of A, and in particular they are all isomorphic.*

Proof (i) Clearly, $S_n \ltimes \mathrm{Aut}(B)^n$ acts on $B^{\oplus n}$, so we need to show that any automorphism g of $B^{\oplus n}$ belongs to this group. By Lemma 1.4, the minimal (nonzero) ideals of $B^{\oplus n}$ are the n copies of B. So they must be permuted by g, inducing an element $s \in S_n$.

Thus gs^{-1} is an automorphism preserving all the copies of B. So $gs^{-1} \in \mathrm{Aut}(B)^n$, as desired.

(ii) Let I be kernel of the projection from A to its maximal semisimple quotient A' as an R_A-module. Then by Lemma 1.2(i), I is a G-invariant ideal in A, and $I \neq A$. Hence $I = 0$, and A is a semisimple R_A-module. So by Lemma 1.2(ii), $A = A_1 \oplus ... \oplus A_m$ is a semisimple C-algebra. Thus by Lemma 1.4, the minimal ideals of A are the A_i. So they are permuted by G. Moreover, the action of G on these ideals must be transitive, as every orbit gives a nonzero G-invariant ideal.

□

Now let B be a simple C-algebra, H a subgroup of G, and $\phi : H \to \mathrm{Aut}(B)$ a homomorphism. Let $A = \mathrm{Fun}_H(G, B)$ be the space of H-invariant functions on G with values in B. Then it is clear that A has a natural structure of a simple G-equivariant C-algebra, isomorphic to $B^{\oplus |G/H|}$ as a usual C-algebra. Note that the stabilizer of any minimal ideal of A is a subgroup of G conjugate to H.

Theorem 2.2 *Any simple G-equivariant C-algebra A is of the form $A = Fun_H(G, B)$. Moreover, the subgroup H is defined by A uniquely up to conjugation in G, and ϕ is defined uniquely up to conjugation in $Aut(B)$.*

Proof By Lemma 2.1(ii), G acts transitively on the set of minimal ideals in A, and they are all isomorphic to some simple C-algebra B. Thus, the result follows from Lemma 2.1(i) and the standard classification of transitive homomorphisms $G \to S_n \ltimes \mathrm{Aut}(B)^n$. Namely, let H be the stabilizer of one of the copies of B. Then H acts on B through some homomorphism $\phi : H \to \mathrm{Aut}(B)$. Moreover, we have a canonical G-equivariant linear map $\psi : A \to \mathrm{Fun}_H(G, B)$ corresponding via Frobenius reciprocity to the H-stable projection $A \to B$ to the chosen copy of B along the direct product of all the other copies. It is easy to check using Lemma 2.1 that ψ is an isomorphism of G-equivariant C-algebras. The rest is easy. □

Remark 2.3 1. Note that in the examples of commutative, associative, and Lie algebras we obtain the classical theorems about classification of simple G-equivariant algebras of these types.

2. Lemma 2.1 and Theorem 2.2 don't hold without the assumption $E_A \neq 0$. E.g., one may take A to be any irreducible representation of G equipped with the zero Lie bracket.
3. The results of this section extend verbatim to the case when G is any group (not necessarily finite), or is an affine algebraic group over F. Namely, as in the finite group case, the classification of simple G-equivariant algebras reduces to classification of transitive homomorphisms $G \to S_n \ltimes \mathrm{Aut}(B)^n$, which are paramertized by finite index subgroups H of G and homomorphisms $\phi : H \to \mathrm{Aut}(B)$ up to conjugation.

Remark 2.4 While the question of classification of G-equivariant simple algebras over operads is natural in its own right, the motivation for writing this note was to provide a more general context for the results of [2]. Namely, Lemma 2.1 and Theorem 2.2 allow one to extend the main result of [2] (Theorem 5.5 on the classification of simple commutative algebras in the Deligne category $\mathrm{Rep}(S_t)$) to algebras over a finitely generated linear operad C over $\mathbb{C}$. Informally speaking, this generalization says that for transcendental t any such algebra is obtained by induction from $\mathrm{Rep}(G) \boxtimes \mathrm{Rep}(S_{t-k})$ of an interpolation B of a family of

$G \times S_{n-k}$-equivariant simple algebras B_n, defined for some strictly increasing sequence of positive integers n and depending algebraically on n.

This gives a classification of simple C-algebras in Rep(S_t) whenever a classification of ordinary simple C-algebras (and their automorphisms) is available. For instance, in the case of associative unital algebras, $B = \mathrm{End}(V)$, where V is an object of Rep(S_t), and in the case of Lie algebras $B = \mathfrak{sl}(V)$, $\mathfrak{o}(V)$, or $\mathfrak{sp}(V)$, where in the second case V is equipped with a nondegenerate symmetric form and in the third case with a nondegenerate skew-symmetric form.

The proof of this generalization is similar to the proof of Theorem 5.5 of [2], which covers the case of commutative unital algebras (in which case $B = \mathbb{C}$), but is somewhat more complicated since in general $\mathrm{Aut}(B) \neq 1$. The finite generation assumption for C is needed to validate the constructibility arguments of [2], Section 4. This will be discussed in more detail elsewhere.

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