Appl Categor Struct (2017) 25:155–186
DOI 10.1007/s10485-015-9416-9

A Generalization of Gabriel's Galois Covering Functors II: 2-Categorical Cohen-Montgomery Duality

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Received: 16 March 2015 / Accepted: 29 October 2015 / Published online: 10 November 2015

Abstract Given a group G, we define suitable 2-categorical structures on the class of all small categories with G-actions and on the class of all small G-graded categories, and prove that 2-categorical extensions of the orbit category construction and of the smash product construction turn out to be 2-equivalences (2-quasi-inverses to each other), which extends the Cohen-Montgomery duality. Further we characterize equivalences in both 2-categories.

Keywords 2-categories · Orbit categories · Smash products

Mathematics Subject Classification (2010) 18D05 · 16W22 · 16W50

1 Introduction

Throughout this paper G is a group and $\Bbbk$ is a commutative ring, and all categories, functors, and algebras considered here are assumed to be $\Bbbk$-linear unless otherwise stated. This is a continuation of the paper [1] and will be applied in subsequent papers [3] and [2].

In [7] Cohen and Montgomery proved the following (called the *Cohen-Montgomery duality*).

This work is partially supported by Grant-in-Aid for Scientific Research (C) 21540036 and Challenging Exploratory Research 25610003 from JSPS

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Theorem 1.1 *Let G be a finite group of order n, A an algebra with a G-action, and B a G-graded algebra. Then we have isomorphisms*

$$\begin{aligned}(A * G)\#G &\cong M_n(A)\\ (B\#G) * G &\cong M_n(B).\end{aligned}$$

In the above, $*$ and # stand for the skew group algebra construction and the smash product construction, respectively and $M_n(A)$ denotes the algebra of all $n \times n$ matrices over A. We can regard each algebra A as a category with a single object, and then $M_n(A)$ can be regarded as a category with precisely n objects that are isomorphic to each other, and A and $M_n(A)$ are equivalent as categories.

Already some attempts have been made to extend this theorem so that it satisfies the following requirements.

(a) Deal with an arbitrary group G;
(b) Replace algebras by categories.

For instance (a) was investigated in [4], [11], and (b) was examined in [6], [1]. To be more precise let $\mathcal{C}$ be a category with a G-action and $\mathcal{B}$ a G-graded category. Then a G-graded category $\mathcal{C}/G$, called the *orbit category* of $\mathcal{C}$ by G is constructed in [1, 6, 8] (this turns out to be also a generalization of the skew group algebra construction); and a category $\mathcal{B}\#G$ with a free G-action, called the *smash product* of $\mathcal{C}$ and G is constructed in [6]; and in [1] we defined a (weakly) G-equivariant equivalence $\varepsilon_{\mathcal{C}} \colon \mathcal{C} \Rightarrow (\mathcal{C}/G)\#G$ and a degree-preserving equivalence $\omega_{\mathcal{B}} \colon \mathcal{B} \Rightarrow (\mathcal{B}\#G)/G$. This seems to give a full categorical generalization of Cohen-Montgomery duality.

Here recall the definition of equivalences between categories: Categories (= objects) $\mathcal{A}$ and $\mathcal{B}$ are said to be *equivalent* if there exist a pair of functors (= 1-morphisms) $E \colon \mathcal{A} \to \mathcal{B}$ and $F \colon \mathcal{B} \to \mathcal{A}$ in mutually reverse directions such that there exist a pair of natural isomorphisms (= 2-isomorphisms) $\epsilon \colon EF \Rightarrow \mathbb{1}_{\mathcal{B}}$ and $\eta \colon \mathbb{1}_{\mathcal{A}} \Rightarrow FE$. Namely, to define equivalences between objects in a categorical sense we need a 2-categorical structure in the class of objects. In our case, the class G-**Cat** of all small $\Bbbk$-categories with G-actions and the class G-**GrCat** of all small G-graded $\Bbbk$-categories should have 2-categorical structures. To insist that the above gives a full categorical generalization of Cohen-Montgomery duality we have to have an affirmative answer to the following question:

(i) Are the G-equivariant equivalence $\varepsilon_{\mathcal{C}}$ and the degree-preserving equivalence $\omega_{\mathcal{B}}$ obtained in [1] equivalences defined by 2-categorical structures on G-**Cat** and G-**GrCat**, respectively?

Once we have 2-categorical structures on G-**Cat** and G-**GrCat**, it also becomes important to consider the following question:

(ii) Are $\varepsilon_{\mathcal{C}}$ and $\omega_{\mathcal{B}}$ 2-natural in $\mathcal{C}$ and in $\mathcal{B}$?

These suggest us the following problem:

(c) Not only give an equivalence for each individual category, but extend it to a 2-equivalence between 2-categories of $\Bbbk$-categories with G-action and of G-graded $\Bbbk$-categories.

In this paper we will give a positive solution to the problem (c) which includes affirmative answers to both (i) and (ii). We also give characterizations of equivalences in 2-categories G-**Cat** and G-**GrCat** in terms of a half of a pair of functors in mutually reverse directions, which give relationships between G-equivariant equivalences and equivalences in G-**Cat** and between degree-preserving equivalences and equivalences in G-**GrCat**. The solution proceeds in the following steps:

- to *suitably* define a 2-category G-**Cat** of all small $\Bbbk$-categories with G-actions (Definition 2.9) and a 2-category G-**GrCat** of all small G-graded $\Bbbk$-categories (Definition 3.3);
- to extend the orbit category construction to a 2-functor

$$?/G\colon G\text{-}\mathbf{Cat} \to G\text{-}\mathbf{GrCat}$$

(Definition 7.1) (this is given by the 2-universality of the canonical functor (P, ψ) that is a generalization of Gabriel's Galois covering functor) and the smash product construction to a 2-functor

$$?\#G\colon G\text{-}\mathbf{GrCat} \to G\text{-}\mathbf{Cat}$$

(Definition 7.3); and
- to prove the following (see Theorem 7.5 for detail):

Theorem 1.2 *$?/G$ is strictly left 2-adjoint to $?\#G$ and they are mutual 2-quasi-inverses* (*in a weak sense*).

Therefore in other words we obtain the following.

Theorem 1.3 *Let $\mathcal{C}, \mathcal{C}' \in G$-**Cat** and $\mathcal{B}, \mathcal{B}' \in G$-**GrCat**. Then*

(1) *there exists an equivalence $\mathcal{C} \simeq (\mathcal{C}/G)\#G$* (*in fact this is given by $\varepsilon_{\mathcal{C}}$ above*) *in the 2-category G-**Cat** that is 2-natural in $\mathcal{C}$;*
(2) *there exists an equivalence $\mathcal{B} \simeq (\mathcal{B}\#G)/G$* (*in fact this is given by $\omega_{\mathcal{B}}$ above*) *in the 2-category G-**GrCat** that is 2-natural in $\mathcal{B}$;*
(3) *there exists an isomorphism*

$$G\text{-}\mathbf{GrCat}(\mathcal{C}/G, \mathcal{B}) \cong G\text{-}\mathbf{Cat}(\mathcal{C}, \mathcal{B}\#G)$$

of $\Bbbk$-categories that is 2-natural in $\mathcal{C}$ and $\mathcal{B}$;
(4) *there exists an equivalence*

$$G\text{-}\mathbf{Cat}(\mathcal{C}, \mathcal{C}') \simeq G\text{-}\mathbf{GrCat}(\mathcal{C}/G, \mathcal{C}'/G)$$

of $\Bbbk$-categories that is 2-natural in $\mathcal{C}$ and $\mathcal{C}'$; and
(5) *there exists an equivalence*

$$G\text{-}\mathbf{GrCat}(\mathcal{B}, \mathcal{B}') \simeq G\text{-}\mathbf{Cat}(\mathcal{B}\#G, \mathcal{B}'\#G)$$

of $\Bbbk$-categories that is 2-natural in $\mathcal{B}$ and $\mathcal{B}'$.

Note that the statements (1) and (2) above give affirmative answers to both questions (i) and (ii). We remark that the definition of *degree-preserving functors* (= 1-morphisms in G-**GrCat**) given here is slightly weakened than that used in [1], where degree-preserving functors were defined as strictly degree-preserving functors in the sense of this paper (see Definition 3.1 (2), (3)). This would be the most important point to establish our 2-equivalences (see Remark 8.9 for the necessity of the weaker definition). The results of this paper are applied at least in papers [2], [3] and [12] so far.

For general 2-categorical notions we refer the reader to [5] or [9]. In this paper 2-categories are strict 2-categories, and we use the word "strictly 2-natural transformation" to mean the 2-natural transformation in a usual sense (e.g., as in [5, 9]), and the word "2-natural transformation" in a weak sense, i.e., we only require that the equalities defining the notion of usual 2-natural transformations hold up to natural isomorphisms. Thus we use the word "2-quasi-inverse" in a weak sense (although in fact a half of the equalities to define this notion hold strictly).

The paper is organized as follows. In Sections 2 and 3 we define the 2-category G-**Cat** and the 2-category G-**GrCat**, respectively. In Sections 4, 5 and 6 we recall from [1] fundamental facts about G-coverings, the definition and characterizations of orbit categories, and fundamental facts about smash products, respectively. In Section 7 we extend the orbit category construction and the smash product construction to 2-functors $?/G$ and $?\#G$, respectively, and give the precise statement of the main result. We also give a characterization of G-covering functors that *induce degree-preserving functors* (Definition 7.7). Section 8 is devoted to the proof of the main theorem. Finally, in Section 9 we characterize equivalences in the 2-categories G-**Cat** and G-**GrCat**.

For categories $\mathcal{A}$ and $\mathcal{B}$ we write $\mathcal{A} \cong \mathcal{B}$ (resp. $\mathcal{A} \simeq \mathcal{B}$) to express that they are isomorphic (resp. equivalent); and the class of objects (resp. morphisms) in $\mathcal{A}$ is denoted by $\mathcal{A}_0$ (resp. $\mathcal{A}_1$). We sometimes write "$x \in \mathcal{A}$" as an abbreviation of "$x \in \mathcal{A}_0$". Natural transformations (and 2-morphisms in 2-categories) are expressed by a double arrow symbol $\Rightarrow$.

2 The 2-Category *G*-Cat

First in this section we define the 2-category of G-categories.

2.1 *G*-Categories

Definition 2.1 A $\Bbbk$-category with a *G-action*, or a *G-category* for short, is a pair $(\mathcal{C}, A)$ of a category $\mathcal{C}$ and a group homomorphism $A\colon G \to \mathrm{Aut}(\mathcal{C})$. We set $A_a := A(a)$ for all $a \in G$. If there is no confusion we always denote G-actions by the same letter A, and simply write $\mathcal{C} = (\mathcal{C}, A)$.

Notation 2.2 We denote by $\Bbbk$-**Cat** the 2-category of small $\Bbbk$-categories, $\Bbbk$-functors between them, and natural transformations between $\Bbbk$-functors.

Example 2.3 Any $\Bbbk$-category $\mathcal{B}$ defines a G-category $\Delta\mathcal{B} := (\mathcal{B}, A)$, where $A\colon G \to \mathrm{Aut}(\mathcal{B})$ is the trivial G-action, namely it is defined by $A_a := \mathbb{1}_{\mathcal{B}}$ for all $a \in G$. We sometimes identify $\Delta\mathcal{B}$ with $\mathcal{B}$.

2.2 *G*-Equivariant Functors

Definition 2.4 ([1, Definition 4.8]) Let $\mathcal{C}$ and $\mathcal{C}'$ be G-categories. Then a *G-equivariant* functor from $\mathcal{C}$ to $\mathcal{C}'$ is a pair (E, ρ) of a $\Bbbk$-functor $E\colon \mathcal{C} \to \mathcal{C}'$ and a family $\rho = (\rho_a)_{a\in G}$ of natural isomorphisms $\rho_a\colon A_aE \Rightarrow EA_a$ $(a \in G)$ such that the diagrams

$$\begin{array}{ccc} A_{ba}E = A_bA_aE & \xRightarrow{A_b\rho_a} & A_bEA_a \\ & \searrow^{\rho_{ba}} & \Downarrow{\scriptstyle \rho_bA_a} \\ & & EA_{ba} = EA_bA_a \end{array}$$

commute for all $a, b \in G$.

A $\Bbbk$-functor $E : \mathcal{C} \to \mathcal{C}'$ is called a *strictly G-equivariant* functor if $(E, (\mathbb{1}_E)_{a\in G})$ is a G-equivariant functor, i.e., if $A_a E = E A_a$ for all $a \in G$.

Remark 2.5 In the above since $A_1 = \mathbb{1}$, we have $\rho_1 x = \rho_1 x \cdot \rho_1 x$, and hence $\rho_1 x = \mathbb{1}_{Ex}$ for all $x \in \mathcal{C}$. Hence the natural requirement $\rho_1 = \mathbb{1}_E$ follows automatically from the defining condition.

Example 2.6 Any $\Bbbk$-functor $F : \mathcal{B} \to \mathcal{B}'$ defines a strictly G-equivariant functor $\Delta F := (F, (\mathbb{1}_F)_{a\in G}) : \Delta\mathcal{B} \to \Delta\mathcal{B}'$.

2.3 Morphisms of *G*-Equivariant Functors

Definition 2.7 Let $(E, \rho), (E', \rho') : \mathcal{C} \to \mathcal{C}'$ be *G-equivariant* functors. Then a *morphism* from (E, ρ) to (E', ρ') is a natural transformation $\eta : E \Rightarrow E'$ such that the diagrams

$$\begin{array}{ccc} A_a E & \overset{\rho_a}{\Longrightarrow} & E A_a \\ {\scriptstyle A_a\eta}\Downarrow & & \Downarrow{\scriptstyle \eta A_a} \\ A_a E' & \underset{\rho'_a}{\Longrightarrow} & E' A_a \end{array}$$

commute for all $a \in G$.

We define a composition of G-equivariant functors.

Lemma 2.8 *Let* $\mathcal{C} \xrightarrow{(E,\rho)} \mathcal{C}' \xrightarrow{(E',\rho')} \mathcal{C}''$ *be G-equivariant functors of G-categories. Then*

(1) $(E'E, ((E'\rho_a)(\rho'_a E))_{a\in G}) : \mathcal{C} \to \mathcal{C}''$ *is a G-equivariant functor, which we define to be the composite* $(E', \rho')(E, \rho)$ *of* (E, ρ) *and* (E', ρ').

(2) *If further* $(E'', \rho'') : \mathcal{C}'' \to \mathcal{C}'''$ *is a G-equivariant functor, then we have*

$$((E, \rho)(E', \rho'))(E'', \rho'') = (E, \rho)((E', \rho')(E'', \rho'')).$$

Proof Straightforward. □

2.4 2-Category *G*-Cat

Definition 2.9 A 2-category G-**Cat** is defined as follows.

- The objects are the small G-categories.
- The 1-morphisms are the G-equivariant functors between objects.
- The identity 1-morphism of an object $\mathcal{C}$ is the 1-morphism $(\mathbb{1}_{\mathcal{C}}, (\mathbb{1}_{\mathbb{1}_{\mathcal{C}}})_{a\in G})$.
- The 2-morphisms are the morphisms of G-equivariant functors.
- The identity 2-morphism of a 1-morphism $(E, \rho) : \mathcal{C} \to \mathcal{C}'$ is the identity natural transformation $\mathbb{1}_E$ of E, which is clearly a 2-morphism.
- The composition of 1-morphisms is the one given in the previous lemma.
- The vertical and the horizontal compositions of 2-morphisms are given by the usual ones of natural transformations.

Proposition 2.10 *The data above determine a 2-category.*
Proof Straightforward. □

Definition 2.11 Let F and F' be functors $\mathcal{B} \to \mathcal{B}'$ in $\Bbbk$-**Cat**, and $\alpha\colon F \to F'$ a natural transformation. Then we define a morphism $\Delta\varepsilon\colon \Delta F \to \Delta F'$ of G-equivariant functors by setting $\Delta\varepsilon := \varepsilon$. This and the constructions given in Examples 2.3 and 2.6 define a 2-functor $\Delta\colon \Bbbk$-**Cat** $\to G$-**Cat**.

3 The 2-Category *G*-GrCat

In this section we cite necessary definitions and statements from [1, §5] and add new concepts and statements to define the 2-category of G-graded categories. Here we modified the definition of degree-preserving functors in order to include the functor H (and hence the functors $\omega'_{\mathcal{B}}$ for all G-graded categories $\mathcal{B}$, see Definition 8.7) in Proposition 6.4 below because H is not degree-preserving in the sense of [1] in general (see [1, Remark 5.9] and Remark 8.9).

Definition 3.1 (1) A *G-graded* $\Bbbk$-category is a category $\mathcal{B}$ together with a family of direct sum decompositions $\mathcal{B}(x, y) = \bigoplus_{a\in G} \mathcal{B}^a(x, y)$ $(x, y \in \mathcal{B})$ of $\Bbbk$-modules such that $\mathcal{B}^b(y, z) \cdot \mathcal{B}^a(x, y) \subseteq \mathcal{B}^{ba}(x, z)$ for all $x, y \in \mathcal{B}$ and $a, b \in G$. If $f \in \mathcal{B}^a(x, y)$ for some $a \in G$, then we set $\deg f := a$.

(2) A *degree-preserving* functor is a pair (H, r) of a $\Bbbk$-functor $H\colon \mathcal{B} \to \mathcal{A}$ of G-graded categories and a map $r\colon \mathcal{B}_0 \to G$ such that

$$H(\mathcal{B}^{r_y a}(x, y)) \subseteq \mathcal{A}^{\,ar_x}(Hx, Hy)$$

(or equivalently $H(\mathcal{B}^a(x, y)) \subseteq \mathcal{A}^{r_y^{-1}ar_x}(Hx, Hy)$) for all $x, y \in \mathcal{B}$ and $a \in G$. This r is called a *degree adjuster* of H.

(3) A $\Bbbk$-functor $H\colon \mathcal{B} \to \mathcal{A}$ of G-graded categories is called a *strictly* degree-preserving functor if $(H, 1)$ is a degree-preserving functor, where 1 denotes the constant map $\mathcal{B}_0 \to G$ with value $1 \in G$, i.e., if $H(\mathcal{B}^a(x, y)) \subseteq \mathcal{A}^a(Hx, Hy)$ for all $x, y \in \mathcal{B}$ and $a \in G$.

(4) Let $(H, r), (I, s)\colon \mathcal{B} \to \mathcal{A}$ be degree-preserving functors. Then a natural transformation $\theta\colon H \Rightarrow I$ is called a *morphism* of degree-preserving functors if $\theta x \in \mathcal{A}^{s_x^{-1}r_x}(Hx, Ix)$ for all $x \in \mathcal{B}$.

The composite of degree-preserving functors can be made into again a degree-preserving functor as follows.

Lemma 3.2 *Let* $\mathcal{B} \xrightarrow{(H,r)} \mathcal{B}' \xrightarrow{(H',r')} \mathcal{B}''$ *be degree-preserving functors. Then*

$$(H'H, (r_x r'_{Hx})_{x\in\mathcal{B}})\colon \mathcal{B} \to \mathcal{B}''$$

is also a degree-preserving functor, which we define to be the composite $(H', r')(H, r)$ *of* (H, r) *and* (H', r').

Proof Straightforward. □

Definition 3.3 A 2-category G-**GrCat** is defined as follows.

- The objects are the small G-graded categories.

- The 1-morphisms are the degree-preserving functors between objects.
- The identity 1-morphism of an object $\mathcal{B}$ is the 1-morphism $(\mathbb{1}_{\mathcal{B}}, 1)$.
- The 2-morphisms are the morphisms of degree-preserving functors.
- The identity 2-morphism of a 1-morphism $(H, r)\colon \mathcal{B} \to \mathcal{A}$ is the identity natural transformation $\mathbb{1}_H$ of H, which is a 2-morphism (because $(\mathbb{1}_H)x = \mathbb{1}_{Hx} \in \mathcal{A}^1(Hx, Hx) = \mathcal{A}^{r_x^{-1} r_x}(Hx, Hx)$ for all $x \in \mathcal{B}$).
- The composition of 1-morphisms is the one given in the previous lemma.
- The vertical and the horizontal compositions of 2-morphisms are given by the usual ones of natural transformations.

Proposition 3.4 *The data above determine a 2-category.*

Proof Straightforward. □

4 Covering Functors

Throughout Sections 4 and 5, $\mathcal{C}$ is a G-category and $\mathcal{B}$ is a $\Bbbk$-category. In this section we cite definitions and statements without proofs from [1, §1].

4.1 G-Invariant Functors

Definition 4.1 ([1, Definition 1.1]) A G-*invariant* functor from $\mathcal{C}$ to $\mathcal{B}$ is a G-equivariant functor

$$(F, \phi)\colon \mathcal{C} \to \Delta\mathcal{B}.$$

We sometimes write this as $(F, \phi)\colon \mathcal{C} \to \mathcal{B}$.

Remark 4.2 In the above the defining condition on $\phi = (\phi_a)_{a\in G}$ becomes as follows: The diagrams

$$\begin{array}{ccc} F & \xRightarrow{\ \phi_a\ } & FA_a \\ & \underset{\phi_{ba}}{\searrow\!\!\searrow} & \Big\Downarrow{\scriptstyle \phi_b A_a} \\ & & FA_{ba} = FA_b A_a \end{array}$$

commute for all $a, b \in G$. In particular, this implies $\phi_a^{-1} = \phi_{a^{-1}} A_a$ for all $a \in G$.

4.2 Morphisms of G-Invariant Functors

Definition 4.3 Let (F, ϕ), (F', ϕ') be G-invariant functors $\mathcal{C} \to \mathcal{B}$. Then a *morphism* of G-invariant functors from (F, ϕ) to (F', ϕ') is just a morphism η of G-equivariant functors, namely η is a natural transformation $F \to F'$ such that the diagrams

$$\begin{array}{ccc} F & \xRightarrow{\ \phi_a\ } & FA_a \\ {\scriptstyle \eta}\Big\Downarrow & & \Big\Downarrow{\scriptstyle \eta A_a} \\ F' & \xRightarrow[\ \phi'_a\ ]{} & F'A_a \end{array}$$

commute for all $a \in G$.

Notation 4.4 All G-invariant functors $\mathcal{C} \to \mathcal{B}$ and all morphisms between them form a category, which we denote by $\mathrm{Inv}(\mathcal{C}, \mathcal{B})$. When both $\mathcal{C}$ and $\mathcal{B}$ are small categories, we have $\mathrm{Inv}(\mathcal{C}, \mathcal{B}) = G\text{-}\mathbf{Cat}(\mathcal{C}, \Delta\mathcal{B})$.

As a special case of Lemma 2.8, the composite of a G-invariant functor and a functor is made into again a G-invariant functor:

Lemma 4.5 ([1, Lemma 1.4]) *Let $(F, \phi)\colon \mathcal{C} \to \mathcal{B}$ be a G-invariant functor and $H\colon \mathcal{B} \to \mathcal{A}$ a functor. Then $(HF, H\phi)\colon \mathcal{C} \to \mathcal{A}$ is again a G-invariant functor, where $H\phi := (H\phi_a)_{a\in G}$. We set $H(F, \phi) := (HF, H\phi)$.*

4.3 *G*-Covering Functors

Notation 4.6 Let $(F, \phi)\colon \mathcal{C} \to \mathcal{B}$ be a G-invariant functor and $x, y \in \mathcal{C}$. Then we define homomorphisms $F^{(1)}_{x,y} := (F, \phi)^{(1)}_{x,y}$ and $F^{(2)}_{x,y} := (F, \phi)^{(2)}_{x,y}$ of $\Bbbk$-modules as follows.

$$F^{(1)}_{x,y}\colon \bigoplus_{a\in G} \mathcal{C}(A_a x, y) \to \mathcal{B}(Fx, Fy),\ (f_a)_{a\in G} \mapsto \sum_{a\in G} F(f_a)\cdot \phi_a x$$
$$F^{(2)}_{x,y}\colon \bigoplus_{b\in G} \mathcal{C}(x, A_b y) \to \mathcal{B}(Fx, Fy),\ (f_b)_{b\in G} \mapsto \sum_{b\in G} \phi_{b^{-1}}(A_b y)\cdot F(f_b)$$

Proposition 4.7 ([1, Proposition 1.6]) *In the above, $F^{(1)}_{x,y}$ is an isomorphism if and only if $F^{(2)}_{x,y}$ is.*

Definition 4.8 ([1, Definition 1.7]) Let $(F, \phi)\colon \mathcal{C} \to \mathcal{B}$ be a G-invariant functor. Then

(1) (F, ϕ) is called a G-*precovering* if for each $x, y \in \mathcal{C}$, $F^{(1)}_{x,y}$ is an isomorphisms (the latter is equivalent to saying that $F^{(2)}_{x,y}$ is an isomorphism by Proposition 4.7);
(2) (F, ϕ) is called a G-*covering* if it is a G-precovering and F is *dense* (i.e., for each $y \in \mathcal{B}$ there is an $x \in \mathcal{C}$ such that $Fx \cong y$ in $\mathcal{B}$).

5 Orbit Categories

In this section we cite necessary definitions and statements without proofs from [1, §2] except for § 5.4. The symbol $\delta_{a,b}$ stands for the Kronecker's delta below.

5.1 Canonical *G*-Covering

Definition 5.1 ([1, Definition 2.1]) The orbit category $\mathcal{C}/G$ of $\mathcal{C}$ by G is a category defined as follows.

- $(\mathcal{C}/G)_0 := \mathcal{C}_0$.
- For each $x, y \in \mathcal{C}/G$, $(\mathcal{C}/G)(x, y)$ is the set of all $f = (f_{b,a}) \in \prod_{(a,b)\in G\times G} \mathcal{C}(A_a x, A_b y)$ such that f is row finite and column finite and that $f_{cb,ca} = A_c f_{b,a}$ for all $c \in G$.

- For any pair $f : x \to y$ and $g : y \to z$ in $\mathcal{C}/G$, $gf := \left(\sum_{c \in G} g_{b,c} f_{c,a}\right)_{(a,b)}$.

Then $\mathcal{C}/G$ becomes a category where the identity $\mathbb{1}_x$ of each $x \in \mathcal{C}/G$ is given by $\mathbb{1}_x = (\delta_{a,b} \mathbb{1}_{A_a x})_{(a,b)}$.

Definition 5.2 ([1, Definition 2.4]) We define a functor $P_{\mathcal{C},G} := P : \mathcal{C} \to \mathcal{C}/G$ as follows.

- For each $x \in \mathcal{C}$, $P(x) := x$;
- For each morphism f in $\mathcal{C}$, $P(f) := (\delta_{a,b} A_a f)_{(a,b)}$.

Then P turns out to be a functor.

Definition 5.3 ([1, Definition 2.5]) For each $c \in G$ and $x \in \mathcal{C}$, set $\psi_c x := (\delta_{a,bc} \mathbb{1}_{A_a x})_{(a,b)} \in (\mathcal{C}/G)(Px, PA_c x)$. Then $\psi_c := (\psi_c x)_{x \in \mathcal{C}} : P \to PA_c$ is a natural isomorphism, and the pair $(P_{\mathcal{C},G}, \psi_{\mathcal{C},G}) := (P, \psi) : \mathcal{C} \to \mathcal{C}/G$ turns out to be a G-invariant functor, where we set $\psi_{\mathcal{C},G} := \psi := (\psi_c)_{c \in G}$. We call (P, ψ) the *canonical* functor.

Proposition 5.4 ([1, Proposition 2.6]) *The following statements hold:*

(1) (P, ψ) *is a* G*-covering functor;*
(2) (P, ψ) *is universal among* G*-invariant functors from* $\mathcal{C}$*, i.e., for any* G*-invariant functor* $(F, \phi) : \mathcal{C} \to \mathcal{B}$ *there exists a unique functor* $H : \mathcal{C}/G \to \mathcal{B}$ *such that* $(F, \phi) = H(P, \psi)$ *as* G*-invariant functors.*

Corollary 5.5 ([1, Corollary 2.7]) *In the above,* (P, ψ) *is 2-universal, i.e., the induced functor*

$$(P, \psi)^* : \ \mathrm{Fun}(\mathcal{C}/G, \mathcal{B}) \to \mathrm{Inv}(\mathcal{C}, \mathcal{B})$$

is an isomorphism of categories, where $\mathrm{Fun}(\mathcal{C}/G, \mathcal{B})$ *is the category of* $\Bbbk$*-functors from* $\mathcal{C}/G$ *to* $\mathcal{B}$*.*

This will be used later in § 7.1.

Lemma 5.6 ([1, Lemma 5.4]) $\mathcal{C}/G$ *is* G*-graded.*

Recall the definition of G-grading of $\mathcal{C}/G$: Let $(P, \psi) : \mathcal{C} \to \mathcal{C}/G$ be the canonical functor. Then the G-grading is given by $(\mathcal{C}/G)(x, y) = \bigoplus_{a \in G} (\mathcal{C}/G)^a(x, y)$, where

$$(\mathcal{C}/G)^a(x, y) := P^{(1)}_{x,y}(\mathcal{C}(A_a x, y)) \tag{5.1}$$

for all $x, y \in \mathcal{C}$ and $a \in G$. Further [1, Remark 5.5] says that for each $x, y \in \mathcal{C}$, $a \in G$, and $f \in (\mathcal{C}/G)(x, y)$ we have $f \in (\mathcal{C}/G)^a(x, y)$ if and only if $f_{c,b} = 0$ whenever $c^{-1}b \neq a$.

Remark 5.7 In Corollary 5.5 if both $\mathcal{C}$ and $\mathcal{B}$ are small categories, then the corollary above gives us an isomorphism of categories

$$(P, \psi)^* : \Bbbk\text{-}\mathbf{Cat}(\mathcal{C}/G, \mathcal{B}) \to G\text{-}\mathbf{Cat}(\mathcal{C}, \Delta\mathcal{B}).$$

In Lemma 7.2 we will define a 2-functor $?/G : G\text{-}\mathbf{Cat} \to G\text{-}\mathbf{GrCat}$. If we consider the composite 2-functor $\mathrm{Fgt} \circ (?/G) : G\text{-}\mathbf{Cat} \to \Bbbk\text{-}\mathbf{Cat}$, where $\mathrm{Fgt} : G\text{-}\mathbf{GrCat} \to \Bbbk\text{-}\mathbf{Cat}$ is the forgetful functor, we see that the isomorphism above is 2-natural in $\mathcal{C}$ and in $\mathcal{B}$. This means that $\mathrm{Fgt} \circ (?/G)$ is a left adjoint to Δ.

5.2 Characterization of *G*-Covering Functors

The following gives a characterization of G-covering functors.

Theorem 5.8 ([1, Theorem 2.9]) *Let $(F, \phi)\colon \mathcal{C} \to \mathcal{B}$ be a G-invariant functor. Then the following are equivalent.*

(1) *(F, ϕ) is a G-covering;*
(2) *(F, ϕ) is a G-precovering that is universal among G-precovering from $\mathcal{C}$;*
(3) *(F, ϕ) is universal among G-invariant functors from $\mathcal{C}$;*
(4) *There exists an equivalence $H\colon \mathcal{C}/G \to \mathcal{B}$ such that $(F, \phi) \cong H(P, \psi)$ as G-invariant functors; and*
(5) *There exists an equivalence $H\colon \mathcal{C}/G \to \mathcal{B}$ such that $(F, \phi) = H(P, \psi)$.*

5.3 Other Isomorphic Forms of Orbit Categories

The orbit category constructed in Definition 5.1 has the form of a "subset of the product", which seems not to match its universality, but it is essentially a left-right symmetrized direct sum as stated below. (Note that the direct sum of modules were also constructed as a "subset of the direct product".)

Definition 5.9 (Cibils-Marcos, Keller) (1) An orbit category $\mathcal{C}/_1 G$ is defined as follows.

- $(\mathcal{C}/_1 G)_0 := \mathcal{C}_0$;
- For any $x, y \in \mathcal{C}$, $\mathcal{C}/_1 G(x, y) := \bigoplus_{\alpha\in G} \mathcal{C}(\alpha x, y)$; and
- For any $x \xrightarrow{f} y \xrightarrow{g} z$ in $\mathcal{C}/_1 G$, $gf := (\sum_{\alpha,\beta\in G;\beta\alpha=\mu} g_\beta \cdot \beta(f_\alpha))_{\mu\in G}$.

(2) Similarly another orbit category $\mathcal{C}/_2 G$ is defined as follows.

- $(\mathcal{C}/_2 G)_0 := \mathcal{C}_0$;
- For any $x, y \in \mathcal{C}$, $(\mathcal{C}/_2 G)(x, y) := \bigoplus_{\beta\in G} \mathcal{C}(x, \beta y)$; and
- For any $x \xrightarrow{f} y \xrightarrow{g} z$ in $\mathcal{C}/_2 G$, $gf := (\sum_{\alpha,\beta\in G;\alpha\beta=\mu} \alpha(g_\beta) \cdot f_\alpha)_{\mu\in G}$.

Note that $\mathcal{C}/_2 G = (\mathcal{C}^{\mathrm{op}}/_1 G)^{\mathrm{op}}$.

Proposition 5.10 ([1, Proposition 2.11]) *We have isomorphisms of categories $\mathcal{C}/_1 G \cong \mathcal{C}/G \cong \mathcal{C}/_2 G$.* □

5.4 Composition of a *G*-Equivariant Functor and a *G*-Invariant Functor

As a special case of Lemma 2.8, the composite of a G-equivariant functor and a G-invariant functor can be made into a G-invariant functor as follows.

Lemma 5.11 (1) *Let $\mathcal{C}' \xrightarrow{(E,\rho)} \mathcal{C} \xrightarrow{(F,\phi)} \mathcal{B}$ be functors with $\mathcal{C}, \mathcal{C}'$ G-categories, (E, ρ) G-equivariant and (F, ϕ) G-invariant. Then*

$$(FE, ((F\rho_a)(\phi_a E))_{a\in G})\colon \mathcal{C}' \to \mathcal{B}$$

is a G-invariant functor, which we define to be the composite $(F, \phi)(E, \rho)$ of (E, ρ) and (F, ϕ).

(2) *In the above if* (E, ρ) *is a* G*-equivariant equivalence and* (F, ϕ) *is a* G*-covering functor, then the composite* $(F, \phi)(E, \rho)$ *is a* G*-covering functor, and hence* $\mathcal{C}'/G$ *is equivalent to* $\mathcal{B}$*.*

Proof (1) This follows from Lemma 2.8
(2) This is shown in the proof of [1, Lemma 4.10].

□

6 Smash Products

In this section we cite necessary definitions and statements from [1, § 5] without proofs.

Definition 6.1 ([1, Definition 5.2]) Let $\mathcal{B}$ be a G-graded category. Then the *smash product* $\mathcal{B}\#G$ is a category defined as follows.

- $(\mathcal{B}\#G)_0 := \mathcal{B}_0 \times G$, we set $x^{(a)} := (x, a)$ for all $x \in \mathcal{B}$ and $a \in G$.
- $(\mathcal{B}\#G)(x^{(a)}, y^{(b)}) := \mathcal{B}^{b^{-1}a}(x, y)$ for all $x^{(a)}, y^{(b)} \in \mathcal{B}\#G$.
- For any $x^{(a)}, y^{(b)}, z^{(c)} \in \mathcal{B}\#G$ the composition is given by the following commutative diagram

$$\begin{array}{ccc} (\mathcal{B}\#G)(y^{(b)}, z^{(c)}) \times (\mathcal{B}\#G)(x^{(a)}, y^{(b)}) & \longrightarrow & (\mathcal{B}\#G)(x^{(a)}, z^{(c)}) \\ \| & & \| \\ \mathcal{B}^{c^{-1}b}(y, z) \times \mathcal{B}^{b^{-1}a}(x, y) & \longrightarrow & \mathcal{B}^{c^{-1}a}(x, z), \end{array}$$

  where the lower horizontal homomorphism is given by the composition of $\mathcal{B}$.

Lemma 6.2 (The first part of [1, Proposition 5.6]) $\mathcal{B}\#G$ *has a free* G*-action.*

Recall the definition of the free G-action on $\mathcal{B}\#G$: For each $c \in G$ and $x^{(a)} \in \mathcal{B}\#G$, $A_c x^{(a)} := x^{(ca)}$. For each $f \in (\mathcal{B}\#G)(x^{(a)}, y^{(b)}) = \mathcal{B}^{b^{-1}a}(x, y) = (\mathcal{B}\#G)(x^{(ca)}, y^{(cb)})$, $A_c f := f$.

Definition 6.3 ([1, Definition 5.7]) Let $\mathcal{B}$ be a G-graded category. Then we define a functor $Q_{\mathcal{B},G} := Q \colon \mathcal{B}\#G \to \mathcal{B}$ as follows.

- $Q(x^{(a)}) = x$ for all $x^{(a)} \in \mathcal{B}\#G$.
- $Q(f) := f$ for all $f \in (\mathcal{B}\#G)(x^{(a)}, y^{(b)}) = \mathcal{B}^{b^{-1}a}(x, y)$.

Proposition 6.4 ([1, Proposition 5.8, Remark 5.9]) $Q = QA_a$ *for all* $a \in G$ *and* $Q = (Q, \mathbb{1}) \colon \mathcal{B}\#G \to \mathcal{B}$ *is a* G*-covering functor. Hence in particular,* Q *factors through the canonical* G*-covering functor* $(P, \psi) \colon \mathcal{B}\#G \to (\mathcal{B}\#G)/G$*, i.e., there exists a unique equivalence* $H \colon (\mathcal{B}\#G)/G \to \mathcal{B}$ *such that* $Q = H(P, \psi)$*.*

7 2-Functors

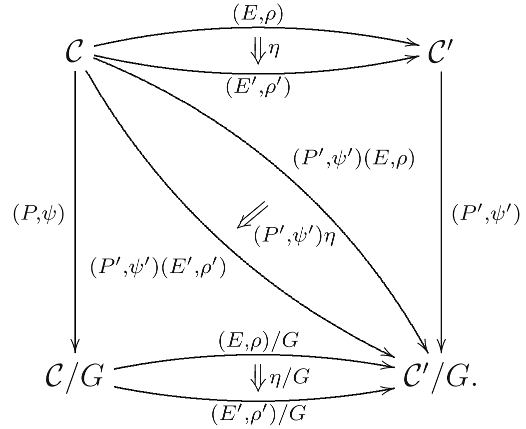
7.1 Orbit 2-Functor

We first extend the orbit category construction to a 2-functor G-**Cat** $\to$ G-**GrCat**.

Definition 7.1 Let $(E, \rho), (E', \rho') : \mathcal{C} \to \mathcal{C}'$ be 1-morphisms and $\eta : (E, \rho) \to (E', \rho')$ a 2-morphism in G-**Cat**. Set $(P, \psi) : \mathcal{C} \to \mathcal{C}/G$, $(P', \psi') : \mathcal{C}' \to \mathcal{C}'/G$ to be the canonical functors. By Proposition 5.11 we have $(P', \psi')\eta : (P', \psi')(E, \rho) \to (P', \psi')(E', \rho')$ is in $\operatorname{Inv}(\mathcal{C}, \mathcal{C}'/G)$. Then using the isomorphism $(P, \psi)^* : \operatorname{Fun}(\mathcal{C}/G, \mathcal{C}'/G) \to \operatorname{Inv}(\mathcal{C}, \mathcal{C}'/G)$ of categories we can define

$$(E, \rho)/G := (P, \psi)^{*-1}((P', \psi')(E, \rho)) \text{ and}$$
$$\eta/G := (P, \psi)^{*-1}((P'\psi')\eta).$$

This construction is visualized in the following diagram:



The explicit form of η/G is given by

$$(\eta/G)Px := P'(\eta x) \in (\mathcal{C}'/G)^1(((E, \rho)/G)Px, ((E', \rho')/G)Px)$$

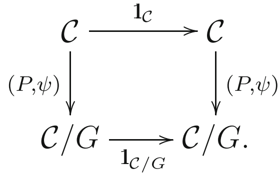
(for $(\mathcal{C}'/G)^1$ see Eq. 5.1) for all $x \in \mathcal{C}$. Then as easily seen, $(E, \rho)/G$ is a strictly degree-preserving functor and η/G is a 2-morphism in G-**GrCat**.

Lemma 7.2 *The definition above extends the orbit category construction to a 2-functor*

$$?/G : G\text{-}\mathbf{Cat} \to G\text{-}\mathbf{GrCat}.$$

Proof (1) $\mathbb{1}_{\mathcal{C}}/G = \mathbb{1}_{\mathcal{C}/G}$ for all $\mathcal{C} \in G$-**Cat**.

Indeed, let $(P, \psi) : \mathcal{C} \to \mathcal{C}/G$ be the canonical functor. Then this follows from the following strict commutative diagram:



(2) For any $\mathcal{C} \xrightarrow{(E,\rho)} \mathcal{C}' \xrightarrow{(E',\rho')} \mathcal{C}''$ in G-**Cat**, $((E', \rho') \cdot (E, \rho))/G = (E', \rho')/G \cdot (E, \rho)/G$.

Indeed, let $(P, \psi) : \mathcal{C} \to \mathcal{C}/G$, $(P', \psi') : \mathcal{C}' \to \mathcal{C}'/G$, $(P'', \psi'') : \mathcal{C}'' \to \mathcal{C}''/G$ be the canonical functors. We can set $(E, \rho)/G = (H, 1) : \mathcal{C}/G \to \mathcal{C}'/G$ and $(E', \rho')/G =$

$(H', 1)\colon \mathcal{C}'/G \to \mathcal{C}''/G$. Then we have the following strictly commutative diagram consisting of solid arrows:

$$\begin{array}{ccccc} & & \overset{(E'E,\rho'')}{\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots>} & & \\ \mathcal{C} & \xrightarrow[(E,\rho)]{} & \mathcal{C}' & \xrightarrow[(E',\rho')]{} & \mathcal{C}'' \\ {\scriptstyle (P,\psi)}\downarrow & & \downarrow{\scriptstyle (P',\psi')} & & \downarrow{\scriptstyle (P'',\psi'')} \\ \mathcal{C}/G & \xrightarrow{H} & \mathcal{C}'/G & \xrightarrow{H'} & \mathcal{C}''/G. \\ & & \underset{H'H}{\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots>} & & \end{array}$$

Comparing the second entries of G-invariant functors this implies the following for all $a \in G$:

$$(P'\rho_a)(\psi'_a E) = H\psi_a \tag{7.1}$$

$$(P''\rho'_a)(\psi''_a E') = H'\psi'_a \tag{7.2}$$

Set $(E'E, \rho'') := (E', \rho') \cdot (E, \rho)$, namely, $\rho'' := ((E'\rho_a)(\rho'_a E))_{a\in G}$. Then the two triangles consisting of dotted arrows and horizontal arrows are strictly commutative. This shows the strict commutativity of the following as a diagram of functors:

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{(E'E,\rho'')} & \mathcal{C}'' \\ {\scriptstyle (P,\psi)}\downarrow & & \downarrow{\scriptstyle (P'',\psi'')} \\ \mathcal{C}/G & \xrightarrow[H'H]{} & \mathcal{C}''/G, \end{array} \tag{7.3}$$

i.e., we have $P''E'E = H'HP$. We have to verify that this is strictly commutative as a diagram of G-invariant functors, i.e., that the following holds:

$$(P'', \psi'') \cdot (E'E, \rho'') = H'H \cdot (P, \psi).$$

Looking at the second entries of G-invariant functors it is enough to show the following for all $a \in G$:

$$(P''\rho''_a)(\psi''_a E'E) = H'H\psi_a. \tag{7.4}$$

From Eq. 7.1 the composition with H' on the left yields

$$(H'P'\rho_a)(H'\psi'_a E) = H'H\psi_a.$$

From Eq. 7.2 the composition with E on the right yields

$$(P''\rho'_a E)(\psi''_a E'E) = H'\psi'_a E.$$

Using these equalities we see that the left hand side of Eq. 7.4 is equal to

$$\begin{aligned} (P''E'\rho_a)(P''\rho'_a E)(\psi'_a E'E) &= (P''E'\rho'_a)(H'\psi'_a E) \\ &= (H'P'\rho_a)(H'\psi'_a E) \\ &= H'H\psi_a, \end{aligned}$$

the right hand side, and the strict commutativity of Eq. 7.3 as a diagram of G-invariant functors is verified, which shows that $((E', \rho')(E, \rho))/G = H'H = (E', \rho')/G \cdot (E, \rho)/G$.

(3) $\mathbb{1}_{(E,\rho)}/G = \mathbb{1}_{(E,\rho)/G}$ for all 1-morphism $(E, \rho)\colon \mathcal{C} \to \mathcal{C}'$ in G-**Cat**.

Indeed, set (P, ψ), (P', ψ'), H to be as in (2) above. For each $Px \in \mathcal{C}/G$ we have $(\mathbb{1}_{(E,\rho)}/G)(Px) = P'((\mathbb{1}_{(E,\rho)})x) = \mathbb{1}_{P'Ex} = \mathbb{1}_{HPx} = (\mathbb{1}_{(E,\rho)/G})(Px)$.

(4) $?/G$ preserves the vertical composition.

Indeed, let $(E, \rho), (E'\rho'), (E'', \rho'') \in (G\text{-}\mathbf{Cat})(\mathcal{C}, \mathcal{C}')$, and let $\eta\colon (E, \rho) \Rightarrow (E', \rho')$, $\eta'\colon (E', \rho') \Rightarrow (E'', \rho'')$ be 2-morphisms in G-**Cat**. Set (P, ψ), (P', ψ') to be as in (2) above. Then for each $Px \in \mathcal{C}/G$ we have

$$((\eta'\eta)/G)(Px) = P'((\eta'\eta)x) = P'(\eta'x)P'(\eta x) = (\eta'/G)(Px) \cdot (\eta/G)(Px).$$

This shows that $(\eta'\eta)/G = (\eta'/G)(\eta/G)$.

(5) $?/G$ preserves the horizontal composition.

Indeed, let $(E, \rho), (E'\rho') \in (G\text{-}\mathbf{Cat})(\mathcal{C}, \mathcal{C}')$, $(F, \tau), (F'\tau') \in (G\text{-}\mathbf{Cat})(\mathcal{C}', \mathcal{C}'')$ and $\eta\colon (E, \rho) \Rightarrow (E', \rho')$, $\eta'\colon (F, \tau) \Rightarrow (F', \tau')$ be 2-morphisms in G-**Cat**. Then we have to show the equality

$$(\eta' * \eta)/G = (\eta'/G) * (\eta/G).$$

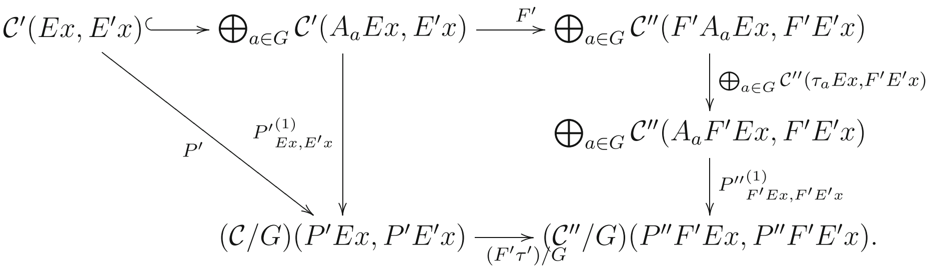
Set (P, ψ), (P', ψ') and (P'', ψ'') to be as in (2) above. Then for each $Px \in \mathcal{C}/G$ we have

$$\begin{aligned} ((\eta' * \eta)/G)(Px) &= P''((\eta' * \eta)x) = P''(((F'\eta)(\eta'E))x) = P''((F'\eta)x \cdot (\eta'E)x) \\ &= P''((F'\eta)x)P''((\eta'E)x) = P''(F'(\eta x)) \cdot P''(\eta'(Ex)), \end{aligned}$$

and

$$\begin{aligned} ((\eta'/G) * (\eta/G))(Px) &= ((F', \tau')/G \cdot \eta/G)(Px) \cdot (\eta'/G \cdot (E, \rho)/G)(Px) \\ &= ((F', \tau')/G)(P'(\eta x)) \cdot (\eta'/G)(P'Ex) \\ &= P''(F'(\eta x)) \cdot P''(\eta'(Ex)), \end{aligned}$$

from which the equality follows, where $((F', \tau')/G)(P'(\eta x)) = P''(F'(\eta x))$ follows from the commutative diagram



As a consequence, $?/G\colon G\text{-}\mathbf{Cat} \to G\text{-}\mathbf{GrCat}$ is a 2-functor. □

7.2 Smash 2-Functor

Next we extend the smash product construction to a 2-functor.

Definition 7.3 Let $(H, r)\colon \mathcal{B} \to \mathcal{B}'$ be in G-**GrCat**. Then the functor $(H, r)\#G\colon \mathcal{B}\#G \to \mathcal{B}'\#G$ is defined as follows.

On Objects For each $x^{(a)} \in \mathcal{B}\#G$ we set

$$((H, r)\#G)(x^{(a)}) := (Hx)^{(ar_x)}.$$

On Morphisms For each $f \in (\mathcal{B}\#G)(x^{(a)}, y^{(b)}) = \mathcal{B}^{b^{-1}a}(x, y)$ we set

$$((H, r)\#G)(f) := H(f),$$

which is an element of $\mathcal{B}'^{r_y^{-1}b^{-1}ar_x}(Hx, Hy) = (\mathcal{B}'\#G)((Hx)^{(ar_x)}, (Hy)^{(br_y)})$. Then as easily seen, $(H, r)\#G$ is a strictly G-equivariant functor, and hence $(H, r)\#G = ((H, r)\#G, 1): \mathcal{B}\#G \to \mathcal{B}'\#G$ is in G-**Cat**.

Next let $(H', r'): \mathcal{B} \to \mathcal{B}'$ be a 1-morphism and $\theta: (H, r) \to (H', r')$ a 2-morphism in G-**GrCat**. We define $\theta\#G: (H, r)\#G \Rightarrow (H', r')\#G$ by

$$(\theta\#G)x^{(a)} := \theta x$$

for all $x^{(a)} \in \mathcal{B}\#G$. Then it is easy to see that $\theta\#G$ is a 2-morphism in G-**Cat**.

Lemma 7.4 *The definition above extends the smash product construction to a 2-functor*

$$?\#G: G\text{-}\mathbf{GrCat} \to G\text{-}\mathbf{Cat}.$$

Proof We only show that $?\#G$ preserves the horizontal composition because the other properties for $?\#G$ to be a 2-functor are immediate from the definition. Let $(H, \xi), (H', \xi') \in G$-**GrCat**$(\mathcal{B}, \mathcal{B}')$, $(F, \zeta), (F'\zeta') \in G$-**GrCat**$(\mathcal{B}', \mathcal{B}'')$ and let $\theta: (H, \xi) \Rightarrow (H', \xi')$, $\theta': (F, \zeta) \Rightarrow (F'\zeta')$ be 2-morphisms in G-**GrCat**. For each $x^{(a)} \in \mathcal{B}\#G$ we have

$$((\theta' * \theta)\#G)(x^{(a)}) = (\theta' * \theta)x = (F'\theta)x \cdot (\theta' H)x = F'(\theta x) \cdot \theta'(Hx),$$

and

$$\begin{aligned} ((\theta\#G) * (\theta\#G))(x^{(a)}) &= (((F', \zeta')\#G)(\theta\#G))((\theta'\#G)((H, \xi)\#G)))(x^{(a)}) \\ &= ((F'\zeta')\#G)(\theta\#G))(x^{(a)}) \cdot ((\theta'\#G)((H, \xi)\#G))(x^{(a)}) \\ &= ((F', \zeta')\#G)(\theta x) \cdot (\theta'\#G)((Hx)^{(a\xi_x)}) \\ &= F'(\theta x) \cdot \theta'(Hx), \end{aligned}$$

which shows that $(\theta' * \theta)\#G = (\theta\#G) * (\theta\#G)$. □

7.3 Main Theorem

We are now in a position to state our main result, which is a precise form of Theorem 1.2.

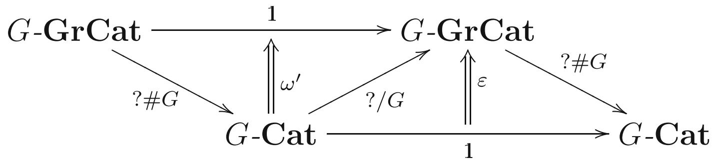
Theorem 7.5 *Both 2-functors $?/G$ and $?\#G$ are 2-equivalences. They are mutual 2-quasi-inverses. Hence the 2-categories G-**Cat** and G-**GrCat** are 2-equivalent. More precisely, we have four 2-natural isomorphisms*

$$\begin{aligned} \varepsilon: \mathbb{1}_{G\text{-}\mathbf{Cat}} &\Rightarrow (?\#G)(?/G) \\ \varepsilon': (?\#G)(?/G) &\Rightarrow \mathbb{1}_{G\text{-}\mathbf{Cat}} \\ \omega: \mathbb{1}_{G\text{-}\mathbf{GrCat}} &\Rightarrow (?/G)(?\#G) \\ \omega': (?/G)(?\#G) &\Rightarrow \mathbb{1}_{G\text{-}GrCat} \end{aligned}$$

with the property that

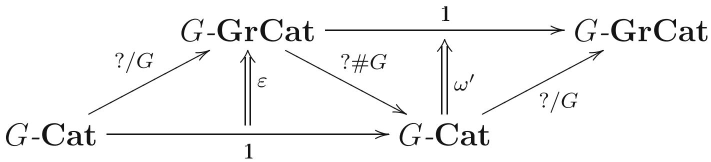
$$\varepsilon'_{\mathcal{C}}\varepsilon_{\mathcal{C}} = \mathbb{1}_{\mathcal{C}}, \tag{7.5}$$
$$\varepsilon_{\mathcal{C}}\varepsilon'_{\mathcal{C}} \cong \mathbb{1}_{(\mathcal{C}/G)\#G}, \tag{7.6}$$
$$\omega'_{\mathcal{B}}\omega_{\mathcal{B}} = \mathbb{1}_{\mathcal{B}}, \tag{7.7}$$
$$\omega_{\mathcal{B}}\omega'_{\mathcal{B}} \cong \mathbb{1}_{(\mathcal{B}\#G)/G}, \tag{7.8}$$

and that $\varepsilon'_{\mathcal{C}}$ *are strictly* G*-equivariant functors and* $\omega_{\mathcal{B}}$ *are strictly degree-preserving functors for all* $\mathcal{C} \in G$-**Cat** *and* $\mathcal{B} \in G$-**GrCat***. Furthermore* ε *and* ω' *are strictly 2-natural transformations, and in particular,* $?/G$ *is strictly left 2-adjoint to* $?\#G$*. Namely the pasting of the diagram*



(7.9)

is equal to the identity $\mathbb{1}_{?\#G}$*, and the pasting of the diagram*



(7.10)

is equal to the identity $\mathbb{1}_{?/G}$

The proof is given in the next section.

7.4 Proof of Theorem 1.3

(1) and (2) These are direct consequences of Eqs. 7.5–7.8.

(3) This follows from Eqs. 7.9 and 7.10 by a general theory of 2-categories (see e.g. [9], [5]; the proof proceeds just the same way as in the usual category case).

(4) $G\text{-}\mathbf{Cat}(\mathcal{C}, \mathcal{C}') \simeq G\text{-}\mathbf{Cat}(\mathcal{C}, (\mathcal{C}'/G)\#G) \cong G\text{-}\mathbf{Cat}(\mathcal{C}/G, \mathcal{C}'/G)$.

(5) A similar proof as above works. □

Theorem 1.3 gives the following.

Corollary 7.6 *Let* $\mathcal{C}, \mathcal{C}' \in G$-**Cat***. Then we have a faithful embedding*

$$G\text{-}\mathbf{Cat}(\mathcal{C}, \mathcal{C}') \to \mathrm{Inv}(\mathcal{C}, \mathcal{C}'/G)$$

of $\Bbbk$*-categories.*

Proof $G\text{-}\mathbf{Cat}(\mathcal{C}, \mathcal{C}') \simeq G\text{-}\mathbf{GrCat}(\mathcal{C}/G, \mathcal{C}'/G) \subseteq \mathrm{Fun}(\mathcal{C}/G, \mathcal{C}'/G) \cong \mathrm{Inv}(\mathcal{C}, \mathcal{C}'/G)$, where the first equivalence is an injection on objects by Eq. 7.5. Indeed, if $(F, \phi), (F', \phi') \in G\text{-}\mathbf{Cat}(\mathcal{C}, \mathcal{C}')$ and $(F, \phi)/G = (F', \phi')/G$, then the naturality of ε shows that

$$\varepsilon_{\mathcal{C}'}(F, \phi) = (((F, \phi)/G)\#G)\varepsilon_{\mathcal{C}} = (((F', \phi')/G)\#G)\varepsilon_{\mathcal{C}} = \varepsilon_{\mathcal{C}'}(F', \phi').$$

Hence by Eq. 7.5 we have $(F, \phi) = (F', \phi')$. □

7.5 Weak Universality of the Canonical Functor of a Smash Product

As an application of Theorem 7.5 we obtain the proposition below, which states that the canonical functor $(Q, \mathbb{1})\colon \mathcal{B}\#G \to \mathcal{B}$ to a G-graded category $\mathcal{B}$ has the weak universality among G-invariant functors from G-categories to $\mathcal{B}$ that *induce degree-preserving functors* (see Definition 7.7 below). (It often does not have the universality as Remark 7.10 shows.)

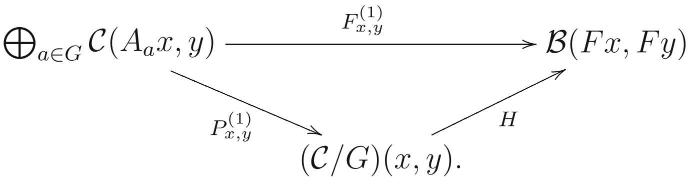
Definition 7.7 Let $\mathcal{C}$ be a G-category with the canonical functor $(P, \psi): \mathcal{C} \to \mathcal{C}/G$, $\mathcal{B}$ a G-graded category, and $r: \mathcal{C}_0 \to G$ a map. Then a G-invariant functor $(F, \phi): \mathcal{C} \to \mathcal{B}$ is said to *induce a degree-preserving functor with* r if the unique functor $H: \mathcal{C}/G \to \mathcal{B}$ such that $(F, \phi) = H(P, \psi)$ (the existence of which is guaranteed by Proposition 5.4) has the property that (H, r) is a degree-preserving functor.

Lemma 7.8 *Let $\mathcal{C}$ be a G-category and $\mathcal{B}$ a G-graded category. Then a G-invariant functor $(F, \phi): \mathcal{C} \to \mathcal{B}$ induces a degree-preserving functor with a map $r: \mathcal{C}_0 \to G$ if and only if for each $x, y \in \mathcal{C}$ and $a \in G$ the restriction of*

$$F^{(1)}_{x,y}: \bigoplus_{b \in G} \mathcal{C}(A_b x, y) \to \mathcal{B}(Fx, Fy)$$

to $\mathcal{C}(A_{r_y a} x, y)$ induces a homomorphism $\mathcal{C}(A_{r_y a} x, y) \to \mathcal{B}^{a r_x}(Fx, Fy)$, or equivalently, for each $f \in \mathcal{C}(A_{r_y a} x, y)$ we have $F(f) \cdot \phi_{r_y a} x \in \mathcal{B}^{a r_x}(Fx, Fy)$.

Proof This follows from the definition Eq. 5.1 of the G-grading of $\mathcal{C}/G$ and the commutativity of the diagram

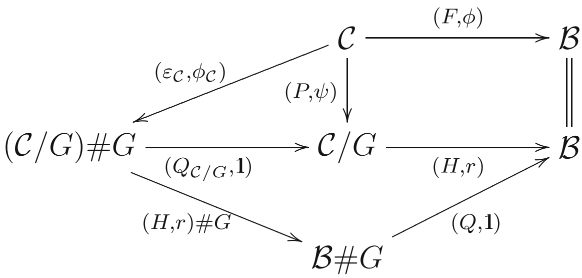


(7.11)

(see Proof of [1, Proposition 2.6 (3)]). □

Proposition 7.9 *Let $\mathcal{C}$ be a G-category, $\mathcal{B}$ a G-graded category, and $(Q, \mathbb{1}): \mathcal{B}\#G \to \mathcal{B}$ the canonical functor. If $(F, \phi): \mathcal{C} \to \mathcal{B}$ is a G-invariant functor inducing a degree-preserving functor, then there exists a G-equivariant functor $(K, \rho): \mathcal{C} \to \mathcal{B}\#G$ such that $(F, \phi) = (Q, \mathbb{1})(K, \rho)$.*

Proof Let $(P, \psi): \mathcal{C} \to \mathcal{C}/G$ be the canonical functor, and assume that a G-invariant functor $(F, \phi): \mathcal{C} \to \mathcal{B}$ induces a degree-preserving functor with a map $r: \mathcal{C}_0 \to G$. Then there exists a unique equivalence $H: \mathcal{C}/G \to \mathcal{B}$ such that $(F, \phi) = H(P, \psi)$ and (H, r) is a degree-preserving functor. It is easy to verify the commutativity of the diagram



using the explicit forms of the functors (see Definition 8.1 for $\varepsilon_{\mathcal{C}} = (\varepsilon_{\mathcal{C}}, \phi_{\mathcal{C}})$). Thus we can take $(K, \rho) := ((H, r)\#G)(\varepsilon_{\mathcal{C}}, \phi_{\mathcal{C}})$, which is G-equivariant by Lemma 2.8. □

Remark 7.10 In the above proposition (K, ρ) is not uniquely determined in general. For instance, consider the case that the center $Z(G)$ of G is not trivial, and take $\mathcal{C} := \mathcal{B}\#G$ and $(F, \phi) := (Q, \mathbb{1})$. Then $(K, \rho) := (A_a, \mathbb{1})$ satisfies the required property for all $a \in Z(G)$.

Also the weak universality of $(Q, \mathbb{1})\colon \mathcal{B}\#G \to \mathcal{B}$ gives us a characterization of a G-covering functor to $\mathcal{B}$ inducing a degree-preserving functor.

Proposition 7.11 *Let $\mathcal{C}$ be a G-category, $\mathcal{B}$ a G-graded category with the canonical functor $(Q, \mathbb{1})\colon \mathcal{B}\#G \to \mathcal{B}$, and $(F, \phi)\colon \mathcal{C} \to \mathcal{B}$ a G-invariant functor inducing a degree-preserving functor. Then (F, ϕ) is a G-covering functor if and only if there exists a G-equivariant equivalence $(K, \rho)\colon \mathcal{C} \to \mathcal{B}\#G$ such that $(F, \phi) = (Q, \mathbb{1})(K, \rho)$.*

Proof ($\Rightarrow$). We keep the notation and the argument used in the proof of the proposition above, which constructed a G-equivariant functor $(K, \rho)\colon \mathcal{C} \to \mathcal{B}\#G$ such that $(F, \phi) = (Q, \mathbb{1})(K, \rho)$. Since $?\#G$ is a 2-functor, $(H, r)\#G$ is an equivalence. In addition $(\varepsilon_{\mathcal{C}}, \phi_{\mathcal{C}})$ is also a G-equivariant equivalence by Theorem 7.5. Hence as the composite of these (K, ρ) is an equivalence.

($\Leftarrow$). This follows by Lemma 5.11(2). □

8 Proof of Theorem 7.5

8.1 $\varepsilon\colon \mathbb{1}_{G\text{-}\mathbf{Cat}} \Rightarrow (?\#G)(?/G)$

Definition 8.1 (see [1, Theorem 5.10]) Let $\mathcal{C}$ be an object of G-**Cat** and $(P, \psi)\colon \mathcal{C} \to \mathcal{C}/G$ the canonical functor. We define a G-equivariant functor $\varepsilon_{\mathcal{C}}\colon \mathcal{C} \to (\mathcal{C}/G)\#G$ as follows.

On Objects For each $x \in \mathcal{C}$ we set

$$\varepsilon_{\mathcal{C}}(x) := (Px)^{(1)}.$$

On Morphisms For each $f\colon x \to y$ in $\mathcal{C}$, we set

$$\varepsilon_{\mathcal{C}}(f) := P^{(1)}_{x,y}(f)\ (= P(f)).$$

Natural Isomorphisms For each $a \in G$ we define a natural transformation $\phi_a\colon A_a\varepsilon_{\mathcal{C}} \to \varepsilon_{\mathcal{C}}A_a$ by $\phi_a x := \psi_a x$ for all $x \in \mathcal{C}$, i.e., by the commutative diagram

$$\begin{array}{ccc} A_a\varepsilon_{\mathcal{C}} & \xrightarrow{\phi_a x} & \varepsilon_{\mathcal{C}}A_a x \\ \| & & \| \\ (Px)^{(a)} & \xrightarrow[\psi_a]{} & (PA_a x)^{(1)}. \end{array}$$

Here note that $((\mathcal{C}/G)\#G)((Px)^{(a)}, (PA_a x)^{(1)}) = (\mathcal{C}/G)^a(Px, PA_a x) \ni \psi_a x$. Set $\phi_{\mathcal{C}} := (\phi_a)_{a\in G}$. Then we have already shown that $\varepsilon_{\mathcal{C}} = (\varepsilon_{\mathcal{C}}, \phi_{\mathcal{C}})$ is a G-equivariant equivalence in [1, Theorem 5.10].

Lemma 8.2 *ε is a strictly 2-natural transformation.*

Proof Let $\mathcal{C}, \mathcal{C}' \in G$-**Cat**.

(1) Let $(E, \rho) \in (G\text{-}\mathbf{Cat})(\mathcal{C}, \mathcal{C}')$. Set $(H, 1) := (E, \rho)/G$. Then we have a strictly commutative diagram

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{(E,\rho)} & \mathcal{C}' \\ {\scriptstyle (P,\psi)}\downarrow & & \downarrow{\scriptstyle (P',\psi')} \\ \mathcal{C}/G & \xrightarrow[(H,1)]{} & \mathcal{C}'/G, \end{array}$$

where the vertical arrows are the canonical functors. For each $x, y \in \mathcal{C}$ we have a commutative diagram

$$\begin{array}{ccccc} \mathcal{C}(x,y) & \hookrightarrow & \bigoplus_{a\in G}\mathcal{C}(A_a x, y) & \xrightarrow{E} & \bigoplus_{a\in G}\mathcal{C}'(EA_a x, Ey) \\ & & \Big\downarrow{\scriptstyle P^{(1)}_{x,y}} & & \downarrow{\scriptstyle \bigoplus_{a\in G}\mathcal{C}'(\rho_a x, Ey)} \\ & & & & \bigoplus_{a\in G}\mathcal{C}'(A_a Ex, Ey) \\ & & & & \downarrow{\scriptstyle P'^{(1)}_{Ex,Ey}} \\ & & (\mathcal{C}/G)(Px, Py) & \xrightarrow[H_{Px,Py}]{} & (\mathcal{C}'/G)(P'Ex, P'Ey) \end{array}$$

by which it is easy to see that the following diagram is strictly commutative:

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{(E,\rho)} & \mathcal{C}' \\ {\scriptstyle \varepsilon_{\mathcal{C}}}\downarrow & & \downarrow{\scriptstyle \varepsilon_{\mathcal{C}'}} \\ (\mathcal{C}/G)\#G & \xrightarrow[(H,1)\#G]{} & (\mathcal{C}'/G)\#G. \end{array}$$

(2) Let $\eta\colon (E, \rho) \to (E', \rho')$ be in $(G\text{-}\mathbf{Cat})(\mathcal{C}, \mathcal{C}')$. Set $(H, 1) := (E, \rho)/G$, $(H', 1) := (E', \rho')/G$ and $\theta\colon = \eta/G$. Then it immediately follows from definition that $\varepsilon_{\mathcal{C}'}\eta = (\eta/G)\#G \cdot \varepsilon_{\mathcal{C}}$.

By (1) and (2) above ε is a strictly 2-natural transformation. □

8.2 $\varepsilon'\colon (?\#G)(?/G) \Rightarrow \mathbb{1}_{G\text{-}\mathbf{Cat}}$

Definition 8.3 Let $\mathcal{C}$ be an object of G-**Cat** and $(P, \psi)\colon \mathcal{C} \to \mathcal{C}/G$ the canonical functor. We define a G-equivariant functor $\varepsilon'_{\mathcal{C}}\colon (\mathcal{C}/G)\#G \to \mathcal{C}$ as follows.

On Objects For each $x \in \mathcal{C}$ and $a \in G$ we set

$$\varepsilon'_{\mathcal{C}}((Px)^{(a)}) := A_a x.$$

On Morphisms Let $f: (Px)^{(a)} \to (Py)^{(b)}$ be in $(\mathcal{C}/G)\#G$. Then we have the diagram

$$\begin{array}{ccc}
((\mathcal{C}/G)\#G)((Px)^{(a)}, (Py)^{(b)}) & \dashrightarrow & \mathcal{C}(A_a x, A_b x) \\
\Vert & & \wr \uparrow A_b \\
(\mathcal{C}/G)^{b^{-1}a}(Px, Py) & \xleftarrow[P^{(1)}_{x,y}]{\cong} & \mathcal{C}(A_{b^{-1}a}x, y).
\end{array}$$

Using this we set

$$\varepsilon'_{\mathcal{C}}(f) := A_b {P^{(1)}_{x,y}}^{-1}(f).$$

Natural Isomorphisms For each $a \in G$ we easily see that $A_a \varepsilon'_{\mathcal{C}} = \varepsilon'_{\mathcal{C}} A_a$. Thus $\varepsilon'_{\mathcal{C}}$ is a strictly G-equivariant functor.

Lemma 8.4 *ε' is a 2-natural transformation.*

Proof Let $(E, \rho): \mathcal{C} \to \mathcal{C}'$ be a 1-morphism in G-**Cat**. We define a natural transformation $\psi_{(E,\rho)}$ in the diagram

$$\begin{array}{ccc}
(\mathcal{C}/G)\#G & \xrightarrow{((E,\rho)/G)\#G} & (\mathcal{C}'/G)\#G \\
\varepsilon'_{\mathcal{C}} \downarrow & \overset{\Psi_{(E,\rho)}}{\underset{\cong}{\swarrow\!\!\!\swarrow}} & \downarrow \varepsilon'_{\mathcal{C}'} \\
\mathcal{C} & \xrightarrow[(E,\rho)]{} & \mathcal{C}'
\end{array}$$

by

$$(\psi_{(E,\rho)})(Px)^{(a)} := \rho_a x$$

for all $(Px)^{(a)} \in (\mathcal{C}/G)\#G$. Then it is not hard to verify that $\Psi_{(E,\rho)}$ is a natural isomorphism. This shows the 1-naturality of ε'. Now let $(E', \rho)': \mathcal{C} \to \mathcal{C}'$ be another 1-morphism and $\eta: (E, \rho) \Rightarrow (E'\rho')$ a 2-morphism in G-**Cat**. Then it is easy to check the commutativity of the diagram

$$\begin{array}{ccc}
\varepsilon'_{\mathcal{C}'} \cdot ((E,\rho)/G)\#G & \xRightarrow[\cong]{\Psi_{(E,\rho)}} & (E,\rho) \cdot \varepsilon'_{\mathcal{C}} \\
\Downarrow \varepsilon'_{\mathcal{C}'} \cdot ((\eta/G)\#G) & & \Downarrow \eta \cdot \varepsilon'_{\mathcal{C}} \\
\varepsilon'_{\mathcal{C}'} \cdot ((E',\rho')/G)\#G & \xRightarrow[\Psi_{(E',\rho')}]{\cong} & (E',\rho') \cdot \varepsilon'_{\mathcal{C}}
\end{array}$$

of natural transformations, which shows the 2-naturality of ε'. □

8.3 $\omega: \mathbb{1}_{G\text{-}\mathbf{GrCat}} \Rightarrow (?/G)(?\#G)$

Definition 8.5 (see [1, Proposition 5.6]) Let $\mathcal{B} \in G$-**GrCat** and let $(P, \psi): \mathcal{B}\#G \to (\mathcal{B}\#G)/G$ be the canonical functor. We define a 1-morphism $\omega_{\mathcal{B}}: \mathcal{B} \to (\mathcal{B}\#G)/G$ in G-**GrCat** as follows.

On Objects For each $x \in \mathcal{B}$ we set

$$\omega_{\mathcal{B}}(x) := P(x^{(1)}).$$

On Morphisms For each $f : x \to y$ in $\mathcal{B}$, we set

$$\omega_{\mathcal{B}}(f) := P^{(1)}_{x^{(1)},y^{(1)}}(f).$$

Then we have already shown that $\omega_{\mathcal{B}}$ is a strictly degree-preserving equivalence of G-graded categories in [1, Proposition 5.6].

Lemma 8.6 *ω is a 2-natural transformation.*

Proof Let $(H, r)\colon \mathcal{B} \to \mathcal{B}'$ be a 1-morphism in G-**GrCat** and $(P', \psi')\colon \mathcal{B}'\#G \to (\mathcal{B}'\#G)/G$ the canonical functor. We define a natural transformation $\Phi_{(H,r)}$ in the diagram

$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{\quad (H,r) \quad} & \mathcal{B}' \\ {\scriptstyle \omega_{\mathcal{B}}}\downarrow & \overset{\Phi_{(H,r)}}{\underset{\cong}{\swarrow\!\!\!\!\swarrow}} & \downarrow{\scriptstyle \omega_{\mathcal{B}'}} \\ (\mathcal{B}\#G)/G & \xrightarrow[((H,r)\#G)/G]{} & (\mathcal{B}'\#G)/G \end{array}$$

by

$$\Phi_{(H,r)}x := \phi'_{r_x}(Hx)^{(1)}$$

for all $x \in \mathcal{B}$. Then it is not hard to verify that $\Phi_{(H,r)}$ is a natural isomorphism. This shows the 1-naturality of ω. Now let $(H', r')\colon \mathcal{B} \to \mathcal{B}'$ be another 1-morphism and $\theta\colon (H, r) \Rightarrow (H'r')$ a 2-morphism in G-**GrCat**. Then it is easy to check the commutativity of the diagram

$$\begin{array}{ccc} \omega_{\mathcal{B}'} \cdot (H,r) & \overset{\Phi_{(H,r)}}{\underset{\cong}{\Longrightarrow}} & ((H,r)\#G)/G \cdot \omega_{\mathcal{B}} \\ \Downarrow{\scriptstyle \omega_{\mathcal{B}'}\cdot\theta} & & \Downarrow{\scriptstyle (\theta\#G)/G\cdot\omega_{\mathcal{B}}} \\ \omega_{\mathcal{B}'} \cdot (H',r') & \overset{\cong}{\underset{\Phi_{(H',r')}}{\Longrightarrow}} & ((H',r')\#G)/G \cdot \omega_{\mathcal{B}} \end{array}$$

of natural transformations, which shows the 2-naturality of ω. □

8.4 $\omega'\colon (?/G)(?\#G) \Rightarrow \mathbb{1}_{G\text{-}GrCat}$

Definition 8.7 (see Proposition 6.4) Let $\mathcal{B} \in G$-**GrCat** and let $(P, \psi)\colon \mathcal{B}\#G \to (\mathcal{B}\#G)/G$ be the canonical functor. We define a functor $\omega'_{\mathcal{B}}\colon (\mathcal{B}\#G)/G \to \mathcal{B}$ as the unique functor that makes the diagram

$$\begin{array}{ccc} \mathcal{B}\#G & \xrightarrow{\qquad Q \qquad} & \mathcal{B} \\ & {\scriptstyle (P,\psi)}\searrow \qquad \nearrow{\scriptstyle \omega'_{\mathcal{B}}} & \\ & (\mathcal{B}\#G)/G & \end{array}$$

strictly commutative, where Q is the canonical G-covering functor associated to the smash product. Namely, $\omega'_{\mathcal{B}}$ is defined as follows.

On Objects For each $P(x^{(a)}) \in (\mathcal{B}\#G)/G$ we set

$$\omega'_{\mathcal{B}}(P(x^{(a)})) := x.$$

On Morphisms For each $P(x^{(a)}), P(y^{(b)}) \in (\mathcal{B}\#G)/G$, we have the following diagram:

$$\begin{array}{ccc}
((\mathcal{B}\#G)/G)(P(x^{(a)}), P(y^{(b)})) & \dashrightarrow & \mathcal{B}(x, y) \\
{\scriptstyle P^{(1)}_{x^{(a)},y^{(b)}}}\big\uparrow \wr & & \big\| \\
\bigoplus_{c\in G}(\mathcal{B}\#G)(x^{(ca)}, y^{(b)}) & = & \bigoplus_{c\in G} \mathcal{B}^{b^{-1}ca}(x, y).
\end{array}$$

Using this we set

$$\omega'_{\mathcal{B}}(u) := {P^{(1)}_{x^{(a)},y^{(b)}}}^{-1}(u)$$

for all $u \in ((\mathcal{B}\#G)/G)(P(x^{(a)}), P(y^{(b)}))$.

Degree Adjuster Finally we define a degree adjuster $r_{\mathcal{B}}$ of $\omega'_{\mathcal{B}}$ by

$$r_{\mathcal{B}}(P(x^{(a)})) := a$$

for all $P(x^{(a)}) \in (\mathcal{B}\#G)/G$.

Lemma 8.8 *$\omega'_{\mathcal{B}} = (\omega'_{\mathcal{B}}, r_{\mathcal{B}})$ is a degree-preserving functor, and hence a 1-morphism in G-***GrCat** *for all $\mathcal{B} \in G$-***GrCat**.

Proof It is not hard to verify that $\omega'_{\mathcal{B}}$ turns out to be a functor. We show that $\omega'_{\mathcal{B}} = (\omega'_{\mathcal{B}}, r_{\mathcal{B}})$ is degree-preserving (see Definition 9). Let $P(x^{(a)}), P(y^{(b)}) \in (\mathcal{B}\#G)/G$ and $c \in G$. Then

$$\begin{aligned}
\omega'_{\mathcal{B}}(((\mathcal{B}\#G/G)^{r_{\mathcal{B}}(y^{(b)})\cdot c}(P(x^{(a)}), P(y^{(b)})) &= \omega'(P^{(1)}_{x^{(a)},y^{(b)}}((\mathcal{B}\#G)(A_{bc}x^{(a)}, y^{(b)}))) \\
&= (\mathcal{B}\#G)(x^{(bca)}, y^{(b)}) \\
&= \mathcal{B}^{b^{-1}bca}(x, y) = \mathcal{B}^{ca}(x, y) \\
&= \mathcal{B}^{c\cdot r_{\mathcal{B}}(x^{(a)})}(\omega'_{\mathcal{B}}(P(x^{(a)}), \omega'_{\mathcal{B}}(P(y^{(b)}))).
\end{aligned}$$

□

Remark 8.9 (cf. [1, Remark 5.9]) As is seen above $\omega'_{\mathcal{B}}$ is not strictly degree-preserving in general. This forced us to extend the definition of degree-preserving functors from a strict version to a weak one.

Lemma 8.10 *ω' is a strictly 2-natural transformation.*

Proof Let $(H, r): \mathcal{B} \to \mathcal{B}'$ be a 1-morphism in G-**GrCat** and $(P, \psi): \mathcal{B}\#G \to (\mathcal{B}\#G)/G$, $(P', \psi'): \mathcal{B}'\#G \to (\mathcal{B}'\#G)/G$ the canonical functors. We first show the 1-naturality of ω', i.e., the commutativity of the diagram

$$\begin{array}{ccc}
(\mathcal{B}\#G)/G & \xrightarrow{((H,r)\#G)/G} & (\mathcal{B}'\#G)/G \\
{\scriptstyle \omega'_{\mathcal{B}}}\big\downarrow & & \big\downarrow{\scriptstyle \omega'_{\mathcal{B}'}} \\
\mathcal{B} & \xrightarrow[(H,r)]{} & \mathcal{B}'.
\end{array}$$

To show this let $u\colon P(x^{(a)}) \to P(y^{(b)})$ be in $(\mathcal{B}\#G)/G$ and $f := {P^{(1)}_{x^{(a)},y^{(b)}}}^{-1}(u)$. Then

$$[\omega'_{\mathcal{B}'} \circ ((H,r)\#G)/G](P(x^{(a)})) = \omega'_{\mathcal{B}'}(P'((Hx)^{(ar_x)})) = Hx = [(H,r) \circ \omega'_{\mathcal{B}}](P(x^{(a)})),$$

and

$$\begin{aligned}[\omega'_{\mathcal{B}'} \circ ((H,r)\#G)/G](u) &\overset{\text{(a)}}{=} [((P',\psi')((H,r)\#G))^{(1)}_{x^{(a)},y^{(b)}}](f)\\ &\overset{\text{(b)}}{=} (P',\psi')^{(1)}_{(Hx)^{(ar_x)},(Hy)^{(br_y)}}(Hf)\\ &= Hf\\ &= [(H,r)\circ\omega'_{\mathcal{B}}](u),\end{aligned}$$

where the equality (a) holds by definition of $((H,r)\#G)/G$ (see (7.11) and Proof of [1, Proposition 2.6 (3)]), and the equality (b) follows from the fact that $(H,r)\#G$ is strictly G-equivariant.

To show the 2-naturality of ω' let $(H',r')\colon \mathcal{B} \to \mathcal{B}'$ be another 1-morphism and $\theta\colon (H,r) \Rightarrow (H',r')$ a 2-morphism in G-**GrCat**. It is enough to verify the following:

$$\omega'_{\mathcal{B}'}((\theta\#G)/G) = \theta\omega'_{\mathcal{B}}.$$

For each $P(x^{(a)}) \in (\mathcal{B}\#G)/G$ we have

$$\begin{aligned}[\omega'_{\mathcal{B}'}((\theta\#G)/G)]P(x^{(a)}) &= \omega'_{\mathcal{B}'}((\theta\#G)/G)P(x^{(a)}))\\ &= \omega'_{\mathcal{B}'}(P'((\theta\#G)(x^{(a)})))\\ &= \omega'_{\mathcal{B}'}(P'(\theta x))\\ &= \omega'_{\mathcal{B}'}(P'^{(1)}{}_{(Hx)^{(ar_x)},(Hy)^{(ar'_x)}}(\theta x))\\ &= \theta x\\ &= \theta\omega'_{\mathcal{B}}(P(x^{(a)})).\end{aligned}$$

□

8.5 Remaining Parts of the Proof of Theorem 7.5

Verification of (7.5) By definitions of ε and ε' the the equality (7.5) is obvious.

Verification of (7.6) Let $\mathcal{C} \in G$-**Cat** and let $(P,\psi)\colon \mathcal{C} \to \mathcal{C}/G$ be the canonical functor. It is easy to see that we can define a natural isomorphism $\Theta\colon \mathbb{1}_{(\mathcal{C}/G)\#G} \to \varepsilon_{\mathcal{C}}\varepsilon'_{\mathcal{C}}$ by

$$\Theta((Px)^{(a)}) := \psi_a x$$

for all $(Px)^{(a)} \in (\mathcal{C}/G)\#G$.

Verification of (7.7) By definitions of ω and ω' the equality (7.7) is obvious.

Verification of (7.8) Let $\mathcal{B} \in G$-**GrCat** and let $(P,\psi)\colon \mathcal{B}\#G \to (\mathcal{B}\#G)/G$ be the canonical functor. It is not hard to see that we can define a natural isomorphism $\Xi\colon \omega_{\mathcal{B}}\omega'_{\mathcal{B}} \to \mathbb{1}_{(\mathcal{B}\#G)/G}$ by

$$\Xi(P(x^{(a)})) := \psi_a(x^{(1)})$$

for all $P(x^{(a)}) \in (\mathcal{B}\#G)/G$.

The verifications that the pasting of Eq. 7.9 is equal to the identity and that the pasting of Eq. 7.10 is equal to the identity are easy and are left to the reader.

This finishes the proof of Theorem 7.5.

9 Equivalences in 2-Categories *G*-Cat and *G*-GrCat

To distinguish several kinds of equivalences (resp. isomorphisms) we call equivalences (resp. isomorphisms) between categories *category equivalences* (resp. *category isomorphisms*). In this section we give characterizations of equivalences in the 2-categories G-**Cat** and G-**GrCat** and examine relationships

(a) between G-equivariant functors that are category equivalences and equivalences in the 2-category G-**Cat** (see Theorem 9.1), and
(b) between degree-preserving functors that are category equivalences and equivalences in the 2-category G-**GrCat**. (See Remark 9.7(2).)

Note that a category equivalence was characterized by a half of a pair of functors in mutually reverse directions, namely a functor is a category equivalence if and only if it is a fully faithful, dense functor. We give similar characterizations of equivalences in both 2-categories G-**Cat** and G-**GrCat**.

9.1 Equivalences in *G*-Cat

First we characterize equivalences in G-**Cat** in the following theorem.

Theorem 9.1 *Let* $(E, \rho)\colon \mathcal{C} \to \mathcal{C}'$ *be a* G*-equivariant functor in* G-**Cat**. *Then the following are equivalent.*

(1) (E, ρ) *is an equivalence in* G-**Cat**;
(2) E *is fully faithful and dense* (*i.e.,* E *is a category equivalence*).

Thus what we called G*-equivariant equivalences in earlier sections are exactly the equivalences in* G-**Cat**.

Proof (1) $\Rightarrow$(2). This is trivial.

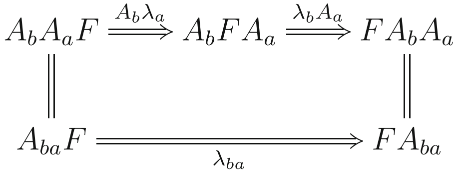
(2) $\Rightarrow$(1). Assume that E is a category equivalence. Then E has a quasi-inverse $F\colon \mathcal{C}' \to \mathcal{C}$, which we may regard as a right adjoint to E, and hence there exist a counit $\varepsilon\colon EF \Rightarrow \mathbb{1}_{\mathcal{C}}$ and a unit $\eta\colon \mathbb{1}_{\mathcal{C}'} \Rightarrow FE$, which are natural isomorphisms. Since (E, ρ) is G-equivariant, ρ_a are natural isomorphisms for all $a \in G$. Therefore we can construct $\lambda = (\lambda_a)_{a\in G}$ by the following commutative diagram:

$$\begin{array}{ccc} A_aF & \xrightarrow{\ \lambda_a\ } & FA_a \\ {\scriptstyle \eta A_aF}\Big\Downarrow \cong & & \cong \Big\Uparrow {\scriptstyle FA_a\varepsilon} \\ FEA_aF & \underset{F\rho_a^{-1}F}{\overset{\cong}{\Longrightarrow}} & FA_aEF. \end{array}$$

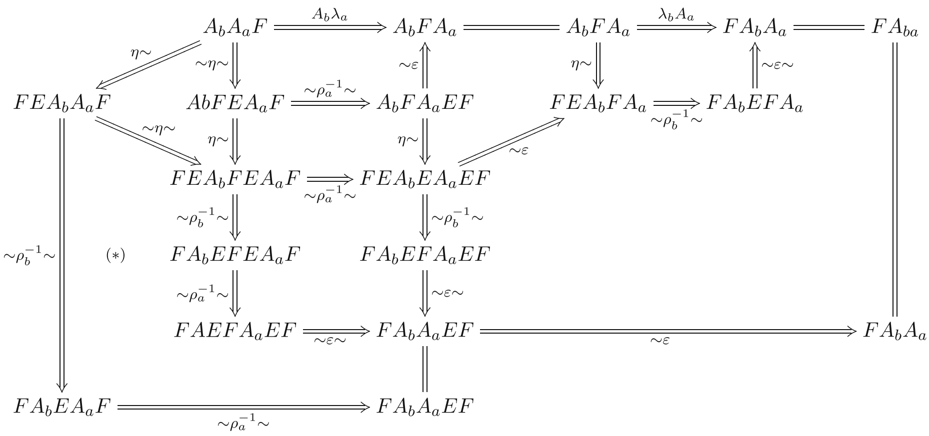
By construction λ_a are natural isomorphisms for all $a \in G$.

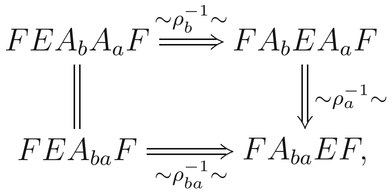
Claim 1 $(F, \lambda)\colon \mathcal{C}' \to \mathcal{C}$ is a 1-morphism in G-**Cat**.

Indeed, let $a, b \in G$. It is enough to show the commutativity of the diagram:

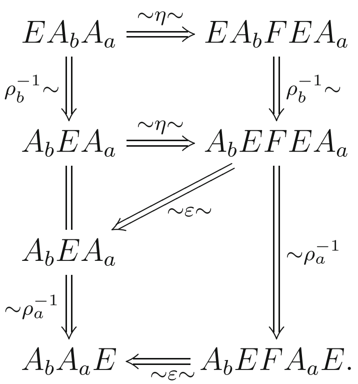


This follows from the following commutative diagrams:





where the commutativity $(*)$ follows from the following commutative diagram:



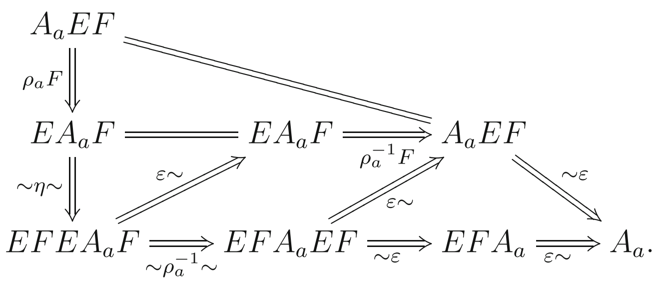
In the above the symbol $\sim$ stands for a functor that is uniquely determined in the diagram.

Claim 2 $\varepsilon\colon (E,\rho)(F,\lambda) \Rightarrow (\mathbb{1}_{\mathcal{C}}, (\mathbb{1}_{A_a})_{a\in G})$ is a 2-isomorphism in G-**Cat**.

Indeed, it is enough to show that ε is a 2-morphism in G-**Cat**, i.e., the following is commutative:

$$\begin{array}{ccc} A_aEF & \overset{A_a\varepsilon}{\underset{\cong}{\Longrightarrow}} & A_a\mathbb{1}_{\mathcal{C}'} \\ {\scriptstyle E\lambda_a\circ\rho_aF}\Downarrow & & \| \\ EFA_a & \overset{\cong}{\underset{\varepsilon A_a}{\Longrightarrow}} & \mathbb{1}_{\mathcal{C}'}A_a. \end{array}$$

This follows from the following commutative diagram:

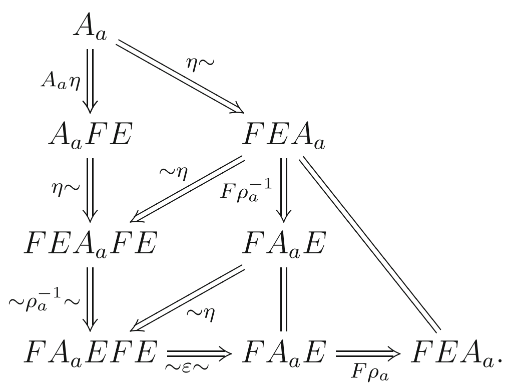


Claim 3 $\eta\colon (\mathbb{1}_{\mathcal{C}'}, (\mathbb{1}_{A'_a})_{a\in G}) \Rightarrow (F,\lambda)(E,\rho)$ is a 2-isomorphism inc G-**Cat**.

Indeed, it is enough to show that η is a 2-morphism in G-**Cat**, i.e., the following is commutative:

$$\begin{array}{ccc} A_aFE & \overset{A_a\eta}{\underset{\cong}{\Longleftarrow}} & A_a\mathbb{1}_{\mathcal{C}} \\ {\scriptstyle F\rho_a\circ\lambda_aE}\Downarrow & & \| \\ FEA_a & \overset{\cong}{\underset{\eta A_a}{\Longleftarrow}} & \mathbb{1}_{\mathcal{C}}A_a. \end{array}$$

This follow from the following commutative diagram:



These three claims show that (E,ρ) is an equivalence in G-**Cat**. □

Remark 9.2 It is now trivial that the G-equivariant equivalence $\varepsilon_{\mathcal{C}}: \mathcal{C} \to (\mathcal{C}/G)\#G$ is an equivalence in G-**Cat** by the theorem above.

9.2 Equivalences in G-GrCat

Next we characterize equivalences in G-**GrCat**. We first define necessary terminologies.

Definition 9.3 Let $\mathcal{A}$ be a category and $\mathcal{B}$ a G-graded category.

(1) Let $E, F: \mathcal{A} \to \mathcal{B}$ be functors. Then a natural transformation $\varepsilon: E \Rightarrow F$ is called *homogeneous* if $\varepsilon_x: Ex \to Fx$ are homogeneous in $\mathcal{B}$ for all $x \in \mathcal{A}_0$.
(2) Let $\mathcal{S}$ be a subclass of $\mathcal{B}_0$ and $\mathcal{B}'$ a full subcategory of $\mathcal{B}$ with $\mathcal{B}'_0 = \mathcal{S}$. Then $\mathcal{S}$ (or $\mathcal{B}'$) is said to be *homogeneously dense* in $\mathcal{B}$ if for each $x \in \mathcal{B}_0$ there exists an $x' \in \mathcal{S}$ such that there exists a homogeneous isomorphism $x \to x'$.
(3) A functor $F: \mathcal{A} \to \mathcal{B}$ is said to be *homogeneously dense* if the object class $F(\mathcal{A}_0)$ is homogeneously dense in $\mathcal{B}$.

We give two examples of homogeneously dense subcategories, the latter will be used to give an alternative proof of the fact that $\omega_{\mathcal{B}}: \mathcal{B} \to (\mathcal{B}\#G)/G$ is an equivalence in G-**GrCat** in Remark 9.7(1).

Recall that a $\Bbbk$-algebra A is called *local* if the sum of non-invertible elements is non-invertible and that if A is local, then 0 and 1 are its only idempotents.

Example 9.4 Let $\mathcal{B}$ be a G-graded $\Bbbk$-category and $(P, \psi): \mathcal{B}\#G \to (\mathcal{B}\#G)/G$ be the canonical functor.

(1) If $\mathcal{B}(x, x)$ are local $\Bbbk$-algebras for all $x \in \mathcal{B}_0$, then any dense full subcategory $\mathcal{B}'$ of $\mathcal{B}$ is homogeneously dense.
(2) Let $\mathcal{B}'$ be the full subcategory of $(\mathcal{B}\#G)/G$ with $\mathcal{B}'_0 := \omega_{\mathcal{B}}(\mathcal{B}_0) = \{P(x^{(1)}) \mid x \in \mathcal{B}\}$ (see Definition 8.5). Then $\mathcal{B}'$ is homogeneously dense in $(\mathcal{B}\#G)/G$. Hence $\omega_{\mathcal{B}}: \mathcal{B} \to (\mathcal{B}\#G)/G$ is homogeneously dense.

Indeed, to show the statement (1) it is enough to show that if $x \cong y$ in $\mathcal{B}$, then there exists a homogeneous isomorphism in $\mathcal{B}(x, y)$. Now let $f: x \to y$ be an isomorphism in $\mathcal{B}$. We may assume that $x \neq 0$. Write f and f^{-1} as finite sums: $f = \sum_{a \in G} f_a$ and $f^{-1} = \sum_{b \in G} g_b$ with $f_a \in \mathcal{B}^a(x, y)$ and $g_b \in \mathcal{B}^b(y, x)$ for all $a, b \in G$. Then $\sum_{a,b \in G} g_b f_a = \mathbb{1}_x$ shows that $h := g_b f_a$ is an automorphism of x for some $a, b \in G$ because $\mathcal{B}(x, x)$ is a local algebra. Thus $(h^{-1} g_b) f_a = \mathbb{1}_x$ and $e := f_a(h^{-1} g_b)$ is an idempotent in $\mathcal{B}(x, x)$, and hence $e = \mathbb{1}_x$ or $e = 0$. But $(h^{-1} g_b) e f_a = \mathbb{1}_x \neq 0$ shows that $e \neq 0$. Hence $f_a: x \to y$ is a homogeneous isomorphism.

The statement (2) follows from the fact that $\psi_{a,x}: P(x^{(1)}) \to P(x^{(a)})$ are homogeneous isomorphisms of degree a in $(\mathcal{B}\#G)/G$ for all $x \in \mathcal{B}_0$ and all $a \in G$ (see proof of [1, p. 131, Claim 4] for deg $\psi_{a,x}$).

We now give a characterization of equivalences in the 2-category G-**GrCat**.

Theorem 9.5 *Let* $(H, r): \mathcal{B} \to \mathcal{A}$ *be a degree-preserving functor in* G-**GrCat**. *Then the following are equivalent.*

(1) (H, r) *is an equivalence in* G-**GrCat**.

(2) *$H: \mathcal{B} \to \mathcal{A}$ is a category equivalence with a quasi-inverse I as a left adjoint both of whose counit $\varepsilon: IH \Rightarrow \mathbb{1}_{\mathcal{A}}$ and unit $\eta: \mathbb{1}_{\mathcal{B}} \Rightarrow HI$ are homogeneous natural isomorphisms.*
(3) *H is fully faithful and homogeneously dense.*

In (2)*, I is made into a quasi-inverse (I, s) of (H, r) with ε the counit and η the unit in a unique way. The degree adjuster s is given by*

$$s = (s_x)_{x \in \mathcal{A}_0} \text{ with } s_x := (\deg \eta_x)^{-1} r_{Ix}^{-1} \in G \text{ for all } x \in \mathcal{A}_0. \tag{9.1}$$

Proof (2) ⇒(1). Assume the statement (2). Set $t_x := \deg \varepsilon_x$ for all $x \in \mathcal{B}_0$, and $t'_x := \deg \eta_x$ for all $x \in \mathcal{A}_0$. Define s as in Eq. 9.1, i.e., $s_x := {t'_x}^{-1} r_{Ix}^{-1} \in G$ for all $x \in \mathcal{A}_0$.

Claim 1 $(I, s): \mathcal{A} \to \mathcal{B}$ is a 1-morphism in G-**GrCat**.

Indeed, let $x, y \in \mathcal{A}_0$, $a \in G$ and $f \in \mathcal{A}^a(x, y)$. It is enough to show that $If \in \mathcal{B}^{s_y^{-1} a s_x}(Ix, Iy)$. Since η is a natural transformation we have $HIf = \eta_y f \eta_x^{-1} \in \mathcal{A}^{t'_y}(y, HIy)\mathcal{A}^a(x, y)\mathcal{A}^{{t'_x}^{-1}}(HIx, x) \subseteq \mathcal{A}^{t'_y a {t'_x}^{-1}}(HIx, HIy)$. Since H is fully faithful, H induces a bijection $\mathcal{B}(Ix, Iy) \to \mathcal{A}(HIx, HIy)$, which also induces bijections

$$\mathcal{B}^b(Ix, Iy) \to \mathcal{A}^{{r_{Iy}}^{-1} b r_{Ix}}(HIx, HIy)$$

for all $b \in G$. Applying this to b with ${r_{Iy}}^{-1} b r_{Ix} = t'_y a {t'_x}^{-1}$, we have

$$If \in \mathcal{B}^{r_{Iy} t'_y a {t'_x}^{-1} r_{Ix}^{-1}}(Ix, Iy) = \mathcal{B}^{s_y^{-1} a s_x}(Ix, Iy).$$

Claim 2 $\varepsilon: (I, s)(H, r) \Rightarrow (\mathbb{1}_{\mathcal{B}}, 1)$ is a 2-isomorphism in G-**GrCat**.

Indeed, it is enough to show that ε is a 2-morphism in G-**GrCat**. This is equivalent to saying that $t_x = r_x s_{Hx}$ for all $x \in \mathcal{B}_0$ because $(I, s)(H, r) = (IH, (r_x s_{Hx})_{x \in \mathcal{B}_0})$. Let $x \in \mathcal{B}_0$. Then since $(H\varepsilon x)(\eta Hx) = \mathbb{1}_{Hx}$, we have $1 = \deg(H\varepsilon x)\deg(\eta Hx) = r_x^{-1} t_x r_{IHx} t'_{Hx}$. Hence $r_x s_{Hx} = r_x {t'_{Hx}}^{-1} r_{IHx}^{-1} = r_x r_x^{-1} t_x = t_x$, as desired.

Claim 3 $\eta: (\mathbb{1}_{\mathcal{A}}, 1) \Rightarrow (H, r)(I, s)$ is a 2-isomorphism in G-**GrCat**.

Indeed, it is enough to show that η is a 2-morphism in G-**GrCat**. This is equivalent to saying that $t'_x = r_{Ix}^{-1} s_x^{-1}$ for all $x \in \mathcal{A}_0$ because $(H, r)(I, s) = (HI, (s_x r_{Ix})_{x \in \mathcal{A}_0})$. By definition $r_{Ix}^{-1} s_x^{-1} = r_{Ix}^{-1} r_{Ix} t'_x = t'_x$, as desired.

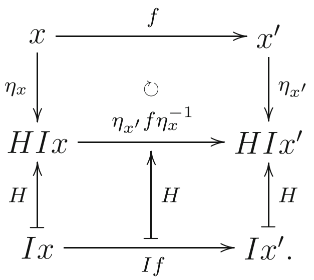
These three claims show that (H, r) is an equivalence in G-**GrCat**. By looking at the proof of Claim 3, we see that the degree adjuster s of I is uniquely determined as in Eq. 9.1 by η and r.

(1) ⇒(3). Assume the statement (1), and let (I, s) be a quasi-inverse of (H, r) with 2-isomorphisms $\varepsilon: (I, s)(H, r) \Rightarrow \mathbb{1}_{\mathcal{B}}$ and $\eta: \mathbb{1}_{\mathcal{A}} \Rightarrow (H, r)(I, s)$. Then $H(\mathcal{B}_0) \supseteq HI(\mathcal{A}_0)$, and the latter is homogeneously dense in $\mathcal{A}$ because $\eta: \mathbb{1}_{\mathcal{A}} \Rightarrow HI$ is a homogeneous natural isomorphism.

(3) ⇒(2). Assume the statement (3). We can imitate the proof of (iii) ⇒(ii) in [10, p. 93, Theorem 1] to construct a quasi-inverse $I: \mathcal{A} \to \mathcal{B}$ as a left adjoint to H and a pair of a counit $\varepsilon: IH \Rightarrow \mathbb{1}_{\mathcal{B}}$ and a unit $\eta: \mathbb{1}_{\mathcal{A}} \Rightarrow HI$. Here we just give definitions of them. Then it is enough to show that both ε and η are homogeneous natural isomorphisms.

Definition of I and η Let $x \in \mathcal{A}_0$. Since H is homogeneously dense, there exists a $y_x \in \mathcal{B}_0$ such that there is a homogeneous isomorphism $\eta_x \colon x \to Hy_x$. Choose a pair (y_x, η_x) once for all x, and define $Ix := y_x$. Then $\eta_x \colon x \to HIx$ is a homogeneous isomorphism, and define $\eta := (\eta_x)_{x\in\mathcal{A}_0}$.

Let $f \in \mathcal{A}(x, x')$. we define If as follows. Since H is fully faithful, H induces a bijection $H_{Ix,Ix'} \colon \mathcal{B}(Ix, Ix') \to \mathcal{A}(HIx, HIx')$. Then define $If := H^{-1}_{Ix,Ix'}(\eta_{x'} f \eta_x^{-1})$ as in the following diagram:



Definition of ε Let $y \in \mathcal{B}_0$. Then $Hy \in \mathcal{A}_0$, and $\eta_{Hy} \in \mathcal{A}(Hy, HIHy)$. H induces a bijection $H_{IHy,\, y} \colon \mathcal{B}(IHy, y) \to \mathcal{A}(HIHy, Hy)$. Then define $\varepsilon_y := H^{-1}_{IHy,\, y}(\eta^{-1}_{Hy})$.

Then the same proof as in [10, p. 93] works (or it is straightforward) to show that I is a left adjoint functor to H with the unit $\eta \colon \mathbb{1}_{\mathcal{A}} \to HI$ and the counit $\varepsilon := (\varepsilon_y)_{y\in\mathcal{B}_0} \colon IH \Rightarrow \mathbb{1}_{\mathcal{B}}$.

Now by definition η is a homogeneous natural isomorphism. It remains to show that ε_y are homogeneous isomorphisms for all $y \in \mathcal{B}_0$. Since η_{Hy} is a homogeneous isomorphism, so is $\eta^{-1}_{Hy} \in \mathcal{A}(HIHy, Hy)$. Set $a := \deg \eta_{Hy}$. Since (H, r) is a degree-preserving functor, the bijection $H_{IHy,\, y}$ induces a bijection

$$\mathcal{B}^{r_y a r^{-1}_{IHy}}(IHy, y) \to \mathcal{A}^a(HIHy, Hy) \ni \eta^{-1}_{Hy}.$$

Hence $\varepsilon_y = H^{-1}_{IHy,\, y}(\eta^{-1}_{Hy}) \in \mathcal{B}^{r_y b r^{-1}_{IHy}}(IHy, y)$ and is a homogeneous isomorphism. □

The following is immediate by Theorem 9.5.

Corollary 9.6 *Let $(H, r) \colon \mathcal{B} \to \mathcal{A}$ be a 1-morphism in G-***GrCat***. Then the following are equivalent.*

(1) *(H, r) is an isomorphism in G-***GrCat***.*
(2) *$H \colon \mathcal{B} \to \mathcal{A}$ is a category isomorphism.*

If this is the case, then the inverse of (H, r) is given by

$$(H, r)^{-1} = (H^{-1}, (r^{-1}_{H^{-1}x})_{x\in\mathcal{A}_0}).$$

Remark 9.7 Let $\mathcal{B} \in G$-**GrCat**$_0$.

(1) Theorem 9.5 and Example 9.4(2) give an immediate alternative proof of the fact that $\omega_{\mathcal{B}} \colon \mathcal{B} \to (\mathcal{B}\#G)/G$ is an equivalence in G-**GrCat**.
(2) Also by Theorem 9.5, a degree-preserving functor $(H, r) \colon \mathcal{A} \to \mathcal{B}$ in G-**GrCat** with H a category equivalence is an equivalence in G-**GrCat** if and only if H is homogeneously dense. In particular, if $\mathcal{B}(x, x)$ are local algebras for all $x \in \mathcal{B}_0$,

then all degree-preserving functors that are category equivalences are equivalences in G-**GrCat** by Example 9.4(1).

Next we will give one more characterization of an equivalence in G-**GrCat** using the composite of degree preserving functors which are surjective, bijective and injective on objects. First we add necessary terminologies.

Definition 9.8 Let $\mathcal{B}$ be a G-graded category.

(1) For each $x, y \in \mathcal{B}$, we say that x and y are *homogeneously isomorphic* (and write $x \cong_H y$) if there exists a homogeneous isomorphism $x \to y$. Since the set of homogeneous isomorphisms in $\mathcal{B}$ is closed under composition and taking inverses, the relation $\cong_H$ on $\mathcal{B}_0$ is an equivalence relation, whose equivalence classes are called *homogeneous isoclasses*.
(2) Let $\mathcal{B}'$ be a full subcategory of $\mathcal{B}$. Then $\mathcal{B}'$ is called a *homogeneous skeleton* of $\mathcal{B}$ if $\mathcal{B}'_0$ forms a complete set of representatives of homogeneous isoclasses in $\mathcal{B}_0$. Note that $\mathcal{B}'$ is homogeneously dense in $\mathcal{B}$ if and only if it contains a homogeneous skeleton of $\mathcal{B}$.

Lemma 9.9 *Let $\mathcal{B} \in G$-**GrCat**$_0$. If $\mathcal{B}'$ is a homogeneously dense full subcategory of $\mathcal{B}$, then the inclusion functor $S\colon \mathcal{B}' \hookrightarrow \mathcal{B}$ induces an equivalence $(S, 1)\colon \mathcal{B}' \to \mathcal{B}$ in G-**GrCat**.*

Proof Note that $\mathcal{B}'$ is again a G-graded category by setting $\mathcal{B}'^a(x, y) := \mathcal{B}^a(x, y)$ for all $x, y \in \mathcal{B}'_0$ and all $a \in G$, and hence $(S, 1)\colon \mathcal{B}' \to \mathcal{B}$ is a degree-preserving functor. Then the assertion following by Theorem 9.5. □

Proposition 9.10 *Let $(H, r)\colon \mathcal{B} \to \mathcal{A}$ be a degree-preserving functor in G-**GrCat**. Then the following are equivalent.*

(1) *(H, r) is an equivalence in G-**GrCat**.*
(2) *There exist homogeneously dense full subcategories $\mathcal{B}'$ and $\mathcal{A}'$ of $\mathcal{B}$ and $\mathcal{A}$, respectively and a homogeneous natural isomorphism*

$$\zeta\colon (H, r) \Rightarrow (S', 1)(H', r')(N, s),$$

*where $S\colon \mathcal{B}' \hookrightarrow \mathcal{B}$ and $S'\colon \mathcal{A}' \hookrightarrow \mathcal{A}$ are inclusion functors, and (N, s) is a quasi-inverse of the equivalence $(S, 1)$ in G-**GrCat**.*

Proof (2) ⇒(1). This immediately follows by Theorem 9.5.

(1) ⇒(2). Assume the statement (1). Then there exist degree-preserving functors $(H, r)\colon \mathcal{B} \to \mathcal{A}$ and $(I, s)\colon \mathcal{A} \to \mathcal{B}$, and 2-isomorphisms $\varepsilon\colon (I, s)(H, r) \Rightarrow (\mathbb{1}_{\mathcal{B}}, 1)$ and $\eta\colon (\mathbb{1}_{\mathcal{B}}, 1) \Rightarrow (H, r)(I, s)$ in G-**GrCat**. Let $\mathcal{B}'$ be a *homogeneous skeleton* of $\mathcal{B}$. Then $\mathcal{B}'$ is homogeneously dense in $\mathcal{B}$. Let $\mathcal{A}'$ be the full subcategory of $\mathcal{A}$ with $\mathcal{A}'_0 := H(\mathcal{B}'_0)$. Then we claim that $\mathcal{A}'$ is a homogeneous skeleton of $\mathcal{A}$. Indeed, let $x \in \mathcal{A}_0$. Then by construction there exist an $x' \in \mathcal{B}'$ and a homogeneous isomorphism $f\colon Ix \to x'$ in $\mathcal{B}$. Hence we have homogeneous isomorphisms $x \xrightarrow{\eta_x} HIx \xrightarrow{Hf} Hx'$ in $\mathcal{A}$. Thus $x \cong_H Hx' \in \mathcal{A}'_0$, which shows that $\mathcal{A}'$ is homogeneously dense in $\mathcal{A}$. Next assume that there exists a homogeneous isomorphism $g\colon Hx \to Hy$ for some $x, y \in \mathcal{B}'_0$. Then we have homogeneous

isomorphisms $x \overset{\varepsilon_x}{\longleftarrow} IHx \overset{Ig}{\longrightarrow} IHy \overset{\varepsilon_y}{\longrightarrow} y$. Thus $x \cong_H y$, and hence $x = y$. As a consequence

$$x \neq y \text{ implies} Hx \ncong_H Hy. \tag{9.2}$$

This proves the claim. Now let $S\colon \mathcal{B}' \hookrightarrow \mathcal{B}$ and $S'\colon \mathcal{A}' \hookrightarrow \mathcal{A}$ be inclusion functors, and as in the proof of Theorem 9.5 construct a quasi-inverse (N, s) of the equivalence $(S, 1)$ in G-**GrCat** as a left adjoint with a counit $\mathbb{1}_{\mathbb{1}_{\mathcal{B}'}}\colon NS = \mathbb{1}_{\mathcal{B}'}$ and a unit $\nu\colon \mathbb{1}_{\mathcal{B}} \Rightarrow SN$. Then $s_x = (\deg \nu_x)^{-1}$ for all $x \in \mathcal{B}_0$. The implication (9.2) also shows that H induces a bijection $\mathcal{B}'_0 \to \mathcal{A}'_0$. As H is fully faithful, H induces a category isomorphism $H'\colon \mathcal{B}' \to \mathcal{A}'$ that satisfies $S'H' = HS$. Let r' be the restriction of r to $\mathcal{B}'_0$. Then $(H', r')\colon \mathcal{B}' \to \mathcal{A}'$ is a degree-preserving functor, which turns out to be an isomorphism in G-**GrCat** by Corollary 9.6. Now $\zeta := H\nu$ is a homogeneous natural isomorphism $H \Rightarrow HSN = S'H'N$. It remains to show that ζ is a 2-morphism $(H, r) \Rightarrow (S', 1)(H', r')(N, s) = (S'H'N, (s_x r'_{Nx})_{x \in \mathcal{B}_0})$ in G-**GrCat**. For this it is enough to show that $\deg H\nu_x = (s_x r'_{Nx})^{-1} r_x$ for all $x \in \mathcal{B}_0$. Now since $\deg \nu_x = s_x^{-1}$ and $\nu_x\colon x \to SNx = Nx$, we have $\deg H\nu_x = r_{Nx}^{-1} s_x^{-1} r_x = (s_x r'_{Nx})^{-1} r_x$, as desired. □

The following is immediate by Proposition 9.10 and Lemma 9.9.

Corollary 9.11 *Let $\mathcal{B}, \mathcal{A} \in G$-**GrCat**$_0$. Then the following are equivalent.*

(1) *$\mathcal{B} \simeq \mathcal{A}$ in G-**GrCat**.*
(2) *There exist homogeneously dense full subcategories $\mathcal{B}'$ and $\mathcal{A}'$ of $\mathcal{B}$ and $\mathcal{A}$, respectively such that $\mathcal{B}' \cong \mathcal{A}'$ in G-**GrCat**.*

Acknowledgments I would like to thank Bernhard Keller and Dai Tamaki for useful conversations. The result was announced at the workshop of LMS Midlands Regional Meeting 2009 held at University of Leicester. Section 9 was added during my stay in Bielefeld in September, 2011. I would like to thank C. M. Ringel, H. Krause and all members of the group of representation theory of algebras for nice conversations and hospitality. Finally, I would like to thank the referee for his/her useful comments to make the paper more readable.

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