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A Decomposition Formula for the Weighted Commutator

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Abstract We decompose the weighted subobject commutator of M. Gran, G. Janelidze and A. Ursini as a join of a binary and a ternary commutator.

Keywords Semi-abelian category · Finitely cocomplete homological category · Weighted · Higher-order commutator · Co-smash product · Weighted commutator

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1 Introduction

In their article [6], M. Gran, G. Janelidze and A. Ursini introduce a *weighted normal commutator* which, depending on the chosen weight, captures classical commutators such as the Huq commutator [1, 3, 9] and the Smith commutator [1, 4, 14, 15]. It is constructed as

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the normal closure of a so-called *weighted subobject commutator*. We show how this latter commutator may be decomposed as a join of a binary and a ternary commutator [7, 8] defined in terms of co-smash products [5]. We moreover explain that the corresponding concept of *weighted centrality* of arrows can be expressed in terms of the *admissibility* of certain diagrams in the first author's sense [12].

2 The Weighted Subobject Commutator

In a finitely cocomplete homological category [1, 10], a **weighted cospan** is a triple of morphisms

$$\begin{array}{ccccc} & & W & & \\ & & \big\downarrow{\scriptstyle w} & & \\ X & \xrightarrow[x]{} & D & \xleftarrow[y]{} & Y \end{array} \tag{A}$$

in which (x, y) plays the role of cospan and w is the weight. Consider the pullback

$$\begin{array}{ccccc} & & W+Y & & \\ & \overset{\pi_2}{\nearrow} & & \searrow^{\left\langle\begin{smallmatrix}1_W\\0\end{smallmatrix}\right\rangle} & \\ (W+X)\times_W(W+Y) & & & & W \\ & \underset{\pi_1}{\searrow} & & \nearrow_{\left\langle\begin{smallmatrix}1_W\\0\end{smallmatrix}\right\rangle} & \\ & & W+X & & \end{array}$$

and the induced outer diagram

$$\begin{array}{ccccc} W+X & \xrightarrow{\left\langle 1_{W+X},\,\iota_W\circ\left\langle\begin{smallmatrix}1_W\\0\end{smallmatrix}\right\rangle\right\rangle} & (W+X)\times_W(W+Y) & \xleftarrow{\left\langle \iota_W\circ\left\langle\begin{smallmatrix}1_W\\0\end{smallmatrix}\right\rangle,\,1_{W+Y}\right\rangle} & W+Y \\ & \searrow_{\left\langle\begin{smallmatrix}w\\x\end{smallmatrix}\right\rangle} & \vdots\,{\scriptstyle\varphi} & \swarrow_{\left\langle\begin{smallmatrix}w\\y\end{smallmatrix}\right\rangle} & \\ & & D. & & \end{array}$$

In [6] the morphisms x and y are said to **commute over** w if and only if there exists a dotted arrow φ (called an **internal multiplication**) such that the above diagram is commutative.

As explained in [6], taking $W = 0$ captures commuting pairs in the Huq sense (x and y commute over 0 if and only if they Huq-commute), and $w = 1_D$ captures centralising equivalence relations in the Smith sense (the respective normalisations x and y of two equivalence relations R and S on D commute over 1_D if and only if R and S Smith-commute).

Now consider the canonical comparison morphism

$$\left\langle\begin{smallmatrix}\iota_W & \iota_W\\ \iota_X & 0\\ 0 & \iota_Y\end{smallmatrix}\right\rangle\colon W+X+Y \to (W+X)\times_W(W+Y)$$

which, being a regular epimorphism [6] as the comparison between a sum and a product in the category of points over an object W in a regular Mal'tsev category, induces a short exact sequence

$$0 \longrightarrow K \rhd\!\!\longrightarrow W+X+Y \xrightarrow{\left\langle\begin{smallmatrix}\iota_W & \iota_W\\ \iota_X & 0\\ 0 & \iota_Y\end{smallmatrix}\right\rangle} (W+X)\times_W(W+Y) \longrightarrow 0. \tag{B}$$

The (W, w)**-weighted subobject commutator** $\kappa\colon [(X, x), (Y, y)]_{(W,w)} \to D$ **of** x **and** y is the direct image of K along the induced arrow to D as in

$$\begin{array}{ccc} K & \rightarrowtriangle & W+X+Y \\ \vdots & & \downarrow{\scriptstyle \left\langle \begin{smallmatrix} w \\ x \\ y \end{smallmatrix} \right\rangle} \\ \triangledown & & \\ [(X,x),(Y,y)]_{(W,w)} & \underset{\kappa}{\rightarrowtail} & D. \end{array}$$

It is clear from the exactness of the above sequence that x and y commute over w if and only if $[(X, x), (Y, y)]_{(W,w)}$ vanishes.

The normal closure of κ is called the (W, w)**-weighted normal commutator** of x and y and denoted by $N[(X, x), (Y, y)]_{(W,w)}$.

3 Admissibility

In order to analyse the weighted subobject commutator in terms of the binary and ternary commutators considered in [7, 8], we pass via an intermediate notion from [12]. An **admissibility diagram** is a diagram of shape

$$\begin{array}{ccccc} A & \underset{r}{\overset{f}{\rightleftarrows}} & B & \underset{s}{\overset{g}{\leftrightarrows}} & C \\ & {\scriptstyle \alpha}\searrow & \downarrow{\scriptstyle \beta} & \swarrow{\scriptstyle \gamma} & \\ & & D & & \end{array} \qquad (\mathbf{C})$$

with $f \circ r = 1_B = g \circ s$ and $\alpha \circ r = \beta = \gamma \circ s$. Note that by taking the pullback of f with g, any admissibility diagram such as (**C**) may be extended to

$$\begin{array}{ccccc} & & C & \xrightarrow{\quad\gamma\quad} & \\ & {\scriptstyle \pi_2}\nearrow\swarrow{\scriptstyle e_2} & & {\scriptstyle s}\nwarrow\searrow{\scriptstyle g} & \\ A \times_B C & & & & B \xrightarrow{\ \beta\ } D \\ & {\scriptstyle e_1}\nwarrow\searrow{\scriptstyle \pi_1} & & {\scriptstyle f}\nearrow\swarrow{\scriptstyle r} & \\ & & A & \xrightarrow{\quad\alpha\quad} & \end{array}$$

in which the pullback square is a double split epimorphism.

The triple (α, β, γ) is said to be **admissible with respect to** (f, r, g, s) if there is a (necessarily unique) morphism $\varphi\colon A \times_B C \to D$ such that $\varphi \circ e_1 = \alpha$ and $\varphi \circ e_2 = \gamma$.

4 Commuting Pairs in Terms of Admissibility

It is immediately clear from the definitions that the morphisms x and y commute over w if and only if the triple $\left(\left\langle \begin{smallmatrix} w \\ x \end{smallmatrix} \right\rangle, w, \left\langle \begin{smallmatrix} w \\ y \end{smallmatrix} \right\rangle\right)$ is admissible with respect to the quadruple $\left(\left\langle \begin{smallmatrix} 1_W \\ 0 \end{smallmatrix} \right\rangle, \iota_W, \left\langle \begin{smallmatrix} 1_W \\ 0 \end{smallmatrix} \right\rangle, \iota_W\right)$ as in the diagram

$$\begin{array}{ccccc} W+X & \underset{\iota_W}{\overset{\left\langle \begin{smallmatrix} 1_W \\ 0 \end{smallmatrix} \right\rangle}{\rightleftarrows}} & W & \underset{\iota_W}{\overset{\left\langle \begin{smallmatrix} 1_W \\ 0 \end{smallmatrix} \right\rangle}{\leftrightarrows}} & W+Y \\ & {\scriptstyle \left\langle \begin{smallmatrix} w \\ x \end{smallmatrix} \right\rangle}\searrow & \downarrow{\scriptstyle w} & \swarrow{\scriptstyle \left\langle \begin{smallmatrix} w \\ y \end{smallmatrix} \right\rangle} & \\ & & D. & & \end{array} \qquad (\mathbf{D})$$

5 Admissibility in Terms of Commuting Pairs

Consider a diagram (C) and the induced weighted cospan

$$\begin{array}{ccccc} & & B & & \\ & & \downarrow{\scriptstyle \beta} & & \\ X = \mathrm{Ker}(f) & \xrightarrow[\alpha \circ \ker(f)]{} & D & \xleftarrow[\gamma \circ \ker(g)]{} & \mathrm{Ker}(g) = Y. \end{array}$$

We claim that the triple (α, β, γ) is admissible with respect to (f, r, g, s) if and only if $x = \alpha \circ \ker(f)$ and $y = \gamma \circ \ker(g)$ commute over $w = \beta \colon W = B \to D$. To see this, it suffices to compare Diagram (C) with the induced Diagram (D). In fact there is a regular epimorphism of admissibility diagrams from the latter to the former which keeps D fixed and makes

$$\begin{array}{ccccc} B + X & \underset{\iota_B}{\overset{\left\langle\begin{smallmatrix} 1_B \\ 0 \end{smallmatrix}\right\rangle}{\rightleftarrows}} & B & \underset{\iota_B}{\overset{\left\langle\begin{smallmatrix} 1_B \\ 0 \end{smallmatrix}\right\rangle}{\leftrightarrows}} & B + Y \\ {\scriptstyle \left\langle\begin{smallmatrix} r \\ \ker(f) \end{smallmatrix}\right\rangle}\downarrow & & \Vert & & \downarrow{\scriptstyle \left\langle\begin{smallmatrix} s \\ \ker(g) \end{smallmatrix}\right\rangle} \\ A & \underset{r}{\overset{f}{\rightleftarrows}} & B & \underset{s}{\overset{g}{\leftrightarrows}} & C \end{array}$$

commute. This already proves the “only if” in our claim. For the “if” suppose that x and y commute over β. For the induced arrow

$$\varphi \colon (B + X) \times_B (B + Y) \to D$$

to factor over the regular epimorphism

$$\left\langle\begin{smallmatrix} r \\ \ker(f) \end{smallmatrix}\right\rangle \times_B \left\langle\begin{smallmatrix} s \\ \ker(g) \end{smallmatrix}\right\rangle \colon (B + X) \times_B (B + Y) \to A \times_B C,$$

we only need that it vanishes on $\mathrm{Ker}\big(\left\langle\begin{smallmatrix} r \\ \ker(f) \end{smallmatrix}\right\rangle\big) \times \mathrm{Ker}\big(\left\langle\begin{smallmatrix} s \\ \ker(g) \end{smallmatrix}\right\rangle\big)$. This does indeed happen, because

$$\begin{aligned} \varphi \circ \Big(\ker\big(\left\langle\begin{smallmatrix} r \\ \ker(f) \end{smallmatrix}\right\rangle\big) \times \ker\big(\left\langle\begin{smallmatrix} s \\ \ker(g) \end{smallmatrix}\right\rangle\big)\Big) \circ \langle 1, 0 \rangle &= \varphi \circ \big\langle 1_{B+X}, \iota_B \circ \left\langle\begin{smallmatrix} 1_B \\ 0 \end{smallmatrix}\right\rangle\big\rangle \circ \ker\big(\left\langle\begin{smallmatrix} r \\ \ker(f) \end{smallmatrix}\right\rangle\big) \\ &= \left\langle\begin{smallmatrix} \beta \\ x \end{smallmatrix}\right\rangle \circ \ker\big(\left\langle\begin{smallmatrix} r \\ \ker(f) \end{smallmatrix}\right\rangle\big) \\ &= \alpha \circ \left\langle\begin{smallmatrix} r \\ \ker(f) \end{smallmatrix}\right\rangle \circ \ker\big(\left\langle\begin{smallmatrix} r \\ \ker(f) \end{smallmatrix}\right\rangle\big) \end{aligned}$$

is trivial. Similarly, one can check that the arrow

$$\varphi \circ \Big(\ker\big(\left\langle\begin{smallmatrix} r \\ \ker(f) \end{smallmatrix}\right\rangle\big) \times \ker\big(\left\langle\begin{smallmatrix} s \\ \ker(g) \end{smallmatrix}\right\rangle\big)\Big) \circ \langle 0, 1 \rangle$$

is trivial.

6 Binary and Ternary Higgins Commutators

If $k \colon K \to X$ and $l \colon L \to X$ are subobjects of an object X in a finitely cocomplete homological category, then the **(Higgins) commutator** $[K, L] \leq X$ is the image of the induced morphism

$$K \diamond L \xrightarrow{\iota_{K,L}} K + L \xrightarrow{\left\langle\begin{smallmatrix} k \\ l \end{smallmatrix}\right\rangle} X,$$

where
$$K \diamond L = \mathrm{Ker}\left(\left\langle\begin{smallmatrix} 1_K & 0 \\ 0 & 1_L \end{smallmatrix}\right\rangle : K + L \to K \times L\right).$$
As explained in [6], the Higgins commutator is another special case of the weighted subobject commutator recalled above. This commutator was first introduced in [7, 11]. Higher-order versions of it exist and are studied in [7, 8].

The object $K \diamond L$, as $K \diamond L \diamond M$ below, is an example of a **co-smash product** [5]. It is worth recalling from [11] that it may be computed as the intersection $K \flat L \wedge L \flat K$, where the object $K \flat L$ from [2] is the kernel in the split exact sequence
$$0 \longrightarrow K\flat L \rightarrowtriangle K + L \underset{\iota_K}{\overset{\left\langle\begin{smallmatrix} 1_K \\ 0 \end{smallmatrix}\right\rangle}{\rightleftarrows}} K \longrightarrow 0.$$
Furthermore, also the sequence
$$0 \longrightarrow K \diamond L \rightarrowtriangle K\flat L \rightleftarrows L \longrightarrow 0 \qquad \textbf{(E)}$$
is split exact.

If $m\colon M \to X$ is another subobject of X, then the **ternary (Higgins) commutator** $[K, L, M] \leqslant X$ is defined as the image of the composite
$$K \diamond L \diamond M \xrightarrow{\iota_{K,L,M}} K + L + M \xrightarrow{\left\langle\begin{smallmatrix} k \\ l \\ m \end{smallmatrix}\right\rangle} X,$$
where $\iota_{K,L,M}$ is the kernel of the morphism
$$K + L + M \xrightarrow{\left\langle\begin{smallmatrix} i_K & i_K & 0 \\ i_L & 0 & i_L \\ 0 & i_M & i_M \end{smallmatrix}\right\rangle} (K + L) \times (K + M) \times (L + M).$$
It is well known that co-smash products are not associative, in general; furthermore, ternary co-smash products or commutators need not be decomposable into iterated binary ones: see [5, 7, 8].

Theorem 1 *Consider a weighted cospan* (**A**) *such that x and y are normal monomorphisms (= kernels) in a finitely cocomplete homological category. Then x and y commute over w precisely when the commutators $[X, Y]$ and $[X, Y, \mathrm{Im}(w)]$ vanish.*

Proof First of all we show that x and y coincide with the images of $\left\langle\begin{smallmatrix} w \\ x \end{smallmatrix}\right\rangle \circ \ker\left(\left\langle\begin{smallmatrix} 1_W \\ 0 \end{smallmatrix}\right\rangle\right)$ and $\left\langle\begin{smallmatrix} w \\ y \end{smallmatrix}\right\rangle \circ \ker\left(\left\langle\begin{smallmatrix} 1_W \\ 0 \end{smallmatrix}\right\rangle\right)$, respectively, as in (**D**). To see this, we consider the diagram with short exact rows
$$\begin{array}{ccccccccc}
 & & & \overset{\eta_X^W}{\swarrow} & X & & & & \\
 & & & & \downarrow \iota_X & & & & \\
0 & \longrightarrow & W\flat X & \xrightarrow[\kappa_{B,X}]{} & W + X & \xrightarrow{\left\langle\begin{smallmatrix} 1_W \\ 0 \end{smallmatrix}\right\rangle} & W & \longrightarrow & 0 \\
 & & \downarrow \xi & & \downarrow \left\langle\begin{smallmatrix} w \\ x \end{smallmatrix}\right\rangle & & \downarrow d\circ w & & \\
0 & \longrightarrow & X & \xrightarrow[x]{} & D & \xrightarrow[d]{} & D_0 & \longrightarrow & 0.
\end{array}$$
It is clear that $\left\langle\begin{smallmatrix} 1_W \\ 0 \end{smallmatrix}\right\rangle \circ \iota_X = 0$ induces the factorisation η_X^W of ι_X over the kernel $\kappa_{B,X}$ of $\left\langle\begin{smallmatrix} 1_W \\ 0 \end{smallmatrix}\right\rangle$. Similarly, since
$$d \circ \left\langle\begin{smallmatrix} w \\ x \end{smallmatrix}\right\rangle \circ \kappa_{B,X} = d \circ w \circ \left\langle\begin{smallmatrix} 1_W \\ 0 \end{smallmatrix}\right\rangle \circ \kappa_{B,X}$$

is trivial we obtain the dotted factorisation ξ. Now

$$x \circ \xi \circ \eta_X^W = \left\langle {w \atop x} \right\rangle \circ \kappa_{B,X} \circ \eta_X^W = \left\langle {w \atop x} \right\rangle \circ \iota_X = x,$$

so $\xi \circ \eta_X^W = 1_X$ because x is a monomorphism. In particular, ξ is a regular epimorphism. It follows that x is the image of $\left\langle {w \atop x} \right\rangle \circ \kappa_{B,X}$.

We know from the above discussion that x and y commute over w precisely when the triple $\big(\left\langle {w \atop x} \right\rangle, w, \left\langle {w \atop y} \right\rangle\big)$ is admissible with respect to $\big(\left\langle {1_W \atop 0} \right\rangle, \iota_W, \left\langle {1_W \atop 0} \right\rangle, \iota_W\big)$. Lemma 4.5 in [8] now tells us that this happens if and only if the commutators $[X, Y]$ and $[X, Y, \mathrm{Im}(w)]$ vanish. □

Via Theorem 4.6 in [8] we now recover the known result that the *Smith is Huq* condition [13] holds if and only if, for any given cospan of normal monomorphisms (x, y), the property of commuting over w is independent of the chosen weight w making (x, y, w) a weighted cospan.

We also see that the (W, w)-weighted normal commutator $N[(X, x), (Y, y)]_{(W,w)}$ of x and y is the normal closure of $[X, Y] \vee [X, Y, \mathrm{Im}(w)]$ in D, since these two normal subobjects satisfy the same universal property. We shall, however, not insist further on this, because we can obtain the following refinement (Theorem 2).

Lemma 1 *If X, Y, and W are objects in a finitely cocomplete homological category, then there is a decomposition*

$$(X + Y) \diamond W \cong \big((X \diamond Y \diamond W) \rtimes (X \diamond W)\big) \rtimes (Y \diamond W).$$

More precisely, there exists an object V and split short exact sequences

$$0 \longrightarrow V \rightarrowtail (X + Y) \diamond W \rightleftarrows Y \diamond W \longrightarrow 0$$

and

$$0 \longrightarrow X \diamond Y \diamond W \rightarrowtail V \rightleftarrows X \diamond W \longrightarrow 0.$$

Proof This Lemma 2.12 in [8], a result which was first obtained by M. Hartl and B. Loiseau. □

Theorem 2 *Given a weighted cospan (**A**) in a finitely cocomplete homological category, the (W, w)-weighted subobject commutator of monomorphisms x and y decomposes as*

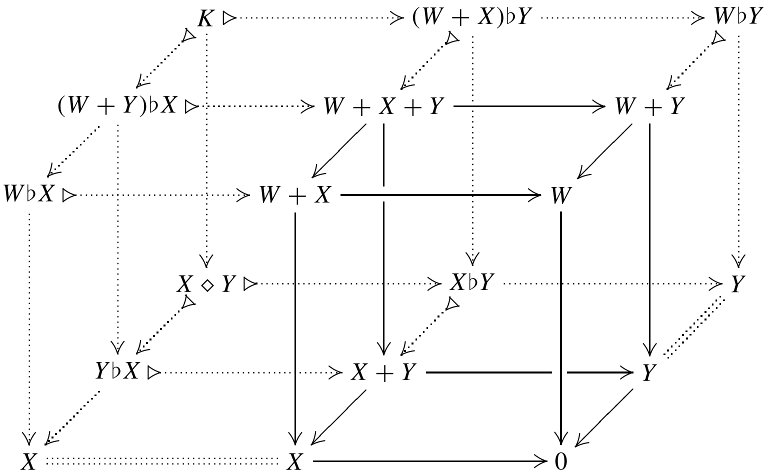
$$[(X, x), (Y, y)]_{(W,w)} = [X, Y] \vee [X, Y, \mathrm{Im}(w)].$$

Proof We decompose the kernel K of the short exact sequence (**B**) into a join of the co-smash products $X \diamond Y$ and $X \diamond Y \diamond W$ considered as subobjects of K. The result then follows from Corollary 2.14 in [7]. Indeed, the image of the composite

$$X \diamond Y \diamond W \to W + X + Y \to D$$

is $[X, Y, \mathrm{Im}(w)]$, which is a subobject of $[(X, x), (Y, y)]_{(W,w)}$. It is easily seen that also $[X, Y] \leq [(X, x), (Y, y)]_{(W,w)}$ and that these two inclusions are jointly regular epic.

Consider the cube of solid split epimorphisms



which, taking kernels horizontally, yields two 3×3 diagrams (or, equivalently, a 3×3 diagram of vertical split epimorphisms). Note that the bottom one has $X \diamond Y$, and the top one K, in its back left corner. It suffices to prove that, taking kernels vertically now, we obtain the split exact sequence

$$0 \longrightarrow X \diamond Y \diamond W \rightarrowtail K \rightleftarrows X \diamond Y \longrightarrow 0$$

in the back left corner of the induced $3 \times 3 \times 3$ diagram. Taking vertical kernels of the front and middle sections of the diagram above, we already obtain a morphism

$$\begin{array}{ccccccccc}
0 & \longrightarrow & U & \rightarrowtail & (X+Y)\flat W & \rightleftarrows & Y\flat W & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & X \diamond W & \rightarrowtail & X\flat W & \rightleftarrows & W & \longrightarrow & 0
\end{array}$$

of short exact sequences. Using (**E**) we see that the sequence

$$0 \longrightarrow U \rightarrowtail (X + Y) \diamond W \rightleftarrows Y \diamond W \longrightarrow 0$$

is split exact. Noting that V in Lemma 1 is the object U, we see that the co-smash product $X \diamond Y \diamond W$ must coincide with the kernel of $U \to X \diamond W$, which we already know coincides with the needed kernel of $K \to X \diamond Y$. $\square$

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