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Compactification: Limit Tower Spaces

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Abstract Convergence approach spaces, defined by E. Lowen and R. Lowen, possess both quantitative and topological properties. These spaces are equipped with a structure which provides information as to whether or not a sequence or filter approximately converges. P. Brock and D. Kent showed that the category of convergence approach spaces with contractions as morphisms is isomorphic to the category of limit tower spaces. It is shown below that every limit tower space has a compactification. Moreover, a characterization of the limit tower spaces which possess a strongly regular compactification is given here. Further, a strongly regular S-compactification of a limit tower space is studied, where S is a limit tower monoid acting on the limit tower space.

Keywords Limit tower space · Strong regularity · Compactification · Completely regular topological reflection · S-compactification

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1 Introduction and Preliminaries

The category AP of "approach spaces" was defined by Lowen in 1989 [11]. The category AP contains the categories TOP and MET as full subcategories and possesses both quantitative

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and topological-like properties. In particular, information as to whether a sequence or filter approximately converges is provided by the approach structure. E. Lowen and R. Lowen [10] embedded AP in the quasitopos CAP of "convergence approach spaces." These and other results and references can be found in the monograph by R. Lowen [12].

The framework of the present paper is the category of LTS of "limit tower spaces." Brock and Kent show in Theorem 9 [2] that CAP and LTS are isomorphic categories. The purpose of our work here is to investigate compactification of objects in LTS. It is shown that each object has a compactification and, moreover, objects which have a strongly regular compactification are characterized. The characterization is given in terms of a "strongly completely regular non-Archimedean approach space."

Let X be a set, $\mathbf{F}(X)$ the set of all filters on X, 2^X the power set of X, and let $\dot{x}$ denote the filter on X whose base is $\{\{x\}\}$.

Definition 1.1 The pair (X, q) is called a **limit space** and q a **limit structure** on X provided $q : \mathbf{F}(X) \to 2^X$ satisfies:

(L1) $x \in q(\dot{x})$ for each $x \in X$
(L2) $\mathcal{F} \subseteq \mathcal{G}$ implies $q(\mathcal{F}) \subseteq q(\mathcal{G})$
(L3) $q(\mathcal{F}) \cap q(\mathcal{G}) \subseteq q(\mathcal{F} \cap \mathcal{G})$.

The more intuitive notation "$\mathcal{F} \xrightarrow{q} x$" ($\mathcal{F}$ q-converges to x) is used in place of $x \in q(\mathcal{F})$. A map $f : (X, q) \to (Y, p)$ between two limit spaces is said to be **continuous** if $f^{\to}\mathcal{F} \xrightarrow{p} f(x)$ whenever $\mathcal{F} \xrightarrow{q} x$, where $f^{\to}\mathcal{F}$ denotes the filter on Y whose base is $\{f(F) : F \in \mathcal{F}\}$. Let **LIM** denote the category consisting of all the limit spaces and continuous maps. The category LIM is a topological construct in the sense of Adámek et al. [1]. Define $LS(X)$ to be the set of all limit structures on X. If $p, q \in LS(X)$, $\mathbf{p} \leq \mathbf{q}$ means that $\mathcal{F} \xrightarrow{p} x$ whenever $\mathcal{F} \xrightarrow{q} x$. Then $(LS(X), \leq)$ is a poset with largest (smallest) member the discrete(indiscrete) topology, respectively. Given $q_j \in LS(X)$, $j \in J$, $q = \bigvee_{j \in J} q_j$ exists, and is defined by $\mathcal{F} \xrightarrow{q} x$ iff for each $j \in J$, $\mathcal{F} \xrightarrow{q_j} x$. Hence $(LS(X), \leq)$ is a complete lattice.

Assume that $(X, q) \in |\text{LIM}|$; then q is called a **pretopology** if for each $x \in X$, the **neighborhood filter** $\mathcal{U}_q(x) := \cap\left\{\mathcal{F} : \mathcal{F} \xrightarrow{q} x\right\} \xrightarrow{q} x$. Further, $A \subseteq X$ is said to be **q-open** if for each $x \in A$, $A \in \mathcal{U}_q(x)$, and $\mathbf{tq} := \{A \subseteq X : A \text{ is } q\text{-open}\}$ is a topology on X. The category TOP is known to be concretely reflective in LIM and for a limit space (X, q), the reflection morphism is $\text{id}_X : (X, q) \to (X, tq)$. An object $(X, q) \in |\text{LIM}|$ is said to be **Hausdorff** if each filter on X has at most one limit. If $A \subseteq X$, define $x \in \text{cl}_q A$ provided there exists $\mathcal{F} \xrightarrow{q} x$ such that $A \in \mathcal{F}$. The limit space (X, q) is said to be **regular** [6] if $\text{cl}_q \mathcal{F} \xrightarrow{q} x$ whenever $\mathcal{F} \xrightarrow{q} x$, where $\text{cl}_q \mathcal{F}$ denotes the filter on X whose base is $\{\text{cl}_q F : F \in \mathcal{F}\}$.

Definition 1.2 The pair $(X, \bar{q})$, where $\bar{q} = (q_\alpha)$, $0 \leq \alpha \leq \infty$, is a family of limit structures on X, is called a **limit tower space** and $\bar{q}$ a **limit tower** on X provided:

(LT1) q_∞ is the indiscrete topology
(LT2) $0 \leq \alpha \leq \beta \leq \infty$ implies that $q_\alpha \geq q_\beta$
(LT3) $\bigvee_{\beta > \alpha} q_\beta = q_\alpha$, for each $0 \leq \alpha < \infty$ (right continuity).

Let **LTS** denote the category of all limit tower spaces and contraction maps. The category LTS is a topological construct in the sense of Adámek et al. [1]. Brock and Kent prove in Theorem 9 [2] that there is an isomorphism $\sigma : CAP \to LTS$. If $(X, \lambda) \in |\text{CAP}|$, then $\sigma(X, \lambda) = (X, \lambda\bar{q})$, where $\mathcal{F} \xrightarrow{(\lambda\bar{q})_\alpha} x$ iff $\lambda(\mathcal{F})(x) \leq \alpha$, $\alpha \in [0, \infty]$. A morphism $f : (X, \lambda) \to (X', \lambda')$ in CAP is called a **contraction**, and is defined as follows: for all $\mathcal{F} \in \mathbf{F}(X)$ and $x \in X$, $\lambda'(f^{\to}\mathcal{F})(f(x)) \leq \lambda(\mathcal{F})(x)$. The morphism $\sigma(f)$ in LTS is simply the map $f : (X, (\lambda\bar{q})_\alpha) \to (X', (\lambda'\bar{q})_\alpha)$ that is continuous in LIM, for each $0 \leq \alpha \leq \infty$. The morphisms in LTS are also referred to as contractions. Given limit towers $\bar{p}, \bar{q}$ on X, define $\bar{q} \leq \bar{p}$ to mean that for each $0 \leq \alpha \leq \infty$, $q_\alpha \leq p_\alpha$ in $LS(X)$.

Definition 1.3 The pair $(X, \bar{q})$ where $\bar{q} = (q_\alpha)$, $0 \leq \alpha \leq \infty$, is a family of limit structures on X, is said to be a **generalized limit tower space** if it obeys (LT1) and (LT2).

A contraction between two generalized limit tower spaces is defined as in LTS. Let **GLTS** denote the category consisting of all the generalized limit tower spaces and contraction maps. The category GLTS is a topological construct in the sense of Adámek et al. [1].

Definition 1.4 An object $(X, \bar{q}) \in |\text{GLTS}|$, where $\bar{q} = (q_\alpha)$, $0 \leq \alpha \leq \infty$ is called **Hausdorff** whenever (X, q_0) is a Hausdorff limit space, and **strongly regular** provided (X, q_α) is a regular limit space, for each $0 \leq \alpha \leq \infty$.

Compactifications, including Hausdorff and strong regularity, are discussed in the remaining sections. The notions of regular and strongly regular limit tower spaces are defined and studied by Brock and Kent [3].

2 Compactification

A generalized limit tower space $(X, \bar{q})$, where $\bar{q} = (q_\alpha)$, $0 \leq \alpha \leq \infty$, is said to be **compact** provided that each ultrafilter on X q_0-converges; equivalently, it follows from axiom LT2 that $(X, \bar{q})$ is compact iff each ultrafilter on X q_α-converges, for each $0 \leq \alpha \leq \infty$. As usual, $((Y, \bar{p}), f)$ is called a **compactification** of $(X, \bar{q})$ in GLTS(LTS) whenever $(Y, \bar{p})$ is compact and $f : (X, \bar{q}) \to (Y, \bar{p})$ is a dense embedding in GLTS(LTS), respectively. An embedding $f : (X, \bar{q}) \to (Y, \bar{p})$ is said to be **dense** whenever $\text{cl}_{p_0} f(X) = Y$.

Definition 2.1 Assume that $(X, \bar{q}) \in |\text{GLTS}|$, define the **limit tower structure** $l\bar{q} = (l\bar{q})_\alpha$, $0 \leq \alpha \leq \infty$, of $\bar{q}$ as follows:

(i) $(l\bar{q})_\infty$ is the indiscrete topology on X

(ii) for $0 \leq \alpha < \infty$, $\mathcal{F} \xrightarrow{(l\bar{q})_\alpha} x$ iff for each $\beta > \alpha$, $\mathcal{F} \xrightarrow{q_\beta} x$.

A straightforward proof of the following lemma is omitted.

Lemma 2.1 (i) *The category LTS is concretely reflective in GLTS, where* $(X, \bar{q}) \to (X, l\bar{q})$ *is the LTS-reflection of* $(X, \bar{q})$.

(ii) *If* $f : (X, \bar{q}) \to (Y, \bar{p})$ *is an embedding in GLTS, then* $f : (X, l\bar{q}) \to (Y, l\bar{p})$ *is an embedding in LTS.*

Given $(X,\bar{q}) \in |\text{GLTS}|$, let η denote the set of all ultrafilters on X which fail to q_0-converge. Define $\langle\mathcal{G}\rangle = \{\mathcal{G}\}$, for each $\mathcal{G} \in \eta$, $X^* = X \cup \{\langle\mathcal{G}\rangle : \mathcal{G} \in \eta\}$, and let $j : X \to X^*$, $j(x) = x$ be the natural injection. For $A, B \subseteq X$, $A^* := A \cup \{\langle\mathcal{G}\rangle : A \in \mathcal{G}\}$, and note that $A^* \cap B^* = (A \cap B)^*$ and $(A \cup B)^* = A^* \cup B^*$. Let $\mathcal{F}^*$ denote the filter on X^* whose base is $\{F^* : F \in \mathcal{F}\}$, where $\mathcal{F} \in \mathbf{F}(X)$.

Definition 2.2 Given $(X,\bar{q}) \in |\text{GLTS}|$, define $\bar{p} = (p_\alpha)$, $0 \le \alpha \le \infty$, on X^* as follows:

(i) p_∞ is the indiscrete topology on X^*

(ii) $\mathcal{H} \xrightarrow{p_\alpha} j(x)$ iff $\mathcal{H} \ge \mathcal{F}^*$ for some $\mathcal{F} \xrightarrow{q_\alpha} x$, $0 \le \alpha < \infty$

(iii) $\mathcal{H} \xrightarrow{p_\alpha} \langle\mathcal{G}\rangle$ iff $\mathcal{H} \ge \mathcal{G}^*$, $0 \le \alpha < \infty$

Theorem 2.1 *Let $(X,\bar{q}) \in |GLTS|(|LTS|)$. Then $((X^*,\bar{p}), j)$ defined above is a compactification of $(X,\bar{q})$ in GLTS(LTS), respectively. Moreover, $(X^*,\bar{p})$ is Hausdorff whenever $(X,\bar{q})$ is Hausdorff.*

Proof Assume that $(X,\bar{q}) \in |\text{GLTS}|$. Note that if $\mathcal{H}_i \xrightarrow{p_\alpha} j(x)$, for $1 \le i \le n$; then $\mathcal{H}_i \ge \mathcal{F}_i^*$ for some $\mathcal{F}_i \xrightarrow{q_\alpha} x$, and thus $\bigcap_{i=1}^n \mathcal{H}_i \ge \bigcap_{i=1}^n \mathcal{F}_i^* = \left(\bigcap_{i=1}^n \mathcal{F}_i\right)^*$. Since $\bigcap_{i=1}^n \mathcal{F}_i \xrightarrow{q_\alpha} x$, $\bigcap_{i=1}^n \mathcal{H}_i \xrightarrow{p_\alpha} j(x)$, and one easily shows that $(X^*, p_\alpha) \in |\text{LIM}|$. It follows from the definition of $\bar{p}$ that $p_\alpha \ge p_\beta$ whenever $\alpha \le \beta$, and thus $(X^*,\bar{p}) \in |\text{GLTS}|$.
Clearly $j : (X,\bar{q}) \to (X^*,\bar{p})$ is an embedding in GLTS. It is shown in [13] that $((X^*, p_\alpha), j)$ is a compactification of (X, q_α) in LIM, and (X^*, p_α) is Hausdorff whenever (X, q_α) is Hausdorff. It follows that $(X^*,\bar{p}), j)$ is a compactification of $(X,\bar{q})$ in GLTS.

Finally, suppose that $(X,\bar{q}) \in |\text{LTS}|$; it is shown that $(X^*,\bar{p}) \in |\text{LTS}|$. Only right continuity remains to be verified. Assume that $0 \le \alpha < \infty$ and $\mathcal{H} \xrightarrow{p_\beta} j(x)$ for each $\beta > \alpha$. Then for each $\beta > \alpha$, there exists $\mathcal{F}_\beta \xrightarrow{q_\beta} x$ such that $\mathcal{H} \ge \mathcal{F}_\beta^*$. Clearly $\bigcup_{\beta>\alpha} \mathcal{F}_\beta$ has the finite intersection property. It follows that $\mathcal{H} \ge \bigvee_{\beta>\alpha} \mathcal{F}_\beta^* = \left(\bigvee_{\beta>\alpha} \mathcal{F}_\beta\right)^*$ and, since $\bigvee_{\beta>\alpha} \mathcal{F}_\beta \xrightarrow{q_\delta} x$ for each $\delta > \alpha$, $\bigvee_{\beta>\alpha} \mathcal{F}_\beta \xrightarrow{q_\alpha} x$. Hence $\mathcal{H} \xrightarrow{p_\alpha} j(x)$. Clearly if $\mathcal{H} \xrightarrow{p_\beta} \langle\mathcal{G}\rangle$ for each $\beta > \alpha$, then $\mathcal{H} \xrightarrow{p_\alpha} \langle\mathcal{G}\rangle$ and thus $(X^*,\bar{p}) \in |\text{LTS}|$ whenever $(X,\bar{q}) \in |\text{LTS}|$. Then$((X^*,\bar{p}), j)$ is a compactification of $(X,\bar{q})$ in LTS. □

An object $(X,\bar{q}) \in |\text{GLTS}|$ is said to be q_0-**regular** if $\text{cl}_{q_0}\mathcal{F} \xrightarrow{q_\alpha} x$ whenever $\mathcal{F} \xrightarrow{q_\alpha} x$, where $\bar{q} = (q_\alpha)$, $0 \le \alpha \le \infty$. Note that this definition is weaker than strong regularity as given in Definition 1.4.

Theorem 2.2 *Assume that $(X,\bar{q}) \in |GLTS|$ and $((X^*,\bar{p}), j)$ is the compactification given in Theorem 2.1. Suppose that $f : (X,\bar{q}) \to (Y,\bar{r})$ is contraction, where $(Y,\bar{r}) \in |GLTS|$*

is compact and r_0-regular. Then there exists a contraction $f^* : (X^*, \bar{p}) \to (Y, \bar{r})$ such that $f^* \circ j = f$.

Proof Let $\langle \mathcal{G} \rangle \in X^*$; define $f^*(\langle \mathcal{G} \rangle) = y$, for some $y \in Y$ such that $f^{\to}\mathcal{G} \xrightarrow{r_0} y$, and $f^*(j(x)) = f(x)$, $x \in X$. Then $f^* \circ j = f$. Observe that if $A \subseteq X$, then $f^*(A^*) \subseteq \mathrm{cl}_{r_0} f(A)$. Indeed, if $\langle \mathcal{G} \rangle \in A^*$, then $A \in \mathcal{G}$, $f^{\to}\mathcal{G} \xrightarrow{r_0} f^*(\langle \mathcal{G} \rangle)$, and thus $f^*(\langle \mathcal{G} \rangle) \in \mathrm{cl}_{r_0} f(A)$. Hence $f^*(A^*) \subseteq \mathrm{cl}_{r_0} f(A)$. If $\mathcal{H} \xrightarrow{p_\alpha} j(x)$, then $\mathcal{H} \geq \mathcal{F}^*$ for some $\mathcal{F} \xrightarrow{q_\alpha} x$. If follows that $f^{*\to}(\mathcal{H}) \geq f^{*\to}(\mathcal{F}^*) \geq \mathrm{cl}_{r_0} f^{\to}\mathcal{F} \xrightarrow{r_\alpha} f(x)$ since $(Y, \bar{r})$ is r_0-regular. Similarly, $\mathcal{H} \xrightarrow{p_\alpha} \langle \mathcal{G} \rangle$ implies that $\mathcal{H} \geq \mathcal{G}^*$ and thus $f^{*\to}(\mathcal{H}) \geq f^{*\to}(\mathcal{G}^*) \geq cl_{r_0} f^{\to}\mathcal{G} \xrightarrow{r_\alpha} f^*(\langle \mathcal{G} \rangle)$. Hence $f^* : (X^*, \bar{p}) \to (Y, \bar{r})$ is a contraction. □

3 Strongly Regular Compactification

Objects in LIM that possess a regular compactification are characterized in [8], and these results are useful in determining which objects in GLTS(LTS) have a strongly regular (Definition 1.4) compactification in GLTS(LTS), respectively. It should be mentioned that a compact, Hausdorff limit space may fail to be regular; limit spaces having a Hausdorff, regular compactification are analyzed in [14]. It is well-known that the category consisting of all regular limit spaces and continuous maps is bireflective in LIM. Given $(X, q) \in |\mathrm{LIM}|$, let (X, rq) denote its regular reflection. More generally, if $(X, \bar{q}) \in |\mathrm{GLTS}|$, define $r\bar{q} := (rq_\alpha)$, $0 \leq \alpha \leq \infty$; then $(X, r\bar{q}) \in |\mathrm{GLTS}|$ is strongly regular. Let **RGLTS**(**RLTS**) denote the full subcategory of GLTS(LTS) consisting of all the strongly regular objects in GLTS(LTS), respectively.

Lemma 3.1 (i) *If $(X, \bar{q}) \in |RGLTS|$, then $(X, l\bar{q}) \in |RLTS|$.*
(ii) *The category RLTS is concretely reflective in GLTS.*

Proof Suppose that $(X, \bar{q}) \in |\mathrm{RGLTS}|$, $0 \leq \alpha < \infty$, and $\mathcal{F} \xrightarrow{(l\bar{q})_\alpha} x$; then for each $\beta > \alpha$, $\mathcal{F} \xrightarrow{q_\beta} x$. It is shown that $\mathrm{cl}_{(l\bar{q})_\alpha} \mathcal{F} \xrightarrow{(l\bar{q})_\alpha} x$. Fix $\beta > \alpha$, $A \subseteq X$, and note that $\mathrm{cl}_{(l\bar{q})_\alpha} A \subseteq \mathrm{cl}_{q_\beta} A$ since $A \in \mathcal{G} \xrightarrow{(l\bar{q})_\alpha} x$ implies that $\mathcal{G} \xrightarrow{q_\beta} x$. Hence $\mathrm{cl}_{(l\bar{q})_\alpha} \mathcal{F} \geq cl_{q_\beta}\mathcal{F} \xrightarrow{q_\beta} x$ and thus $\mathrm{cl}_{(l\bar{q})_\alpha} \mathcal{F} \xrightarrow{q_\beta} x$ for each $\beta > \alpha$. Therefore $\mathrm{cl}_{(l\bar{q})_\alpha} \mathcal{F} \xrightarrow{(l\bar{q})_\alpha} x$ and thus $(X, l\bar{q})$ is strongly regular. It follows from Lemma 2.1 (i) that $(X, l\bar{q}) \in |\mathrm{RLTS}|$.

(ii) Assume that $(X, \bar{q}) \in |\mathrm{GLTS}|$; then $(X, r\bar{q}) \in |\mathrm{RGLTS}|$ and by (i) above, $(X, lr\bar{q}) \in |\mathrm{RLTS}|$. Note that $\mathrm{id}_X : (X, \bar{q}) \to (X, lr\bar{q})$ is a contraction, and suppose that $f : (X, \bar{q}) \to (Y, \bar{p}) \in |\mathrm{RLTS}|$ is a contraction. Since RLIM is concretely reflective in LIM, it follows that $f : (X, r\bar{q}) \to (Y, \bar{p})$ is also a contraction in RGLTS. Applying (i) above and Lemma 2.1(i), $f : (X, lr\bar{q}) \to (Y, \bar{p})$ is a contraction in RLTS. Hence RLTS is concretely reflective in GLTS. □

Given $(X, q) \in |\mathrm{LIM}|$; let $\mathcal{C}_b(X, q)$ denote the set of all bounded, continuous, real-valued functions defined on (X, q), and let δq be the completely regular topology on X determined by $\mathcal{C}_b(X, q)$. The subconstruct of all completely regular topological spaces is concretely reflective in LIM and for a limit space (X, q), the reflection morphism is given by $\mathrm{id}_X : (X, q) \to (X, \delta q)$. Further, $(X, q) \in |\mathrm{LIM}|$ is said to be **δ-regular** in LIM if $\mathrm{cl}_{\delta q} \mathcal{F} \xrightarrow{q} x$ whenever $\mathcal{F} \xrightarrow{q} x$.

Definition 3.1 An object $(X, \bar{q}) \in |\mathrm{GLTS}|$ is called **strongly δ-regular** if for each $0 \leq \alpha \leq \infty$, $(X, q_\alpha) \in |\mathrm{LIM}|$ is δ-regular with respect to the family $\mathcal{C}_b(X, q_\alpha)$.

Let $(X, \bar{q}) \in |\mathrm{GLTS}|$, the following structure on X^* is used to show the existence of a strongly regular compactification of $(X, \bar{q})$.

Definition 3.2 Given $(X, \bar{q}) \in |\mathrm{GLTS}|$ and $(X^*, \bar{p})$ of Definition 2.2, define $\bar{s} = (s_\alpha)$, $0 \leq \alpha \leq \infty$, on X^* as follows:

(i) s_∞ is the indiscrete topology
(ii) $\mathcal{H} \xrightarrow{s_\alpha} j(x)$ iff $\mathcal{H} \geq \mathrm{cl}_{\delta p_\alpha} j^{\rightarrow}\mathcal{F}$ for some $\mathcal{F} \xrightarrow{q_\alpha} x$, $0 \leq \alpha < \infty$
(iii) $\mathcal{H} \xrightarrow{s_\alpha} \langle \mathcal{G} \rangle$ iff $\mathcal{H} \geq \mathrm{cl}_{\delta p_\alpha} j^{\rightarrow}\mathcal{G}$, $\mathcal{G} \in \eta$, $0 \leq \alpha < \infty$.

Lemma 3.2 *Assume that $(X, \bar{q}) \in |GLTS|$ and consider $(X^*, \bar{p}) \in |GLTS|$ and $(X^*, \bar{s})$ given in Definitions 2.2 and 3.2. Then $p_\alpha \geq s_\alpha \geq \delta p_\alpha$, $0 \leq \alpha \leq \infty$, and $(X^*, \bar{s}) \in |RGLTS|$.*

Proof It easily follows that $(X^*, s_\alpha) \in |\mathrm{LIM}|$, $0 \leq \alpha < \infty$. Suppose that $0 \leq \alpha \leq \beta < \infty$ and $\mathcal{H} \xrightarrow{s_\alpha} j(x)$; then $\mathcal{H} \geq \mathrm{cl}_{\delta p_\alpha} j^{\rightarrow}\mathcal{F}$ for some $\mathcal{F} \xrightarrow{q_\alpha} x$. Then $\mathcal{F} \xrightarrow{q_\beta} x$ and since $p_\alpha \geq p_\beta$, $\mathcal{H} \geq \mathrm{cl}_{\delta p_\alpha} j^{\rightarrow}\mathcal{F} \geq \mathrm{cl}_{\delta p_\beta} j^{\rightarrow}\mathcal{F}$. Hence $\mathcal{H} \xrightarrow{s_\beta} j(x)$. Similarly, $\mathcal{H} \xrightarrow{s_\alpha} \langle \mathcal{G} \rangle$ implies that $\mathcal{H} \xrightarrow{s_\beta} \langle \mathcal{G} \rangle$, and thus $s_\alpha \geq s_\beta$. Then $(X^*, \bar{s}) \in |\mathrm{GLTS}|$. Observe that if $\mathcal{F} \in \mathbf{F}(X)$ and $0 \leq \alpha < \infty$, $\mathcal{F}^* \geq \mathrm{cl}_{p_\alpha} j^{\rightarrow}\mathcal{F} \geq \mathrm{cl}_{\delta p_\alpha} j^{\rightarrow}\mathcal{F}$, and it follows that $p_\alpha \geq s_\alpha \geq \delta p_\alpha$. Hence $(X^*, \bar{s})$ is strongly regular. □

Theorem 3.1 *Let $(X, \bar{q}) \in |GLTS|(|LTS|)$; then $(X, \bar{q})$ has a strongly regular compactification in GLTS(LTS) iff $(X, \bar{q})$ is strongly δ-regular, respectively.*

Proof Verification is given for the case whenever $(X, \bar{q}) \in |\mathrm{LTS}|$. According to Theorem 3.2 [8], (X, q_α) has a regular compactification in LIM iff it is δ-regular. Hence a necessary condition for $(X, \bar{q})$ to possess a strongly regular compactification in LTS is that it be strongly δ-regular.

Conversely, assume that $(X, \bar{q}) \in |\mathrm{LTS}|$ is strongly δ-regular and let $((X^*, \bar{p}), j)$ be the compactification of $(X, \bar{q})$ given in Theorem 2.1. It follows from Theorem 2.2 that each bounded, continuous, real-valued function $f : (X, q_\alpha) \to \mathbb{R}$, $0 \leq \alpha < \infty$, has a unique continuous extension $f^* : (X^*, p_\alpha) \to \mathbb{R}$ such that $f^* \circ j = f$. This implies that $(X, \delta q_\alpha)$ is a subspace of $(X^*, \delta p_\alpha)$. Since $p_\alpha \geq s_\alpha$, $j : (X, q_\alpha) \to (X^*, s_\alpha)$ is continuous. Conversely, assume that $\mathcal{F} \in \mathbf{F}(X)$ such that $j^{\rightarrow}\mathcal{F} \xrightarrow{s_\alpha} j(x)$; then there exists $\mathcal{K} \xrightarrow{q_\alpha} x$ such that $j^{\rightarrow}\mathcal{F} \geq \mathrm{cl}_{\delta p_\alpha} j^{\rightarrow}\mathcal{K}$. Since $(X, \delta q_\alpha)$ is a topological subspace of $(X^*, \delta p_\alpha)$, $\mathcal{F} \geq j^{\leftarrow}(\mathrm{cl}_{\delta p_\alpha} j^{\rightarrow}\mathcal{K}) = \mathrm{cl}_{\delta q_\alpha} \mathcal{K} \xrightarrow{q_\alpha} x$ because (X, q_α) is δ-regular. Hence $\mathcal{F} \xrightarrow{q_\alpha} x$ and $j : (X, q_\alpha) \to (X^*, s_\alpha)$ is a dense embedding in LIM. According to Lemma 3.2, (X^*, s_α) is regular, and hence $j : (X, \bar{q}) \to (X^*, \bar{s})$ is a dense embedding in RGLTS. It follows from Lemma 2.1 (ii) and Lemma 3.1(i) that $j : (X, \bar{q}) \to (X^*, l\bar{s})$ is a dense embedding and $(X^*, l\bar{s})$ is strongly regular. Therefore $((X^*, l\bar{s}), j)$ is a strongly regular compactification of $(X, \bar{q})$ in LTS. □

Theorem 3.2 *Suppose that $(X, \bar{q}) \in |GLTS|(|LTS|)$ is strongly δ-regular and $((X^*, \bar{s}), j)$ $(((X^*, l\bar{s}), j))$ is the strongly regular compactification of $(X, \bar{q})$ given in Theorem 3.1, respectively. If $f : (X, \bar{q}) \to (Y, \bar{r})$ is a contraction in GLTS(LTS), where $(Y, \bar{r})$*

is compact and strongly regular, then there exists a contraction $f^* : (X^*, \bar{s}) \to (Y, \bar{r})$ $(f^* : (X^*, l\bar{s}) \to (Y, \bar{r}))$ *in GLTS(LTS), respectively.*

Proof Assume that $(X, \bar{q}) \in |\text{GLTS}|$ is strongly δ-regular and let $0 \leq \alpha < \infty$. According to Theorem 2.2, f has a continuous extension $f^* : (X^*, p_\alpha) \to (Y, r_\alpha)$. Since the full subcategory of all regular objects in LIM is bireflective in LIM, $f^* : (X^*, rp_\alpha) \to (Y, r_\alpha)$ is also continuous, where rp_α denotes the regular reflection of p_α. Since (X^*, rp_α) is compact and regular, it follows from Proposition 3.1 [8] that for each $B \subseteq X^*$, $\text{cl}_{\delta p_\alpha} B = \text{cl}^2_{rp_\alpha} B$. If $\mathcal{H} \xrightarrow{s_\alpha} j(x)$, then there exists $\mathcal{F} \xrightarrow{q_\alpha} x$ such that $\mathcal{H} \geq \text{cl}_{\delta p_\alpha} j^{\to}\mathcal{F} = \text{cl}^2_{rp_\alpha} j^{\to}\mathcal{F} \xrightarrow{rp_\alpha} j(x)$. Since $f^* : (X^*, rp_\alpha) \to (Y, r_\alpha)$ is continuous, $f^{*\to}\mathcal{H} \xrightarrow{r_\alpha} f^*(j(x))$. Likewise, if $\mathcal{H} \xrightarrow{s_\alpha} \langle\mathcal{G}\rangle$, then $f^{*\to}\mathcal{H} \xrightarrow{r_\alpha} f^*(\langle\mathcal{G}\rangle)$, and thus $f^* : (X, \bar{s}) \to (Y, \bar{r})$ is a contraction in GLTS.

Next, suppose that $(X, \bar{q}) \in |\text{LTS}|$ and $f : (X, \bar{q}) \to (Y, \bar{r})$ is a contraction in LTS. It follows from the preceding argument that $f^* : (X^*, \bar{s}) \to (Y, \bar{r})$ is a contraction, and thus by Lemma 2.1(i), $f^* : (X^*, l\bar{s}) \to (Y, \bar{r})$ is a contraction in LTS. □

Given $(X, q) \in |\text{LIM}|$ and $A \subseteq X$, q_A denotes the subspace limit structure on A.

A limit tower space for which all its level components are topological is a non-Archimedean approach space in the sense of [3]. Slightly generalizing this notion, define $(X, \bar{q}) \in |\text{GLTS}|$ to be **non-Archimedean** provided that each of its level components is topological. Further, $(X, \bar{q}) \in |\text{GLTS}|$ is called **strongly completely regular non-Archimedean** whenever each of its level components is a completely regular topology. In particular, if $(X, \bar{q}) \in |\text{GLTS}|$, then $(X, \delta\bar{q}) \in |\text{GLTS}|$ is strongly completely regular non-Archimedean, where $\delta\bar{q} = (\delta q_\alpha)$, $0 \leq \alpha \leq \infty$.

Theorem 3.3 *Assume that* $(X, \bar{q}) \in |LTS|$ *is a non-Archimedean approach space, and let* $((X^*, \bar{p}), j)$ *be the compactification in LTS given in Theorem 2.1. Then* $((X^*, lt\bar{p}), j)$ *is a compactification of* $(X, \bar{q})$ *in LTS which is non-Archimedean.*

Proof Observe that for each $0 \leq \alpha < \infty$, p_α is a pretopology with neighborhood filters of the form: $\mathcal{U}_{p_\alpha}(j(x)) = (\mathcal{U}_{q_\alpha}(x))^*$ and $\mathcal{U}_{p_\alpha}(\langle\mathcal{G}\rangle) = \mathcal{G}^*$, where $x \in X$ and $\mathcal{G} \in \eta$. Since $j : (X, q_\alpha) \to (X^*, p_\alpha)$ is an embedding, it follows that $j : (X, tq_\alpha) \to (X^*, tp_\alpha)$ is continuous. Further, if $U \in tq_\alpha$, it is shown that $j(U) \in (tp_\alpha)_{j(X)}$. Indeed, if $j(x) \in U^*$, then $U^* \in (\mathcal{U}_{q_\alpha}(x))^* = \mathcal{U}_{p_\alpha}(j(x))$ and $\langle\mathcal{G}\rangle \in U^*$ implies that $U \in \mathcal{G}$ and thus $U^* \in \mathcal{G}^* = \mathcal{U}_{p_\alpha}(\langle\mathcal{G}\rangle)$. Therefore $U^* \in tp_\alpha$ and hence $j(U) = U^* \cap j(X) \in (tp_\alpha)_{j(X)}$. It follows that $j : (X, q_\alpha) \to (X^*, tp_\alpha)$ is an embedding and thus $j : (X, \bar{q}) \to (X^*, t\bar{p})$ is an embedding.

Note that if $0 \leq \alpha \leq \beta$, then $p_\alpha \geq p_\beta$ and thus $tp_\alpha \geq tp_\beta$; consequently, $(X^*, t\bar{p}) \in |\text{GLTS}|$ and it follows from Lemma 2.1(ii) that $j : (X, \bar{q}) \to (X^*, lt\bar{p})$ is an embedding in LTS. Recall that $\mathcal{H} \xrightarrow{(lt\bar{p})_\alpha} z$ iff for each $\beta > \alpha$, $\mathcal{H} \xrightarrow{tp_\beta} z$; equivalently, $(lt\bar{p})_\alpha = \bigvee_{\beta > \alpha} tp_\beta$, and thus $(lt\bar{p})_\alpha$ is a topology on X^*. Hence $((X^*, lt\bar{p}), j)$ is non-Archimedean and is a compactification of $(X, \bar{q})$ in LTS. □

Theorem 3.4 *Suppose that* $(X, \bar{q}) \in |LTS|$. *Then* $(X, \bar{q})$ *has a strongly regular compactification in LTS which is non-Archimedean iff* $\bar{q} = \delta\bar{q}$.

Proof Assume that $\delta\bar{q} = \bar{q}$ and since $(X, \delta\bar{q})$ is a subspace of $(X^*, \delta\bar{p})$, $((X^*, \delta\bar{p}), j)$ is a strongly regular compactification of $(X, \bar{q})$ in GLTS. It follows from Lemma 2.1(ii) that $((X^*, l\delta\bar{p}), j)$ is a strongly regular compactification of $(X, \bar{q})$ in LTS which is non-Archimedean. □

4 Strongly Regular *S*-Compactification

An "*S*-compactification" of a limit tower space is investigated in this section, where S is a limit tower monoid acting on the limit tower space. Basic properties of "*S*-spaces" in the context of convergence approach spaces can be found in [4]; moreover, a study of *S*-compactifications in the realm of convergence spaces is given in [9].

Definition 4.1 The pair $(S, \bar{\gamma})$ is called a **limit tower monoid** provided:

(LTM1) $S = (S, \cdot)$ is a monoid
(LTM2) $(S, \bar{\gamma}) \in |\text{LTS}|$
(LTM3) The binary operation $(x, y) \to x \cdot y$ is continuous.

Let **LTM** denote the category whose objects consist of all the limit tower monoids and whose morphisms are all the maps between objects which are both contractions and homomorphisms.

Definition 4.2 Let $X \in |\text{LTS}|$, $S \in |\text{LTM}|$ with identity e, $\lambda : X \times S \to X$ a map, and consider the following axioms:

(A1) $\lambda(x, e) = x$ for all $x \in X$
(A2) $\lambda(\lambda(x, s), t) = \lambda(x, s \cdot t)$ for all $x \in X$ and all $s, t \in S$
(A3) λ is a contraction.

Then λ is called an **action** provided (A1) and (A2) are satisfied. If in addition, λ obeys (A3), then λ is said to be a **c-action**.

Definition 4.3 Denote **C** to be the category consisting of all triples (X, S, λ), where $X \in |\text{LTS}|$, $S \in |\text{LTM}|$, $\lambda : X \times S \to X$ is a c-action, and whose morphisms are all pairs of maps $(f, k) : (X, S, \lambda) \to (Y, T, \mu)$ satisfying:

(C1) $f : X \to Y$ is a morphism in LTS
(C2) $k : S \to T$ is a morphism in LTM
(C3) $\mu \circ (f \times k) = f \circ \lambda$.

A compactification $((Y, \bar{r}), f)$ of $(X, \bar{q})$ in GLTS is called **strict** if for each $\mathcal{H} \xrightarrow{r_\alpha} y$ there exists an $\mathcal{F} \in \mathbf{F}(X)$ such that $\mathcal{H} \geq \text{cl}_{r_\alpha} f^{\to}\mathcal{F}$ and $f^{\to}\mathcal{F} \xrightarrow{r_\alpha} y$. Let **CHY** denote the category of all Cauchy spaces and Cauchy-continuous maps. Keller [7] proved that a limit space (X, q) is induced by a Cauchy space iff $\mathcal{F} \xrightarrow{q} x, y$ implies that all the q-convergent filters to x coincide with all the q-convergent filters to y. Here these limit spaces are called **reciprocal** since they are isomorphic to the category of all complete Cauchy spaces. Further, $(X, \bar{q}) \in |\text{GLTS}|$ is called **strongly reciprocal** provided each (X, q_α) is reciprocal, $0 \leq \alpha \leq \infty$.

Definition 4.4 $((Y, S, \mu), f)$ is said to be an S-**compactification** of (X, S, λ) in C whenever:

(COM1) (Y, f) is a compactification of X in LTS
(COM2) $(f, \mathrm{id}_S) : (X, S, \lambda) \to (Y, S, \mu)$ is a morphism in C.

Lemma 4.1 *Assume the following:*

(i) *$((Y, \bar{r}), f)$ is a strict, strongly regular compactification of $(X, \bar{q})$ in GLTS, with (Y, r_0) Hausdorff*
(ii) *$(S, \bar{\gamma}) \in |LTM|$ and $\lambda : X \times S \to X$ is an action*
(iii) *for each $f^{\to}\mathcal{F} \xrightarrow{r_0} y$ and $s \in S$, $(f \circ \lambda)^{\to}(\mathcal{F} \times \dot{s}) \xrightarrow{r_0} z \equiv \mu(y, s)$*
(iv) *for each $f^{\to}\mathcal{F} \xrightarrow{r_\alpha} y$, $\mathcal{G} \xrightarrow{\gamma_\alpha} s$, $(f \circ \lambda)^{\to}(\mathcal{F} \times \mathcal{G}) \xrightarrow{r_\alpha} \mu(y, s)$, $0 \le \alpha \le \infty$.*
Then $\mu : (Y, \bar{r}) \times (S, \bar{\gamma}) \to (Y, \bar{r})$ is a contraction.

Proof Since (Y, r_0) is Hausdorff, it is straightforward to show that μ defined in (iii) is well-defined. Also, observe that $\mu(f(x), s) = f(\lambda(x, s))$, for each $x \in X$ and $s \in S$. Next, it is shown that if $A \subseteq X$ and $B \subseteq S$, then $\mu(\mathrm{cl}_{r_\alpha} f(A) \times B) \subseteq \mathrm{cl}_{r_\alpha}(f \circ \lambda)(A \times B)$. Indeed, assume that $y \in \mathrm{cl}_{r_\alpha} f(A)$ and $s \in B$; then there exists $f^{\to}\mathcal{F} \xrightarrow{r_\alpha} y$ such that $A \in \mathcal{F}$. Using (iv), $(f \circ \lambda)(A \times B) \in (f \circ \lambda)^{\to}(\mathcal{F} \times \dot{s}) \xrightarrow{r_\alpha} \mu(y, s)$, and thus $\mu(y, s) \in \mathrm{cl}_{r_\alpha}(f \circ \lambda)(A \times B)$. Therefore $\mu(\mathrm{cl}_{r_\alpha} f(A) \times B) \subseteq \mathrm{cl}_{r_\alpha}(f \circ \lambda)(A \times B)$. Next, suppose that $\mathcal{H} \xrightarrow{r_\alpha} y$ and $\mathcal{G} \xrightarrow{\gamma_\alpha} s$. Since (Y, f) is a strict compactification of X, there exists an $\mathcal{F} \in \mathbf{F}(X)$ such that $\mathcal{H} \ge \mathrm{cl}_{r_\alpha} f^{\to}\mathcal{F}$ and $f^{\to}\mathcal{F} \xrightarrow{r_\alpha} y$. Then $\mu^{\to}(\mathcal{H} \times \mathcal{G}) \ge \mu^{\to}(\mathrm{cl}_{r_\alpha} f^{\to}\mathcal{F} \times \mathcal{G}) \ge \mathrm{cl}_{r_\alpha}(f \circ \lambda)^{\to}(\mathcal{F} \times \mathcal{G})$, and by (iv) and the regularity of (Y, r_α), $\mathrm{cl}_{r_\alpha}(f \circ \lambda)^{\to}(\mathcal{F} \times \mathcal{G}) \xrightarrow{r_\alpha} \mu(y, s)$. Hence $\mu^{\to}(\mathcal{H} \times G) \xrightarrow{r_\alpha} \mu(y, s)$ and μ is a contraction. □

Theorem 4.1 *Suppose that $(X, S, \lambda) \in |C|$, $((Y, \bar{r}), f)$ is a strict, strongly regular compactification of $(X, \bar{q})$ in LTS, and $(Y, \bar{r})$ is strongly reciprocal with (Y, r_0) Hausdorff. For each $0 \le \alpha \le \infty$, denote $C_\alpha = \{\mathcal{F} \in \mathbf{F}(X) : f^{\to}\mathcal{F}\ r_\alpha\text{-converges}\}$. Assume that $(S, \bar{\gamma}) \in |LTM|$ is strongly reciprocal and (S, γ_α) is induced by the complete Cauchy space (S, D_α), $0 \le \alpha \le \infty$. Then there exists a c-action $\mu : Y \times S \to Y$ such that $((Y, S, \mu), f)$ is a strongly regular S-compactification of (X, S, λ) in C iff for each $0 \le \alpha \le \infty$, $\lambda : (X, C_\alpha) \times (S, D_\alpha) \to (X, C_\alpha)$ is Cauchy-continuous.*

Proof First, observe that for each $0 \le \alpha \le \infty$, $(X, C_\alpha) \in |\mathrm{CHY}|$. Clearly, $\dot{x} \in C_\alpha$ and $\mathcal{G} \ge \mathcal{F} \in C_\alpha$ implies that $\mathcal{G} \in C_\alpha$, $0 \le \alpha \le \infty$. Suppose that $\mathcal{F}_i \in C_\alpha$ such that $\mathcal{F}_1 \vee \mathcal{F}_2$ exists, and $f^{\to}\mathcal{F}_i \xrightarrow{r_\alpha} y_i$, $i = 1, 2$. Then $f^{\to}\mathcal{F}_1 \vee f^{\to}\mathcal{F}_2$ exists and $f^{\to}\mathcal{F}_1 \vee f^{\to}\mathcal{F}_2 \xrightarrow{r_\alpha} y_1, y_2$. Since (Y, r_α) is reciprocal, $f^{\to}(\mathcal{F}_1 \cap \mathcal{F}_2) \xrightarrow{r_\alpha} y_1, y_2$ and thus $\mathcal{F}_1 \cap \mathcal{F}_2 \in C_\alpha$. Hence $(X, C_\alpha) \in |\mathrm{CHY}|$.
Assume that $((Y, S, \mu), f)$ is an S-compactification of $(X, S, \lambda) \in C$. Suppose that $\mathcal{F} \in C_\alpha$ and $\mathcal{G} \in D_\alpha$; then $f^{\to}\mathcal{F} \xrightarrow{r_\alpha} y$ and $\mathcal{G} \xrightarrow{\gamma_\alpha} s$, for some $y \in Y$ and $s \in S$. Since μ is a c-action, $f^{\to}(\lambda^{\to}(\mathcal{F} \times \mathcal{G})) = (\mu \circ (f \times \mathrm{id}_S))^{\to}(\mathcal{F} \times \mathcal{G}) = \mu^{\to}(f^{\to}\mathcal{F} \times \mathcal{G}) \xrightarrow{r_\alpha} \mu(y, s)$, and thus $\lambda^{\to}(\mathcal{F} \times \mathcal{G}) \in C_\alpha$. Hence $\lambda : (X, C_\alpha) \times (S, D_\alpha) \to (X, C_\alpha)$ is Cauchy-continuous, for each $0 \le \alpha \le \infty$.

Conversely, suppose that $\lambda : (X, C_\alpha) \times (S, D_\alpha) \to (X, C_\alpha)$ is Cauchy-continuous, for each $0 \le \alpha \le \infty$. Define $\mu : Y \times S \to Y$ by $\mu(y, s) = \lim_{r_0}(f \circ \lambda)^{\to}(\mathcal{F} \times \dot{s})$, where $f^{\to}\mathcal{F} \xrightarrow{r_0} y$. Observe that μ is well-defined. Indeed, if $f^{\to}\mathcal{F}_1 \xrightarrow{r_0} y$ and $\mathcal{G}_1 \xrightarrow{\gamma_0} s$, then $f^{\to}(\mathcal{F} \cap \mathcal{F}_1) = f^{\to}\mathcal{F} \cap f^{\to}\mathcal{F}_1 \xrightarrow{r_0} y$ and $\dot{s} \cap \mathcal{G}_1 \xrightarrow{\gamma_0} s$. Recall that λ is Cauchy-continuous, $\mathcal{F} \cap \mathcal{F}_1 \in C_0$, $\dot{s} \cap \mathcal{G}_1 \in D_0$, and thus $\lambda^{\to}((\mathcal{F} \cap \mathcal{F}_1) \times (\dot{s} \cap \mathcal{G}_1)) \in C_0$. Therefore, $f^{\to}(\lambda^{\to}((\mathcal{F} \cap \mathcal{F}_1) \times (\dot{s} \cap \mathcal{G}_1)))$ r_0-converges. Since (Y, r_0) is Hausdorff, $(f \circ \lambda)^{\to}(\mathcal{F} \times \dot{s})$ and $(f \circ \lambda)^{\to}(\mathcal{F}_1 \times \mathcal{G}_1)$ r_0-converge to $\mu(y, s)$, and hence μ is well-defined.

Lemma 4.1 is used to verify that $\mu : Y \times S \to Y$ is a contraction; it remains to verify (iv). Assume that $f^{\to}\mathcal{F} \xrightarrow{r_\alpha} y$ and $\mathcal{G} \xrightarrow{\gamma_\alpha} s$. There exist an $f^{\to}\mathcal{F}_1 \xrightarrow{r_0} y$ and $\mathcal{G}_1 \xrightarrow{\gamma_0} s$, and thus $r_0 \ge r_\alpha$ implies that $f^{\to}(\mathcal{F} \cap \mathcal{F}_1) \xrightarrow{r_\alpha} y$ and $\mathcal{G} \cap \mathcal{G}_1 \xrightarrow{\gamma_\alpha} s$. Since $\lambda : (X, C_\alpha) \times (S, D_\alpha) \to (X, C_\alpha)$ is Cauchy-continuous, $(f \circ \lambda)^{\to}((\mathcal{F} \cap \mathcal{F}_1) \times (\mathcal{G} \cap \mathcal{G}_1))$ r_α-converges. However, $(f \circ \lambda)^{\to}(\mathcal{F}_1 \times \mathcal{G}_1) \xrightarrow{r_\alpha} \mu(y, s)$, and since (Y, r_α) is reciprocal, $(f \circ \lambda)^{\to}(\mathcal{F} \times \mathcal{G}) \xrightarrow{r_\alpha} \mu(y, s)$. Therefore, by Lemma 4.1, μ is a contraction. Since (Y, r_0) is Hausdorff, the proof given in Theorem 3.5[9] for the convergence space setting shows that μ is also an action. Hence $((Y, \bar{r}), S, \mu)$ is a strongly regular S-compactification of (X, S, λ) in C. □

The following assumptions and notations are used in Lemma 4.2 below: $(X, \bar{q}) \in |\mathrm{LTS}|$, (X, q_0) is Hausdorff, $(S, \cdot)$ is a monoid, and $\lambda : X \times S \to X$ is an action. Let $((X^*, \bar{p}), j))$ denote the compactification of $(X, \bar{q})$ in LTS as defined in Section 2. Recall that $X^* = X \cup \{\langle \mathcal{G} \rangle : \mathcal{G} \in \eta\}$ and $j : X \to X^*$ is the natural injection $j(x) = x$, $x \in X$. Define $\lambda^* : X^* \times S \to X^*$ as follows:

$$\begin{aligned} &\lambda^*(j(x), s) = j(\lambda(x, s)), x \in X, s \in S \\ &\lambda^*(\langle \mathcal{G} \rangle, s) = \lim_{p_0}(j \circ \lambda)^{\to}(\mathcal{G} \times \dot{s}), \mathcal{G} \in \eta. \end{aligned} \tag{4.1}$$

It is shown in Theorem 2.1 that (X^*, p_0) is Hausdorff whenever (X, q_0) is Hausdorff. Also δp_α denotes the completely regular topological reflection associated with (X^*, p_α), $0 \le \alpha \le \infty$.

Lemma 4.2 *Using the notations and assumptions given above, suppose that for each* $0 \le \alpha \le \infty$, $\lambda : (X, q_\alpha) \times \{s\} \to (X, q_\alpha)$ *is continuous, for each fixed* $s \in S$. *Let* $A \subseteq X$ *and* $B \subseteq S$; *then*

(i) $\lambda^*(A^* \times \{s\}) \subseteq \mathrm{cl}_{p_0}(j \circ \lambda)(A \times \{s\})$, *for each fixed* $s \in S$
(ii) $\lambda^* : (X^*, \delta p_\alpha) \times \{s\} \to (X^*, \delta p_\alpha)$ *is continuous*
(iii) $\lambda^*(\mathrm{cl}_{\delta p_\alpha} j(A) \times B) \subseteq \mathrm{cl}_{\delta p_\alpha}(j \circ \lambda)(A \times B)$.

Proof (i): Let $\langle \mathcal{G} \rangle \in A^*$; then $A \in \mathcal{G}$. If $\lambda^{\to}(\mathcal{G} \times \dot{s}) \xrightarrow{q_0} x$, then $(j \circ \lambda)^{\to}(\mathcal{G} \times \dot{s}) \xrightarrow{p_0} j(x)$ and thus $\lambda^*(\langle \mathcal{G} \rangle, s) = j(x) \in \mathrm{cl}_{p_0}(j \circ \lambda)(A \times \{s\})$. Moreover if $\lambda^{\to}(\mathcal{G} \times \dot{s}) \in \eta$, then $\lambda^*(\langle \mathcal{G} \rangle, s) = \langle \lambda^{\to}(\mathcal{G} \times \dot{s}) \rangle \in \mathrm{cl}_{p_0}(j \circ \lambda)(A \times \{s\})$ and the result follows.

(ii) First, it is shown that $\lambda^* : (X^*, p_\alpha) \times \{s\} \to (X^*, \delta p_\alpha)$ is continuous. Assume that $\mathcal{H} \xrightarrow{p_\alpha} j(x)$; then $\mathcal{H} \ge \mathcal{F}^*$ for some $\mathcal{F} \xrightarrow{q_\alpha} x$. Since $\lambda^{\to}(\mathcal{F} \times \dot{s}) \xrightarrow{q_\alpha} \lambda(x, s)$, it follows from (i) that $\lambda^{*\to}(\mathcal{H} \times \dot{s}) \ge \lambda^*(\mathcal{F}^* \times \dot{s}) \ge \mathrm{cl}_{p_\alpha}(j \circ \lambda)^{\to}(\mathcal{F} \times \dot{s}) \xrightarrow{\delta p_\alpha} j(\lambda(x, s)) = \lambda^*(j(x), s)$. Suppose that $\mathcal{H} \xrightarrow{p_\alpha} \langle \mathcal{G} \rangle$; then $\mathcal{H} \ge \mathcal{G}^*$ and from (i), $\lambda^{*\to}(\mathcal{H} \times \dot{s}) \ge \lambda^{*\to}(\mathcal{G}^* \times \dot{s}) \ge$

$\mathrm{cl}_{p_0}(j \circ \lambda)^{\to}(\mathcal{G} \times \dot{s}) \xrightarrow{\delta p_0} \lambda^*(\langle \mathcal{G} \rangle, s)$. Then $\lambda^* : (X^*, p_\alpha) \times \{s\} \to (X^*, \delta p_\alpha)$ is continuous, and it follows that for each $0 \leq \alpha \leq \infty$, $\lambda^* : (X^*, \delta p_\alpha) \times \{s\} \to (X^*, \delta p_\alpha)$ is continuous.

(iii) Employing (ii),

$$\begin{aligned} \lambda^*(\mathrm{cl}_{\delta p_\alpha} j(A) \times B) &= \bigcup_{s \in B} \lambda^*(\mathrm{cl}_{\delta p_\alpha} j(A) \times \{s\}) \\ &\subseteq \bigcup_{s \in B} \mathrm{cl}_{\delta p_\alpha} \lambda^*(j(A) \times \{s\}) \\ &\subseteq \mathrm{cl}_{\delta p_\alpha} \lambda^*(j(A) \times B) \\ &= \mathrm{cl}_{\delta p_\alpha} (j \circ \lambda)(A \times B). \end{aligned}$$

□

Lemma 4.3 *Given the assumptions and the notations listed in Lemma 4.2, let $((X^*, \bar{p}), j)$ and $((X^*, \bar{s}), j)$ be the compactifications of $(X, \bar{q})$ defined in Sections 2 and 3. Define $\bar{\gamma} = (\gamma_\alpha)$, $0 \leq \alpha \leq \infty$ as follows:*

$$\begin{aligned} &\text{(a)} \quad \gamma_\infty \text{ is the indiscrete structure on } S \\ &\text{(b)} \quad \text{for } 0 \leq \alpha < \infty, \mathcal{K} \xrightarrow{\gamma_\alpha} s \text{ iff for each } \beta > \alpha \text{ and } j^{\to}\mathcal{F} \xrightarrow{s_\beta} y, \\ &\qquad (j \circ \lambda)^{\to}(\mathcal{F} \times \mathcal{K}) \xrightarrow{(l\bar{s})_\beta} \lambda^*(y, s). \end{aligned} \tag{4.2}$$

Then

(i) *$\lambda^* : X^* \times S \to X^*$ defined in (4.1) is an action*
(ii) *$(S, \cdot, \bar{\gamma}) \in |LTM|$*

Proof The straightforward proof of (i) is omitted.

(ii) First, it is shown that $(S, \bar{\gamma}) \in$ |LTS|. Fix $0 \leq \alpha < \infty$. Observe that $\dot{s} \xrightarrow{\gamma_\alpha} s$; indeed, if $\beta > \alpha$ and $j^{\to}\mathcal{F} \xrightarrow{s_\beta} j(x)$, then $\mathcal{F} \xrightarrow{q_\beta} x$. Since $\lambda : (X, q_\beta) \times \{s\} \to (X, q_\beta)$ is continuous, $(j \circ \lambda)^{\to}(\mathcal{F} \times \dot{s}) \xrightarrow{s_\beta} j(\lambda(x, s)) = \lambda^*(j(x), s)$. Moreover, if $j^{\to}\mathcal{F} \xrightarrow{s_\beta} \langle \mathcal{G} \rangle$, then $j^{\to}\mathcal{F} \geq \mathrm{cl}_{\delta p_\beta} j^{\to}\mathcal{G}$ and, since $(X, \delta q_\beta)$ is a subspace of $(X^*, \delta p_\beta)$, $\mathcal{F} \geq \mathrm{cl}_{\delta q_\beta} \mathcal{G}$. Then $(j \circ \lambda)^{\to}(\mathcal{F} \times \dot{s}) \geq (j \circ \lambda)^{\to}(\mathrm{cl}_{\delta q_\beta} \mathcal{G} \times \dot{s}) = (\lambda^* \circ (j \times \mathrm{id}_S))^{\to}(\mathrm{cl}_{\delta q_\beta} \mathcal{G} \times \dot{s}) = \lambda^{*\to}(j^{\to}(\mathrm{cl}_{\delta q_\beta} \mathcal{G}) \times \dot{s})$. According to Lemma 4.2 (iii), $\lambda^{*\to}(j^{\to}(\mathrm{cl}_{\delta q_\beta} \mathcal{G}) \times \dot{s}) \geq \lambda^{*\to}(\mathrm{cl}_{\delta p_\beta} j^{\to}\mathcal{G} \times \dot{s}) \geq \mathrm{cl}_{\delta p_\beta}(j \circ \lambda)^{\to}(\mathcal{G} \times \dot{s}) \xrightarrow{s_\beta} \lambda^*(\langle \mathcal{G} \rangle, s)$. Then $\dot{s} \xrightarrow{\gamma_\alpha} s$.
Suppose that $\mathcal{K}_i \xrightarrow{\gamma_\alpha} s$, $i = 1, 2$, $\beta > \alpha$, and $j^{\to}\mathcal{F} \xrightarrow{s_\beta} y$. Then $(j \circ \lambda)^{\to}(\mathcal{F} \times \mathcal{K}_1 \cap \mathcal{K}_2) = (j \circ \lambda)^{\to}(\mathcal{F} \times \mathcal{K}_1) \cap (j \circ \lambda)^{\to}(\mathcal{F} \times \mathcal{K}_2) \xrightarrow{(l\bar{s})_\beta} \lambda^*(y, s)$ since $(X^*, (l\bar{s})_\beta) \in$ |LIM|. Hence $\mathcal{K}_1 \cap \mathcal{K}_2 \xrightarrow{\gamma_\alpha} s$. It easily follows that if $\alpha \leq \beta$ then $\gamma_\alpha \geq \gamma_\beta$. Next, it is shown that $(S, \bar{\gamma})$ is right-continuous. Assume that for each $\delta > \alpha$, $\mathcal{K} \xrightarrow{\gamma_\delta} s$. Given $\beta > \alpha$, choose $\alpha < \alpha_1 < \beta$ and since $\mathcal{K} \xrightarrow{\gamma_{\alpha_1}} s$, it follows from the definition of γ_{α_1} that $(j \circ \lambda)^{\to}(\mathcal{F} \times \mathcal{K}) \xrightarrow{(l\bar{s})_\beta} \lambda^*(y, s)$ whenever $j^{\to}\mathcal{F} \xrightarrow{s_\beta} y$. Then $\mathcal{K} \xrightarrow{\gamma_\alpha} s$ and thus $(S, \bar{\gamma}) \in$ |LTS|.
Finally, it is shown that the operation on $(S, \cdot, \bar{\gamma})$ is a contraction. Fix $0 \leq \alpha < \infty$ and suppose that $\mathcal{K}_i \xrightarrow{\gamma_\alpha} s_i$, $i = 1, 2$. It is shown that $\mathcal{K}_1 \cdot \mathcal{K}_2 \xrightarrow{\gamma_\alpha} s_1 \cdot s_2$. Let $\beta > \alpha$ and $j^{\to}\mathcal{F} \xrightarrow{s_\beta} y$. Then $(j \circ \lambda)^{\to}(\mathcal{F} \times \mathcal{K}_1 \cdot \mathcal{K}_2) = j^{\to}(\lambda^{\to}(\lambda^{\to}(\mathcal{F} \times \mathcal{K}_1) \times \mathcal{K}_2))$. Since $\mathcal{K}_1 \xrightarrow{\gamma_\alpha} s_1$, $(j \circ \lambda)^{\to}(\mathcal{F} \times \mathcal{K}_1) \xrightarrow{(l\bar{s})_\beta} \lambda^*(y, s_1)$, and thus $(j \circ \lambda)^{\to}(\mathcal{F} \times \mathcal{K}_1) \xrightarrow{s_\delta} \lambda^*(y, s_1)$, for

each $\delta > \beta$. However, $\delta > \beta > \alpha$, $\mathcal{K}_2 \xrightarrow{\gamma_\alpha} s_2$, and $j^{\rightarrow}(\lambda^{\rightarrow}(\mathcal{F} \times \mathcal{K}_1)) \xrightarrow{s_\delta} \lambda^*(y, s_1)$ implies that $(j \circ \lambda)^{\rightarrow}(\lambda^{\rightarrow}(\mathcal{F} \times \mathcal{K}_1) \times \mathcal{K}_2) \xrightarrow{(l\bar{s})_\delta} \lambda^*(\lambda^*(y, s_1), s_2) = \lambda^*(y, s_1 \cdot s_2)$, for each $\delta > \beta$. Since $(S, l\bar{s}) \in |\text{LTS}|$, $(j \circ \lambda)^{\rightarrow}(\lambda^{\rightarrow}(\mathcal{F} \times \mathcal{K}_1) \times \mathcal{K}_2) \xrightarrow{(l\bar{s})_\beta} \lambda^*(y, s_1 \cdot s_2)$; that is, $(j \circ \lambda)^{\rightarrow}(\mathcal{F} \times \mathcal{K}_1 \cdot \mathcal{K}_2) \xrightarrow{(l\bar{s})_\beta} \lambda^*(y, s_1 \cdot s_2)$, and thus $\mathcal{K}_1 \cdot \mathcal{K}_2 \xrightarrow{\gamma_\alpha} s_1 \cdot s_2$. Therefore $(S, \bar{\gamma}) \in |\text{LTM}|$. □

Theorem 4.2 *Assume that $(X, \bar{q}) \in |LTS|$, (X, q_0) is Hausdorff, $(S, \cdot)$ is a monoid, and $\lambda : X \times S \to X$ is an action such that $\lambda : (X, q_\alpha) \times \{s\} \to (X, q_\alpha)$ is continuous, for each fixed $s \in S$ and $0 \le \alpha \le \infty$. Let $((X^*, \bar{p}), j)$ and $((X^*, \bar{s}), j)$ be the compactifications defined in Sections 2 and 3. Then $\bar{\gamma}$, defined in (4.2), is the coarsest structure on S such that $((X^*, l\bar{s}), (S, \bar{\gamma}), \lambda^*)$ is a strongly regular S-compactification of $((X, \bar{q}), (S, \bar{\gamma}), \lambda)$ in C.*

Proof Fix $0 \le \alpha < \infty$ and suppose that $\mathcal{H} \xrightarrow{(l\bar{s})_\alpha} j(x)$ and $\mathcal{K} \xrightarrow{\gamma_\alpha} s$; it is shown that $\lambda^{*\rightarrow}(\mathcal{H} \times \mathcal{K}) \xrightarrow{(l\bar{s})_\alpha} \lambda^*(j(x), s)$. Let $\beta > \alpha$; then $\mathcal{H} \xrightarrow{s_\beta} j(x)$ and thus there exists $\mathcal{F} \xrightarrow{q_\beta} x$ such that $\mathcal{H} \ge \text{cl}_{\delta p_\beta} j^{\rightarrow}\mathcal{F}$. Using the definition of γ_α, since $j^{\rightarrow}\mathcal{F} \xrightarrow{s_\beta} j(x)$, $(j \circ \lambda)^{\rightarrow}(\mathcal{F} \times \mathcal{K}) \xrightarrow{(l\bar{s})_\beta} \lambda^*(j(x), s)$, and by Lemma 4.2(iii), $\lambda^{*\rightarrow}(\mathcal{H} \times \mathcal{K}) \ge \lambda^{*\rightarrow}(\text{cl}_{\delta p_\beta} j^{\rightarrow}\mathcal{F} \times \mathcal{K}) \ge \text{cl}_{\delta p_\beta}(j \circ \lambda)^{\rightarrow}(\mathcal{F} \times \mathcal{K})$. Since $(j \circ \lambda)^{\rightarrow}(\mathcal{F} \times \mathcal{K}) \xrightarrow{s_\delta} \lambda^*(j(x), s)$ for each $\delta > \beta$, $\text{cl}_{\delta p_\beta}(j \circ \lambda)^{\rightarrow}(\mathcal{F} \times \mathcal{K}) \xrightarrow{s_\delta} \lambda^*(j(x), s)$ for each $\delta > \beta$. Hence $\text{cl}_{\delta p_\beta}(j \circ \lambda)^{\rightarrow}(\mathcal{F} \times \mathcal{K}) \xrightarrow{(l\bar{s})_\beta} \lambda^*(j(x), s)$, and thus $\lambda^{*\rightarrow}(\mathcal{H} \times \mathcal{K}) \xrightarrow{(l\bar{s})_\beta} \lambda^*(j(x), s)$ for each $\beta > \alpha$. Therefore $\lambda^{*\rightarrow}(\mathcal{H} \times \mathcal{K}) \xrightarrow{(l\bar{s})_\alpha} \lambda^*(j(x), s)$. Next, if $\mathcal{H} \xrightarrow{(l\bar{s})_\alpha} \langle\mathcal{G}\rangle$ and $\beta > \alpha$, then $\mathcal{H} \xrightarrow{s_\beta} \langle\mathcal{G}\rangle$ and thus $\mathcal{H} \ge \text{cl}_{\delta p_\beta} j^{\rightarrow}\mathcal{G}$. An argument similar to the previous case shows that $\lambda^{*\rightarrow}(\mathcal{H} \times \mathcal{K}) \xrightarrow{(l\bar{s})_\alpha} \lambda^*(\langle\mathcal{G}\rangle, s)$, and thus $\lambda^* : (X, l\bar{q}) \times (S, \bar{\gamma}) \to (X^*, l\bar{s})$ is a contraction. Since $\lambda^* \circ (j \times \text{id}_S) = j \circ \lambda$, $((X^*, l\bar{s}), (S, \bar{\gamma}), \lambda^*)$ is a strongly regular S-compactification of $((X, \bar{q}), (S, \bar{\gamma}), \lambda)$ in C.

Next, assume that $(S, \bar{\sigma}) \in |\text{LTM}|$ such that $\lambda^* : (X^*, l\bar{s}) \times (S, \bar{\sigma}) \to (X^*, l\bar{s})$ is a contraction. Fix $0 \le \alpha < \infty$ and suppose that $\mathcal{K} \xrightarrow{\sigma_\alpha} s$. Let $\beta > \alpha$ and $j^{\rightarrow}\mathcal{F} \xrightarrow{s_\beta} y$; then $j^{\rightarrow}\mathcal{F} \xrightarrow{(l\bar{s})_\beta} y$. Since $\mathcal{K} \xrightarrow{\sigma_\beta} s$, the continuity of $\lambda^* : (X^*, (l\bar{s})_\beta) \times (S, \sigma_\beta) \to (X^*, (l\bar{s})_\beta)$ implies that $\lambda^{*\rightarrow}(j^{\rightarrow}\mathcal{F} \times \mathcal{K}) \xrightarrow{(l\bar{s})_\beta} \lambda^*(y, s)$. However, $(j \circ \lambda)^{\rightarrow}(\mathcal{F} \times \mathcal{K}) = (\lambda^* \circ (j \times \text{id}_S))^{\rightarrow}(\mathcal{F} \times \mathcal{K}) = \lambda^{*\rightarrow}(j^{\rightarrow}\mathcal{F} \times \mathcal{K}) \xrightarrow{(l\bar{s})_\beta} \lambda^*(y, s)$, and thus $\mathcal{K} \xrightarrow{\gamma_\alpha} s$. Therefore $\sigma_\alpha \ge \gamma_\alpha$ and thus $\bar{\sigma} \ge \bar{\gamma}$. □

Remark 4.1 Under the assumptions made in Theorem 4.2, if $\bar{\sigma}$ denotes the discrete structure on S, then $((X^*, l\bar{s}), (S, \bar{\sigma}), \lambda^*)$ is a strongly regular S-compactification of $((X, \bar{q}), (S, \bar{\sigma}), \lambda)$ in C.

5 Conclusion

Some comments pertaining to strict compactifications are made. The compactification $((X^*, \bar{p}), j)$ given in Theorem 2.1 is easily shown to be strict. Strictness of $((X^*, \bar{s}), j)$ listed in Theorem 3.1 seems to fail. However, $\mathcal{H} \xrightarrow{s_\alpha} y$ implies that there exists $\mathcal{F} \in \mathbf{F}(X)$

such that $\mathcal{H} \geq \text{cl}_{\delta p_\alpha} j^{\rightarrow}\mathcal{F} = \text{cl}_{\delta s_\alpha} j^{\rightarrow}\mathcal{F} = \text{cl}^2_{s_\alpha} j^{\rightarrow}\mathcal{F}$ and $j^{\rightarrow}\mathcal{F} \xrightarrow{s_\alpha} y$. If each (X^*, s_α), $0 \leq \alpha \leq \infty$, is symmetric (regular and reciprocal), then $\text{cl}^2_{s_\alpha} j^{\rightarrow}\mathcal{F} = \text{cl}_{s_\alpha} j^{\rightarrow}\mathcal{F}$ and, in this case, $((X^*, \bar{s}), j)$ is strict. Strictness does not seem to carry over to the l-modification $((X^*, l\bar{s}), j) \in |\text{LTS}|$. Further, a general definition of strictness has been defined by Colebunders et al. [5]. Assume that $((Y, r), f)$ is a compactification of $(X, \bar{q})$ in LTS. In particular, $\text{cl}_{r_0} f(X) = Y$. Suppose that $f(X)$ is an $\alpha = 0$ strict subspace of Y in the sense of [5]. It follows from this definition that if $\mathcal{H} \xrightarrow{r_\beta} y$, $\beta \in [0, \infty]$, then there exists $\mathcal{F} \in \mathbf{F}(X)$ such that $\text{cl}_{r_0} f^{\rightarrow}\mathcal{F} \leq \mathcal{H}$ and $f^{\rightarrow}\mathcal{F} \xrightarrow{r_\beta} y$. Then $\text{cl}_{r_\beta} f^{\rightarrow}\mathcal{F} \leq \mathcal{H}$ and thus $((Y, r), f)$ is a strict compactification of $(X, \bar{q})$ in our sense.

Two open questions are listed below; the second question was raised by the referee:

(i) Characterize the limit tower spaces possessing a regular (as defined in [3]) compactification.
(ii) When does a limit tower space have a strict, regular (strongly regular) compactification, respectively?

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